Notes on Complex Geometry

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1 Introduction

In this note we give an outline of the very basics of complex geometry. References:

1. Griffiths, Harris. *Principles of Algebraic Geometry.*
2. Huybrechts. *Complex Geometry.*
3. Voisin. *Hodge Theory and Complex Algebraic Geometry.*

2 Holomorphic Functions of Several Variables

2.1 Holomorphicity and Analyticity

Let $f : U \to \mathbb{C}$ be a C^1 map on an open set $U \subset \mathbb{C}^n$. For each $u \in U$, there is canonical isomorphism $T_uU \cong \mathbb{C}^n$.

Theorem 2.1. *f is **holomorphic** if any of the following equivalent conditions holds.*

- *The (real) differential*
$$df_u \in \mathrm{Hom}(T_uU, \mathbb{C}) \cong \mathrm{Hom}(\mathbb{C}^n, \mathbb{C})$$
is $\mathbb{C}$-linear at each $u \in U$.
- *f satisfies the Cauchy-Riemann equations*
$$\frac{\partial f}{\partial \bar{z_i}} = \frac{1}{2}\left(\frac{\partial f}{\partial x_i} + i\frac{\partial f}{\partial y_i}\right) = 0, \quad \forall i = 1, ..., n.$$
- *f admits a power series expansion in a neighborhood of each $z_0 \in U$:*
$$f(z_0 + z) = \sum_I \alpha_I z^I,$$
and the series converges absolutely: $\exists R_1 > 0, ..., R_n > 0$ such that
$$\sum_I |\alpha_I| r^I < \infty$$
for every $r_1 < R_1, ..., r_n < R_n$.
- *f satsifies the Cauchy integral formula: for each (open) polydisk $D_r(a)$ whose closure is contained in U, one has for each $z \in D_r(z_0)$,*
$$f(z) = \frac{1}{(2\pi i)^n} \int_{T_r(a)=\{\zeta||\zeta_i - a_i|=r_i\}} f(\zeta)\frac{d\zeta_1}{\zeta_1 - z_1} \wedge \cdots \wedge \frac{d\zeta_n}{\zeta_n - z_n},$$
where the orientation on the torus $T_r(a)$ is the product of natural orientaions on circles.

A C^1 map $f : U \subset \mathbb{C}^n \to V \subset \mathbb{C}^m$ is holomorphic if each component function $f_1, ..., f_m : U \to \mathbb{C}$ is holomorphic.

2.2 Properties of Holomorphic Functions

Using Cauchy's integral formula, one gets

Theorem 2.2 (Maximum Principle)**.** *Let $f : U \subset \mathbb{C}^n \to \mathbb{C}$ be holomorphic. If $|f|$ admits a local maximum at some point $u \in U$, then f is constant in a neighborhood of u.*

Theorem 2.3 (Analytic Continuation)**.** *Let $U \subset \mathbb{C}^n$ be a connected open subset. If $f : U \to \mathbb{C}$ is holomorphic and vanish on an open subset of U, then $f \equiv 0$ on U.*

Theorem 2.4 (Riemann Extension)**.** *Let f be a holomorphic function defined on $U \setminus \{z \mid z_1 = 0\}$ for some open set $U \subset \mathbb{C}^n$. If f is locally bounded on U, then f extends (uniquely) to a holomorphic function on U.*

Theorem 2.5 (Hartogs' Extension)**.** *Let f be a holomorphic function defined on $U \setminus \{z \mid z_1 = z_2 = 0\}$ for some open set $U \subset \mathbb{C}^n$. Then f extends (uniquely) to a holomorphic function on U.*

Theorem 2.6 (Global Extensions)**.** *On a complex manifold X, every holomorphic function that is defined on the complement of an analytic subset of codimension ≥ 1 and is locally bounded on X extends (uniquely) to a holomorphic function on all of X. Similarly, every holomorphic function defined on the complement of an analytic subset of codimension ≥ 2 extends (uniquely) to a holomorphic function on all of X.*

Theorem 2.7 (Weierstrass Preparation Theorem)**.** *Let $f : D_\varepsilon(0) \to \mathbb{C}$ be a holomorphic function on a polydisk $D_\varepsilon(0)$. If $f(0) = 0$ and $f_0(z_1) = f(z_1, 0, ..., 0) \not\equiv 0$, then there exists a smaller polydisk $D_\delta(0) \subset D_\varepsilon(0)$, and a Weierstrass polynomial $g(z_1, w) = g_w(z_1)$ and a holomorphic function h on $D_\delta(0)$ such that*

$$f = g \cdot h \qquad \text{on } D_\delta(0),$$

and $h(0) \neq 0$. Such Weierstrass polynomial g is unique.

Here a Weierstrass polynomial g has the form $g(z_1, w) = z_1^d + a_{d-1}(w) z_1^{d-1} + \cdots + a_0(w)$, where the coefficients $a_j(w)$ are holomorphic functions on the open subset $\mathbb{C}^{n-1} \cap D_\delta(0)$, and $a_j(0) = 0$.

Theorem 2.8 (Inverse Function Theorem)**.** *Let $f : U \to V$ be a holomorphic map between open subsets $U, V \subset \mathbb{C}^n$. If f is regular at $z \in U$, then there exist open subsets $z \in U' \subset U$ and $f(z) \subset V' \subset V$ such that $f : U' \to V'$ is biholomorphism.*

Theorem 2.9 (Implicit Function Theorem)**.** *Let $f : U \subset \mathbb{C}^m \to \mathbb{C}^n$ be a holomorphic map, where $m \geq n$. If $z_0 \in U$ is a point such that*

$$\det \left(\partial_j f_i(z_0) \right)_{1 \leq i,j \leq n} \neq 0,$$

then there exists open subsets $U_1 \subset \mathbb{C}^{m-n}$, $U_2 \subset \mathbb{C}^n$, and a holomorphic map $g : U_1 \to U_2$ such that $z_0 \in U_1 \times U_2 \subset U$ and

$$\{z \in U_1 \times U_2 \mid f(z) = f(z_0)\} = \{(w, g(w)) \mid w \in U_1\}.$$

Corollary 2.10. *Let $f : U \subset \mathbb{C}^m \to \mathbb{C}^n$ be a holomorphic map. Suppose f is regular at $z_0 \in U$, i.e.,* $\operatorname{rank} J(f)(z_0) = \min(m, n)$*. Then*

- *If $m \geq n$, then f is projection up to change of domain coordinate: there exists a biholomorphism $h : V \to U'$ for some $V \subset \mathbb{C}^m$ and $z_0 \in U' \subset U$ such that $f(h(z_1, ..., z_m)) = (z_1, ..., z_n)$ for all $z \in V$.*
- *If $m \leq n$, then f is inclusion up to change of image coordinate: there exists a biholomorphism $h : V \to V'$ for some $f(z_0) \in V \subset \mathbb{C}^n$ and $V' \subset \mathbb{C}^n$ such that $h(f(z_1, ..., z_m)) = (z_1, ..., z_m, 0, ..., 0)$ for all $z \in f^{-1}(V)$.*

Theorem 2.11. *The Jacobian of a biholomorphism is everywhere invertible.*

More local theory of holomorphic functions are given in the language of the stalk of sheaf of holomorphic functions.

Theorem 2.12 (Weierstrass Division Theorem)**.** *Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$ and $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ a Weierstrass polynomial of degree d. Then there exists $r \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ of degree $< d$ and $h \in \mathcal{O}_{\mathbb{C}^n,0}$ such that*

$$f = g \cdot h + r,$$

and such functions h and r are uniquely determined.

As a consequence,

Theorem 2.13. *$\mathcal{O}_{\mathbb{C}^n,0}$ is a local UFD: every element in $\mathcal{O}_{\mathbb{C}^n,0}$ can be factorized (up to a unit) as a product of Weierstrass poynomials irreducible in $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$, which are also irreducible in $\mathcal{O}_{\mathbb{C}^n,0}$. The local UFD $\mathcal{O}_{\mathbb{C}^n,0}$ is Noetherian.*

Theorem 2.14. *Let $g \in \mathcal{O}_{\mathbb{C}^n,0}$ be irreducible. If $f \in \mathcal{O}_{\mathbb{C}^n,0}$ such that f vanishes on $Z(g)$, then $g \mid f$.*

Definition 2.15. A germ $X \subset \mathbb{C}^n$ in 0 is analytic if there exists $f_1, ..., f_k \in \mathcal{O}_{\mathbb{C}^n,0}$ such that $X = Z(f_1, ..., f_k)$ as germs. An analytic germ X is irreducible if whenever $X = X_1 \cup X_2$ where X_1, X_2 are germs, then $X = X_1$ or $X = X_2$.

Proposition 2.16. *For each ideal $I \subset \mathcal{O}_{\mathbb{C}^n,0}$, $Z(I)$ is analytic. An analytic germ X is irreducible if and only if $I(X) \subset \mathcal{O}_{\mathbb{C}^n,0}$ is a prime ideal.*

Theorem 2.17 (Nullstellensatz)**.** *For each ideal $I \subset \mathcal{O}_{\mathbb{C}^n,0}$, $\sqrt{I} = I(Z(I))$.*

When an analytic germ X has codimension 1, i.e. $X = Z(f)$ for a single non-trivial function $f \in \mathcal{O}_{\mathbb{C}^n,0}$, we have $I(X) = \sqrt{(f)} = (g)$, where g is the product of irreducible factors of $f \in \mathcal{O}_{\mathbb{C}^n,0}$. We call g the defining function for the analytic germ X.

Proposition 2.18. *Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$ be irreducible. Then for all $z \in \mathbb{C}^n$ sufficiently close to 0, $f \in \mathcal{O}_{\mathbb{C}^n,z}$ is irreducible.*

3 Complex Manifolds

3.1 Manifolds and Tensor Bundles

Definition 3.1. A complex manifold is a differentiable manifold of even real dimension equipped with a complex structure: there exists a covering by open sets, which are diffeomorphic to open sets of $\mathbb{C}^n$, such that the transition diffeomorphisms are holomorphic.

Definition 3.2. An almost complex structure on a differentiable manifold is a vector bundle morphism $I : T_{X,\mathbb{R}} \to T_{X,\mathbb{R}}$ such that $I^2 = -\operatorname{Id}$. Equivalently, it is the structure of a complex vector bundle on $T_{X,\mathbb{R}}$.

Let (X, I) be an almost complex manifold. Then there is decomposition of $T_{X,\mathbb{C}}$ into eigenspaces of I with eigenvalues $\pm i$:

$$T_{X,\mathbb{C}} = T_X^{1,0} \oplus T_X^{0,1},$$
$$u \mapsto \left(\frac{u - iI(u)}{2}, \frac{u + iI(u)}{2} \right).$$

Proposition 3.3. *Let X be a complex manifold. Then $T_X^{1,0}$ is isomorphic to the holomorphic tangent bundle T_X as complex vector bundles.*

Every complex manifold induces an almost complex structure. In holomorphic coordinates ($z_i = x_i + iy_i$), I is multiplication by i, or equivalently I sends ∂_{x_i} to ∂_{y_i} and ∂_{y_i} to $-\partial_{x_i}$. We can thus identify the complex vector bundles

$$(T_{X,\mathbb{R}}, I) \cong T_X^{1,0} \cong T_X,$$
$$(\partial_{x_i}/2, \partial_{y_i}/2) \leftrightarrow (\partial_{z_i}, i\partial_{z_i}) \leftrightarrow (\partial_{z_i}, i\partial_{z_i}).$$

Theorem 3.4 (Newlander-Nirenberg)**.** *Let (X, I) be an almost complex manifold. The almost complex structure I is integrable, i.e. induced by a complex structure on X, if and only if one of the following equivalent conditions holds:*

- $[T_X^{0,1}, T_X^{0,1}] \subset T_X^{0,1}$
- $d\alpha = \partial\alpha + \overline{\partial}\alpha$ *for all* $\alpha \in \mathcal{A}^*(X)$*, i.e.* d *maps* $\mathcal{A}^{p,q}(X)$ *into* $\mathcal{A}^{p+1,q}(X) \oplus \mathcal{A}^{p,q+1}(X)$
- $d\alpha$ *has no* $(0, 2)$*-part for all* $\alpha \in \mathcal{A}^{1,0}(X)$.
- $\overline{\partial}^2\alpha = 0$ *for all* $\alpha \in \mathcal{A}^*(X)$.

Here $\mathcal{A}^{p,q}$ denotes the sheaf of (differentiable) sections of the complex vector bundle $\Omega_X^{p,q}$. See below. Some objects defined with respect to the real tensor bundles, e.g. complex structure, metric, exterior derivative, covariant derivative, connection, Lie bracket, has natural $\mathbb{C}$-linear extensions to complexified bundles $(\cdot) \otimes_{\mathbb{R}} \mathbb{C}$, and in the sequal we use them without furthur notice.

Let (X, I) be an almost complex manifold. Dual to the decomposition of $T_{X,\mathbb{C}}$, we have

$$\Omega_{X,\mathbb{C}} = \Omega_X^{1,0} \oplus \Omega_X^{0,1},$$

and hence the decomposition of complex k-forms:

$$\bigwedge^k \Omega_{X,\mathbb{C}} = \bigoplus_{p+q=k} \Omega_X^{p,q} = \bigoplus_{p+q=k} \left(\bigwedge^p \Omega_X^{1,0} \otimes \bigwedge^q \Omega_X^{0,1} \right).$$

Definition 3.5. Let X^n be a complex manifold. Define the holomorphic cotangent bundle Ω_X as the dual of the holomorphic tangent bundle T_X. Define the holomorphic vector bundle of holomorphic p-forms as $\Omega_X^p = \wedge^p \Omega_X$. The canonical bundle is $K_X := \wedge^n \Omega_X = \det(\Omega_X)$.

Definition 3.6. A complex submanifold Y of a complex manifold X is a differentiable submanifold whose tangent space, identified with a subspace of the tangent space of X, is stable under the almost complex structure on X.

By Newlander-Nirenberg and theory of holomorphic functions, this is equivalent to defining complex submanifolds of codimension k as subsets $Y \subset X$ such that for each $y \in Y$, there exists a neighborhood of y in Y that equals the vanishing set of k holomorphic functions defined in a neighborhood of y in X with $\mathbb{C}$-linearly independent differentials. When Y is embedded, this is equivalent to the existence of a holomorphic atlas $\{(U_i, \varphi_i)\}$ on X such that $\varphi_i : U_i \cap Y \cong \varphi_i(U_i) \cap \mathbb{C}^{n-k}$ for each i.

Proposition 3.7 (Adjunction Formula)**.** *Let $Y \subset X$ be a complex submanifold. Then there is natural isomorphism*

$$K_Y \cong (K_X)|_Y \otimes \det\big(N_{Y|X}\big)$$

Theorem 3.8 (Construction of Complex Manifolds via Quotient)**.** *Let $G \curvearrowright X$ be a free and proper action by a complex Lie group G on a complex manifold X. Then X/G is a complex manifold such that the quotient map $X \to X/G$ is holomorphic.*

Definition 3.9 (Analytic Subvariety)**.** An analytic subvariety of a complex manifold X is a closed subset $Y \subset X$ such that for each $x \in X$, there exists a neighborhood $x \in U \subset X$ such that $Y \cap U = Z(f_1, ..., f_k)$ for some $f_1, ..., f_k \in \mathcal{O}_X(U)$. $y \in Y$ is regular if the holomorphic map $(f_1, ..., f_k) : U \to \mathbb{C}^k$ is regular at y.

Theorem 3.10. *The regular part of an analytic subvariety of X is a complex submanifold of X. The singular part of an analytic subvariety is also an analytic subvariety of X, of strictly smaller dimension.*

Theorem 3.11. *An analytic subvariety $Z \subset X$ can be locally written as a finite union of irreducible analytic subsets. If Z is compact, then Z can be written as a finite union of irreducible analytic subvarieties.*

Theorem 3.12. *Every analytic hypersurface $Y \subset X$ can be written uniquely as the union of irreducible analytic hypersurfaces $Y = Y_1 \cup \cdots \cup Y_m$, where Y_i are closures of the connected components of Y_{reg}.*

Theorem 3.13. *An analytic subvariety $Y \subset X$ is irreducible if and only if Y_{reg} is connected.*

Theorem 3.14 (Proper Mapping Theorem)**.** *Let $f : X \to Y$ be a holomorphic map between complex manifolds. Suppose $V \subset X$ is an analytic subvariety and $f|_V$ is proper map, then $f(V) \subset Y$ is analytic subvariety.*

3.2 $\overline{\partial}$-Poincaré and Dolbeault Complex

First recall the d-Poincaré:

Theorem 3.15 (d-Poincaré)**.** *Let α be a d-closed differentiable form of strictly positive degree on a differentiable manifold. Then locally there exists a differential form β such that $d\beta = \alpha$.*

Theorem 3.16 ($\overline{\partial}$-Poincaré)**.** *Let $\alpha \in \mathcal{A}^{p,q}(X)$ be a $\overline{\partial}$-closed differential form of type (p, q) on a complex manifold X, where $q > 0$. Then locally there exists a differential form β of type $(p, q-1)$ such that $\overline{\partial}\beta = \alpha$.*

Let $\mathcal{A}_E^{p,q}$ denote the sheaf of differentiable sections of $\Omega_X^{p,q} \otimes E$, where $E \to X$ is any holomorphic vector bundle. There is Dolbeault operator

$$\overline{\partial}_E : \mathcal{A}_E^{p,q} \to \mathcal{A}_E^{p,q+1}$$

defined such that

$$\overline{\partial}_E^2 = 0,$$

$$\overline{\partial}_E(f\alpha) = \overline{\partial}f \wedge \alpha + f\overline{\partial}_E(\alpha)$$

for all smooth function $f : U \to \mathbb{C}$ and section $\alpha \in \mathcal{A}_E^{p,q}(U)$, for all open sets $U \subset X$. For any local holomorphic frame e_j of E, one has

$$\overline{\partial}_E(\alpha_j \otimes e_j) = \overline{\partial}(\alpha_j) \otimes e_j.$$

The Dolbeault complex

$$0 \to \mathcal{A}_E^{0,0} \to \mathcal{A}_E^{0,1} \to \mathcal{A}_E^{0,2} \to \dots$$

is thus a resolution of the sheaf E by $\overline{\partial}$-Poincaré.

Theorem 3.17. *Let E be a complex vector bundle over a complex manifold X. A holomorphic structure on E is uniquely determined by a $\mathbb{C}$-linear operator $\overline{\partial}_E : \mathcal{A}_E^{0,0} \to \mathcal{A}_E^{0,1}$ satisfying the Leibniz rule and the integrability condition $\overline{\partial}_E^2 = 0$.*

3.3 Kähler Metrics

Let X be a complex manifold with induced almost complex structure I.

Definition 3.18. A Hermitian metric h on X is a collection of (positive definite) Hermitian metrics h_x on complex vector spaces $(T_{x,\mathbb{R}}, I_x)$ that varies smoothly on x.

In coordinates, using $\mathbb{C}$-basis $\{\partial_{x_i}\}$ for $(T_{x,\mathbb{R}}, I)$, we can represent h by

$$h_{ij} = h(\partial_{x_i}, \partial_{x_j}),$$

where (h_{ij}) is a Hermitian matrix and h_{ij} are smooth $\mathbb{C}$-valued functions. The associated fundamental form is defined as

$$\omega = -\operatorname{Im} h \in \Omega_X^{1,1} \cap \Omega_{X,\mathbb{R}}^2.$$

In coordinates above,

$$\omega = \frac{i}{2} h_{ij} dz^i \wedge d\overline{z}^j.$$

ω is Kähler if it is d-closed. Similarly,

$$g = \operatorname{Re} h,$$

defines a Riemannian metric

$$g = \operatorname{Re}(h_{ij}) dx^i \otimes dx^j + \operatorname{Im}(h_{ij}) dx^i \otimes dy^j - \operatorname{Im}(h_{ij}) dy^i \otimes dx^j + \operatorname{Re}(h_{ij}) dy^i \otimes dy^j.$$

Moreover, h, ω, g are compatible with the almost complex structure, and

$$g(Iu, v) = \omega(u, v), \quad g(u, v) = \omega(u, Iv), \quad \forall u, v \in T_{x,\mathbb{R}}.$$

The volume form is

$$\sqrt{\det(g)} dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n = \frac{\omega^n}{n!}.$$

Theorem 3.19 (Existence of Normal Coordinate)**.** *Let (X, ω) be a Kähler manifold. For each $x \in X$, there exists a normal holomorphic coordinate near x, where ω is given by*

$$\omega = \frac{i}{2} \omega_{i\overline{j}} dz^i \wedge d\overline{z}^j,$$

such that

$$\omega_{i\overline{j}}(x) = \delta_{ij},$$

$$\frac{\partial \omega_{i\overline{j}}}{\partial z_k}(x) = \frac{\partial \omega_{i\overline{j}}}{\partial \overline{z}_k}(x) = 0.$$

3.4 Connections

Let $E \to X$ be a holomorphic vector bundle. A connection on E is an operator

$$\nabla : \mathcal{A}_E^0 \to \mathcal{A}_E^1$$

satisfying the Leibniz rule:

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla(\sigma)$$

for all smooth function $f : U \to \mathbb{C}$ and section $\sigma \in \mathcal{A}_E^0(U)$, for all open sets $U \subset X$.

Theorem 3.20 (Chern Connection)**.** *Let $(E, h) \to X$ be a holomorphic vector bundle equipped with a Hermitian metric h. There exists a unique connection ∇ on E satisfying:*

- *∇ is compatible with h, i.e.*

$$d\left(h(\sigma, \tau)\right) = h(\nabla(\sigma), \tau) + h(\sigma, \nabla(\tau)), \quad \forall \sigma, \tau \in \mathcal{A}_E^0,$$

- $\nabla^{0,1} = \overline{\partial}_E : \mathcal{A}_E^0 \to \mathcal{A}_E^{0,1}$.

Theorem 3.21. *Let h be a Hermitian metric on the holomorphic tangent bundle T_X. The following are equivalent:*

- *h is Kähler,*
- *The almost complex structure I is flat for the Levi-Civita connection:*

$$\nabla(IV) = I\nabla V, \quad \forall V \in \mathcal{A}^0_{T_{X,\mathbb{R}}},$$

- *The Chern connection and the Levi-Civita connection coincide under identification $T_X \cong (T_{X,\mathbb{R}}, I)$, identified via the map taking the real part.*

4 Sheaf and Cohomology

4.1 Presheaf and Sheaf

Definition 4.1. Let X be a topological space. A presheaf $\mathcal{F}$ of abelian groups over X is given by an abelian group $\mathcal{F}(U)$ for each open set $U \subset X$, together with a restriction morphism $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for each pair of open sets $V \subset U$, such that for each triple of open sets $W \subset V \subset U$, one has $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

A presheaf morphism consist of group morphisms over sections which commute with restrictions.

Definition 4.2. A sheaf of abelian groups is a presheaf satisfying the gluing condition: for each open set $U \subset X$ and each covering of U by open sets $V \in \mathcal{V}$, the natural map

$$\prod_V \rho_{UV} : \mathcal{F}(U) \to \prod_V \mathcal{F}(V)$$

induces an isomorphism of $\mathcal{F}(U)$ onto

$$\{(\sigma_V)_{V\in\mathcal{V}} \mid \sigma_V|_{W\cap V} = \sigma_W|_{W\cap V} \quad \forall W, V \in \mathcal{V}\}.$$

Proposition 4.3 (Universality of Sheafification)**.** *For each presheaf $\mathcal{F}$, there exists a unique sheaf $\mathcal{F}_f$ such that*

- *There is a presheaf morphism*
$$\phi : \mathcal{F} \to \mathcal{F}_f$$
- *For every presheaf morphism*
$$\psi : \mathcal{F} \to \mathcal{G}$$
where $\mathcal{G}$ is a sheaf, there exists a unique sheaf morphism $\Psi : \mathcal{F}_f \to \mathcal{G}$ such that

$$\begin{array}{ccc} \mathcal{F}_f & & \\ {\scriptstyle\phi}\uparrow & \searrow^{\Psi} & \\ \mathcal{F} & \xrightarrow[\psi]{} & \mathcal{G} \end{array}$$

In this case the sheafification induces isomorphisms on stalks:

$$\phi_x : \mathcal{F}_x \cong \mathcal{F}_{f\,x}, \quad \forall x \in X.$$

Definition 4.4. Let $\mathcal{A}$ be a sheaf of rings over X. A sheaf $\mathcal{F}$ of $\mathcal{A}$-modules is a sheaf such that

- each $\mathcal{F}(U)$ is equipped with the structure of an $\mathcal{A}(U)$-module compatible with its group structure
- the restriction morphisms $\mathcal{F}(U) \to \mathcal{F}(V)$ are morphisms of $\mathcal{A}(U)$-modules, where $\mathcal{F}(V)$ is equipped with the structure of an $\mathcal{A}(U)$-module via the restriction morphism $\mathcal{A}(U) \to \mathcal{A}(V)$.

Definition 4.5. A sheaf $\mathcal{F}$ of $\mathcal{A}$-modules is a sheaf of free $\mathcal{A}$-modules of rank n if there exists a covering of X by open sets U and isomorphisms of sheaves of $\mathcal{A}$-modules $\tau_U : \mathcal{F}|_U \cong \mathcal{A}|_U^n$ over each U.

The correspondence between a vector bundle and the sheaf of its sections establishes a bijection (in fact an equivalence of categories) between vector bundles and sheaves of free $\mathcal{A}$-modules, for some appropriate choice of the type of vector bundles and $\mathcal{A}$. For example, between holomorphic vector bundles over complex manifold X and sheaves of free $\mathcal{O}_X$-modules.

Proposition 4.6 (Injective Sheaf Morphism)**.** *Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. The following are equivalent:*

- *ϕ is injective*
- *the kernel sheaf $\ker\phi$ is zero*
- *the morphisms on stalks $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$.*

Moreover, the natural inclusion $i : \ker\phi \to \mathcal{F}$ always induces isomorphisms

$$i_x : (\ker\phi)_x \cong \ker(\phi_x), \quad \forall x \in X.$$

Proposition 4.7 (Surjective Sheaf Morphism)**.** *Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. The following are equivalent:*

- *ϕ is surjective*
- *the image sheaf $\operatorname{Im}\phi$ (sheafification of the image presheaf) is equal to $\mathcal{G}$*
- *the morphisms on stalks $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective for all $x \in X$.*

The inclusion $j : \operatorname{Im}\phi \to \mathcal{G}$ induced by the inclusion of the image presheaf into $\mathcal{G}$ always induces isomorphisms

$$j_x : (\operatorname{Im}\phi)_x \cong \operatorname{Im}(\phi_x), \quad \forall x \in X.$$

There is sheaf isomorphism

$$\mathcal{F}/\ker\phi \cong \operatorname{Im}\phi.$$

The cokernel sheaf $\operatorname{coker}\phi$ of $\phi : \mathcal{F} \to \mathcal{G}$ is the sheafification of the cokernel presheaf. We have

$$\operatorname{coker}\phi \cong \mathcal{G}/\operatorname{Im}\phi,$$

$$(\operatorname{coker}\phi)_x \cong \mathcal{G}_x/\operatorname{Im}(\phi_x).$$

The exactness of a sequence of sheaves

$$\mathcal{F} \xrightarrow[\phi]{} \mathcal{G} \xrightarrow[\psi]{} \mathcal{H}$$

at $\mathcal{G}$ means $\operatorname{Im}\phi = \ker\psi$, or equivalently the exactness of sequences

$$\mathcal{F}_x \xrightarrow[\phi_x]{} \mathcal{G}_x \xrightarrow[\psi_x]{} \mathcal{H}_x$$

for all $x \in X$.

These definitions make the category of sheaves of abelian groups an abelian category, and we can treat them just like the category of abelian groups when we talk about sequences, complexes, resolutions, etc.

4.2 Examples of Resolutions

Define the Čech resolution of a sheaf $\mathcal{F}$ as follows. Fix a countable open covering $\mathcal{U} = \{U_i\}$ of X. For each finite set $I \subset \mathbb{N}$, let $U_I := \cap_{i \in I} U_i$. Define sheaves

$$\mathcal{C}^k(\mathcal{U}, \mathcal{F}) := \prod_{|I|=k+1} \mathcal{F}_I,$$

where the sheaves $\mathcal{F}_I$ are defined via $\mathcal{F}_I(U) := \mathcal{F}(U \cap U_I)$. Define sheaf morphisms $d : \mathcal{C}^k(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^{k+1}(\mathcal{U}, \mathcal{F})$ by

$$(d\sigma)_{j_0,\ldots,j_{k+1}} = \sum_i (-1)^i \sigma_{j_0,\ldots,\hat{j}_i,\ldots,j_{k+1}}|_{U \cap U_{j_0,\ldots,j_{k+1}}}, \quad j_0 < \cdots < j_{k+1},$$

for each section $\sigma = (\sigma_I)$, $I \subset \mathbb{N}$, $|I| = k+1$, $\sigma_I \in \mathcal{F}_I(U) = \mathcal{F}(U \cap U_I)$. We then have the Čech resolution

$$0 \longrightarrow \mathcal{F} \xrightarrow{j} \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d} \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \xrightarrow{d} \mathcal{C}^2(\mathcal{U}, \mathcal{F}) \longrightarrow \ldots$$

for $\mathcal{F}$, where the injection $j : \mathcal{F} \to \mathcal{C}^0(\mathcal{U}, \mathcal{F})$ is given by

$$j(\sigma)_i = \sigma|_{U \cap U_i}, \quad \sigma \in \mathcal{F}(U).$$

Define the de Rham resolution as follows. Let X be a differentiable manifold, and $\mathbb{R}$ denote the sheaf of locally constant real-valued functions. Then we have de Rham resolution of $\mathbb{R}$:

$$0 \longrightarrow \mathbb{R} \xrightarrow{j} \Omega^0_{X,\mathbb{R}} \xrightarrow{d} \Omega^1_{X,\mathbb{R}} \xrightarrow{d} \Omega^2_{X,\mathbb{R}} \longrightarrow \ldots$$

where j is the natural inclusion, and d are exterior differentials. The exactness is given by d-Poincaré.

Define the Dolbeault resolution as follows. Let E be a holomorphic vector bundle over complex manifold X. Using the definition of $\overline{\partial}_E$ and $\overline{\partial}$-Poincaré, we have a resolution for E, the sheaf of holomorphic sections of E:

$$0 \longrightarrow E \xrightarrow{j} \mathcal{A}_E^{0,0} \xrightarrow{\overline{\partial}_E} \mathcal{A}_E^{0,1} \xrightarrow{\overline{\partial}_E} \mathcal{A}_E^{0,2} \longrightarrow \ldots$$

More generally, we have resolutions for holomorphic vector bundles $\Omega^p_X \otimes E$:

$$0 \longrightarrow \Omega^p_X \otimes E \xrightarrow{j} \mathcal{A}_E^{p,0} \xrightarrow{\overline{\partial}_E} \mathcal{A}_E^{p,1} \xrightarrow{\overline{\partial}_E} \mathcal{A}_E^{p,2} \longrightarrow \ldots$$

4.3 Functors and Derived Functors

Here are some abstract theory for the abelian category useful to the construction of sheaf cohomology.

Definition 4.8. An object I of an abelian category is injective if for every injective morphism $j : A \to B$ and for every morphism $\phi : A \to I$, there exists a morphism $\psi : B \to I$ such that $\psi \circ j = \phi$.

An abelian category has sufficiently many injective objects if every object A admits an injective morphism $j : A \to I$ for some injective object I.

Proposition 4.9. *In an abelian category having sufficiently many injective objects, every object admits an injetive resolution, i.e. a resolution $I^\cdot$ consisting of injective objects.*

Injective resolution is unique up to homotopy equivalence.

Proposition 4.10. *Let $(I^\cdot, i : A \to I^0)$ and $(J^\cdot, j : B \to J^0)$ be resolutions of A, B respectively. Let $\phi : A \to B$ be a morphism. If the second resolution is injective, there exists a morphism of complexes $\phi^\cdot : I^\cdot \to J^\cdot$ such that $\phi^0 \circ i = j \circ \phi$:*

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & A & \xrightarrow{i} & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi^0} & & \downarrow{\scriptstyle \phi^1} & & \downarrow{\scriptstyle \phi^2} & & \\
0 & \longrightarrow & B & \xrightarrow[j]{} & J^0 & \longrightarrow & J^1 & \longrightarrow & J^2 & \longrightarrow & \dots
\end{array}$$

Moreover, if we have two such morphisms $\phi^\cdot$ and $\psi^\cdot$, then there exists a homotopy $H^\cdot$ between $\phi^\cdot$ and $\psi^\cdot$.

From now on, let $\mathcal{C}, \mathcal{C}'$ be two abelian categories. Assume $\mathcal{C}$ has sufficiently many injective objects. Let F be a left-exact functor from $\mathcal{C}$ to $\mathcal{C}'$.

Theorem 4.11 (Derived Functor)**.** *For every object M of $\mathcal{C}$, there exist objects $R^iF(M)$, $i \geq 0$, in $\mathcal{C}'$, determined up to isomorphism, satisfying the following conditions:*

- $R^0F(M) = F(M)$
- *For each short exact sequence*

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

  in $\mathcal{C}$, we can construct a long exact sequence in $\mathcal{C}'$:

$$0 \longrightarrow F(A) \xrightarrow{\phi} F(B) \xrightarrow{\psi} F(C) \longrightarrow R^1F(A) \longrightarrow R^1F(B) \longrightarrow R^1F(C) \longrightarrow \dots$$

- *For every injective object I of $\mathcal{C}$, we have $R^iF(I) = 0$ for all $i > 0$.*

In this case, we can define $R^iF(A) = H^i(F(I^\cdot))$, the cohomology of the complex $F(I^\cdot)$, where $I^\cdot$ is any injective resolution of A.

Proposition 4.12 (Functoriality of R^iF)**.** *If $\phi : A \to B$ is a morphism in $\mathcal{C}$, and $I^\cdot$, $J^\cdot$ are injective resolutions of A and B respectively. Then there exists a canonical morphism induced by ϕ,*

$$R^iF(\phi) : R^iF(A) \to R^iF(B),$$

where $R^iF(A)$, $R^iF(B)$ are given by $I^\cdot$, $J^\cdot$ respectively.

We can use acyclic resolutions (weaker than injective) to compute the cohomology.

Definition 4.13. An object M of $\mathcal{C}$ is acyclic for the functor F if $R^iF(M) = 0$ for all $i > 0$.

Proposition 4.14. *Let $(M^\cdot, i : A \to M^0)$ be a resolution of A, where M^i are F-acyclic. Then $R^iF(A) = H^i(F(M^\cdot))$.*

4.4 Sheaf Cohomology

We now consider the category $\mathcal{C}$ of sheaves of abelian groups over a topological space X, and the category $\mathcal{C}'$ of abelian groups.

Lemma 4.15. *The following are true:*

- *$\mathcal{C}$ is an abelian category.*
- *$\mathcal{C}$ has sufficiently many injective objects.*
- *The functor $\Gamma : \mathcal{C} \to \mathcal{C}'$ taking the global section is left-exact. Hence write $R^i\Gamma(\mathcal{F}) =: H^i(X, \mathcal{F})$.*

Definition 4.16. A sheaf $\mathcal{F}$ is flasque if all restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ are surjective.

Proposition 4.17. *Flasque sheaves are Γ-acyclic.*

Definition 4.18. A fine sheaf $\mathcal{F}$ is a sheaf of $\mathcal{A}$-modules, where $\mathcal{A}$ is a sheaf of rings over X satisfying the partition of unity property: for each open cover $\{U_i\}$ of X, there exists $f_i \in \mathcal{A}(X)$ with compact support inside U_i, such that $\sum_i f_i = 1$ and the sum is locally finite.

Proposition 4.19. *A fine sheaf $\mathcal{F}$ is Γ-acyclic, i.e. $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.*

Theorem 4.20 (De Rham Cohomology)**.** *Let X be a C^∞ manifold. Then*

$$H^k(X, \mathbb{R}) = \frac{\ker\left(d : \Omega^k_{X,\mathbb{R}}(X) \to \Omega^{k+1}_{X,\mathbb{R}}(X)\right)}{\operatorname{Im}\left(d : \Omega^{k-1}_{X,\mathbb{R}}(X) \to \Omega^k_{X,\mathbb{R}}(X)\right)}.$$

A similar equality holds for $H^k(X, \mathbb{C})$. Hence $H^k(X, \mathbb{C}) = H^k(X, \mathbb{R}) \otimes \mathbb{C}$.

Theorem 4.21 (Dolbeault Cohomology)**.** *Let E be a holomorphic vector bundle over a complex manifold X. Then*

$$H^{p,q}(X, E) := H^q(X, \Omega^p_X \otimes E) = \frac{\ker\left(\overline{\partial}_E : \mathcal{A}^{p,q}_E(X) \to \mathcal{A}^{p,q+1}_E(X)\right)}{\operatorname{Im}\left(\overline{\partial}_E : \mathcal{A}^{p,q-1}_E(X) \to \mathcal{A}^{p,q}_E(X)\right)}.$$

In particular,

$$H^{p,q}(X) = H^{p,q}(X, \mathcal{O}_X) = \frac{\ker\left(\overline{\partial} : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X)\right)}{\operatorname{Im}\left(\overline{\partial} : \mathcal{A}^{p,q-1}(X) \to \mathcal{A}^{p,q}(X)\right)}.$$

Cohomology for general sheaves can be computed via Čech cohomology as follows. Let $\mathcal{U} = \{U_i\}$ be a countable ordered open covering of X.

Theorem 4.22 (Čech Cohomology for Nice Covering)**.** *If for each $I \subset \mathbb{N}$, $|I| < \infty$, the open sets U_I satisfy $H^q(U_I, \mathcal{F}) = 0$ for all $q > 0$, then*

$$H^q(X, \mathcal{F}) = \check{H}^q(\mathcal{U}, \mathcal{F}) := H^q(\mathcal{C}^\cdot(\mathcal{U}, \mathcal{F})(X)),$$

where $\mathcal{C}^\cdot(\mathcal{U}, \mathcal{F})$ is the Čech resolution for $\mathcal{F}$ with respect to $\mathcal{U}$ constructed above:

$$\mathcal{C}^q(\mathcal{U}, \mathcal{F})(X) = \prod_{|I|=q+1} \mathcal{F}(U_I).$$

When the open cover $\mathcal{U}$ does not satisfy the assumption above, note that by choosing an injective resolution for $\mathcal{F}$, Proposition 4.10 gives a canonical morphism

$$\check{H}^q(\mathcal{U}, \mathcal{F}) \to H^q(X, \mathcal{F}).$$

Ordering open covering by refinement, we have

Theorem 4.23 (Čech Cohomology for Arbitrary Covering)**.** *If X is separable, the morphisms $\check{H}^q(\mathcal{U},\mathcal{F}) \to H^q(X,\mathcal{F})$ for each open covering $\mathcal{U}$ induce an isomorphism*

$$\varinjlim_{\mathcal{U}} \check{H}^q(\mathcal{U},\mathcal{F}) \cong H^q(X,\mathcal{F}).$$

Theorem 4.24 (De Rham Theorems)**.** *Let X be a locally contractible topological space. There is canonical isomorphism*

$$H^q_{\text{sing}}(X,\mathbb{Z}) \cong H^q(X,\mathbb{Z}),$$

and the same result holds with $\mathbb{Z}$ replaced by any commutative ring R, identified with the locally constant sheaf R.

Moreover, when X is a differentiable manifold, there is isomorphism

$$H^q_{\text{sing}}(X,\mathbb{R}) \cong H^q_{dR}(X,\mathbb{R}) \cong \text{Hom}(H_q^{\text{sing}}(X,\mathbb{Z}),\mathbb{R}),$$

and the second isomorphism is induced by sending a closed q-form ω to the linear form

$$\int \omega : \phi \mapsto \int_{\Delta_q} \phi^*\omega,$$

for each singular q-chain $\phi : \Delta_q \to X$.

4.5 The Group H^1 and Cocycle

Let $\mathcal{F}$ be a sheaf of abelian groups over a separable topological space X. Then every element of $H^1(X,\mathcal{F})$ can be represented by a Čech cocycle for a suitable open covering of X. To see this, fix an injective sheaf morphism $\mathcal{F} \to I$, where I is injective. Then there is a short exact sequence

$$0 \to \mathcal{F} \to I \to \mathcal{G} \to 0,$$

whose associated long exact sequence of cohomology gives isomorphism

$$H^1(X,\mathcal{F}) \cong \operatorname{coker}\left(H^0(X,I) \to H^0(X,\mathcal{G})\right),$$

since $H^1(X,I) = 0$. Fix $\alpha \in H^1(X,\mathcal{F})$, which thus can be represented by a global section $\beta \in \mathcal{G}(X)$. By surjectivity of $I \to \mathcal{G}$, there exists a countable open covering $\mathcal{U} = \{U_i\}$ of X such that β lifts to sections $\beta_i \in I(U_i)$. By exactness, $\beta_{ij} := \beta_i - \beta_j$ is a section of $\mathcal{F}$ over U_{ij}. Then $\{\beta_{ij} \in \mathcal{F}(U_{ij})\}$ is a cocyle in $\mathcal{C}^1(\mathcal{U},\mathcal{F})(X)$,

$$\beta_{ij} - \beta_{ik} + \beta_{jk} = 0 \in \mathcal{F}(U_{ijk}),$$

and thus determines a class $\gamma \in \check{H}^1(\mathcal{U},\mathcal{F})$. This class does not depend on the liftings or on the representative β.

Theorem 4.25. *Let $\mathcal{A}$ be one of the sheaves of rings $\mathbb{C}$, $\mathcal{C}^0_{X,\mathbb{C}}$, $\mathcal{O}_X$ (the last one in the case where X is a complex manifold), and $\mathcal{A}^*$ the sheaf of corresponding multiplicative groups. The group $H^1(X,\mathcal{A}^*)$ is in bijection with*

- *the set of isomorphism classes of sheaves of free $\mathcal{A}$-modules of rank 1, and also with*
- *the set of isomorphism classes of complex line bundles equipped with flat, continuous, or holomorphic structures according to $\mathcal{A}$.*

In particular, this bijection is a group isomorphism $\operatorname{Pic}(X) \cong H^1(X,\mathcal{O}_X^*)$, where $\operatorname{Pic}(X)$ is the group of isomorphism classes of holomorphic line bundles over X with group operation $\otimes$.

Theorem 4.26. *Let E be a holomorphic vector bundle over a complex manifold, identified with the sheaf of its holomorphic sections. Then group $H^1(X,E)$ is in bijection with the isomorphism classes of extensions of E by the trivial bundle, i.e. of holomorphic vector bundles F containing E as a holomorphic vector subbundle such that the quotient bundle is the trivial line bundle $\mathcal{O}_X$:*

$$0 \to E \to F \to \mathcal{O}_X \to 0.$$

5 Holomorphic Vector Bundles

5.1 Divisors and Line Bundles

Let X be a complex manifold.

Definition 5.1. The (Weil) divisor group $\mathrm{Div}(X)$ is the set of locally finite formal $\mathbb{Z}$-linear combinations of irreducible analytic hypersurfaces of X, equipped with the natural group structure. Locally finite means: writing $D \in \mathrm{Div}(X)$ as $D = \sum a_i Y_i$, then for each $x \in X$, there exists an open neighborhood $x \in U \subset X$ such that there are only finitely many coefficients $a_i \neq 0$ such that $Y_i \cap U \neq \emptyset$.

Definition 5.2. A divisor $D = \sum a_i Y_i$ is effective if $a_i \geq 0$ for all i.

Let $Y \subset X$ be any irreducible analytic hypersurface, and $y \in Y$ any point. Pick any local defining function $g \in \mathcal{O}_{X,y}$ for Y near y, which is unique up to a unit. Then the order along Y at y of a holomorphic function f defined near y, $\mathrm{ord}_{Y,y}(f)$, is the largest integer a such that $g^a \mid f$ in $\mathcal{O}_{X,y}$. This order is locally independent of the point y, so that by connectedness of Y_{reg}, $\mathrm{ord}_Y(f)$ is well-defined for any $f \in \mathcal{O}_X(X)$. Similarly, there is well-defined order along Y for each global meromorphic function $f \in \mathcal{K}_X(X)$, such that

$$\mathrm{ord}_Y(f_1 f_2) = \mathrm{ord}_Y(f_1) + \mathrm{ord}_Y(f_2), \quad \forall f_1, f_2 \in \mathcal{K}_X(X).$$

For each $f \in \mathcal{K}_X^*(X)$, the associated divisor is

$$(f) = \sum_Y \mathrm{ord}_Y(f) \cdot Y \in \mathrm{Div}(X),$$

where Y ranges over all irreducible hypersurfaces of X.

The connection between meromorphic functions and irreducible hypersurfaces via order gives

Proposition 5.3 (Cartier Divisor and Weil Divisor)**.** *There exists a natural isomorphism*

$$H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \cong \mathrm{Div}(X).$$

The exact sequence

$$0 \to \mathcal{O}_X^* \to \mathcal{K}_X^* \to \mathcal{K}_X^*/\mathcal{O}_X^* \to 0$$

gives the natural group homomorphism

$$\mathcal{O} : \mathrm{Div}(X) \to H^1(X, \mathcal{O}_X^*) \cong \mathrm{Pic}(X),$$

whose kernel, called principal divisors, is the image of

$$\mathcal{K}_X^*(X) = H^0(X, \mathcal{K}_X^*) \to H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*).$$

The map $\mathcal{O}$ takes a global section of $\mathcal{K}_X^*/\mathcal{O}_X^*$ represented by $f_i \in \mathcal{K}_X^*(U_i)$ to the line bundle given by transition maps $\{\varphi_{ij} = f_i/f_j \in \mathcal{O}_X^*(U_{ij})\}$.

From above, principal divisors consist exactly of $(f) \in \mathrm{Div}(X)$ where $f \in \mathcal{K}_X^*(X)$. Two divisors are linearly equivalent if their difference is principal.

Proposition 5.4 (Pullback of Divisors)**.** *Let $f : X \to Y$ be a holomorphic map between connected complex manifolds, and suppose that f is dominant, i.e. $f(X)$ is dense in Y. The the pullback defines a group homomorphism*

$$f^* : \mathrm{Div}(Y) \to \mathrm{Div}(X),$$

such that the diagram commutes:

$$\begin{array}{ccc} \mathrm{Div}(Y) & \xrightarrow{\mathcal{O}_Y} & \mathrm{Pic}(Y) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathrm{Div}(X) & \xrightarrow[\mathcal{O}_X]{} & \mathrm{Pic}(X) \end{array}$$

Similarly, viewing a non-zero global section of a holomorphic line bundle as a global section of $\mathcal{K}_X^*/\mathcal{O}_X^*$ via line bundle trivialization, we have a map

$$Z : H^0(X, L) \setminus \{0\} \to \mathrm{Div}(X)$$

for each holomorphic line bundle $L \to X$, such that each $Z(s)$ is effective divisor, and

$$Z(s_1 \otimes s_2) = Z(s_1) + Z(s_2), \quad \forall s_i \in H^0(X, L_i) \setminus \{0\}.$$

Proposition 5.5. *Let $0 \neq s \in H^0(X, L)$, then $O(Z(s)) \cong L$.*

Proposition 5.6. *For any effective divisor $D \in \mathrm{Div}(X)$ there exists $0 \neq s \in H^0(X, \mathcal{O}(D))$ such that $Z(s) = D$.*

Corollary 5.7. *Non-trivial sections $s_1 \in H^0(X, L_1)$ and $s_2 \in H^0(X, L_2)$ define linearly equivalent divisors $Z(s_1) \sim Z(s_2)$ if and only if $L_1 \cong L_2$.*

Corollary 5.8. *The image of $\mathcal{O} : \mathrm{Div}(X) \to \mathrm{Pic}(X)$ is generated by line bundles $L \in \mathrm{Pic}(X)$ with $H^0(X, L) \neq 0$.*

Proposition 5.9. *Let Y be a smooth hypersurface of a complex manifold X, defined by a section $0 \neq s \in H^0(X, L)$ for some holomorphic line bundle $L \to X$. Then $N_{Y|X} \cong L|_Y$ and thus $K_Y \cong (K_X \otimes L)|_Y$.*

Proposition 5.10. *Let $Y \subset X$ be an irreducible hypersurface. For any $0 \neq s \in H^0(X, \mathcal{O}(Y))$ such that $Z(s) = Y$, the sheaf morphism $\mathcal{O}(-Y) \to \mathcal{O}_X$ given by $(\cdot) \otimes s$ is injective, and the image is the ideal sheaf $\mathcal{I}_Y$ of holomorphic functions vanishing on Y.*

Combined with the short exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

defining $\mathcal{O}_Y$, we have

$$0 \to \mathcal{O}(-Y) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0.$$

More generally, for each effective divisor $D = \sum a_i Y_i$, we have short exact sequences

$$0 \to \mathcal{I}_D \to \mathcal{O}_X \to \mathcal{O}_D \to 0,$$

$$0 \to \mathcal{O}(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0,$$

where $\mathcal{I}_D$ is the ideal sheaf of holomorphic functions vanishing of order at least a_i on Y_i for each i.

Proposition 5.11. *Suppose $L \to X$ is a holomorphic line bundle and $s_0, ..., s_N \in H^0(X, L)$ is a basis. Then the holomorphic map*

$$\varphi_L : X \setminus \mathrm{Bs}(L) \to \mathbb{P}^N, \quad x \mapsto [s_0(x) : \cdots : x_N(x)]$$

satisfies

$$\varphi_L^* \mathcal{O}_{\mathbb{P}^N}(1) \cong L_{X \setminus \mathrm{Bs}(L)}.$$

L is called very ample if $\mathrm{Bs}(L) = \emptyset$ and the map $\varphi_L : X \to \mathbb{P}^N$ is a holomorphic embedding. L is called ample if L^k is very ample for some $k > 0$.

5.2 Example: Projective Space and Blow-up

In this note the transition maps for a vector bundle is always written by $\varphi_{ij} = \psi_i \circ \psi_j^{-1} : U_i \cap U_j \to \mathrm{GL}(n, \mathbb{C})$.

Let $\mathbb{P}^n = \mathbb{C}^{n+1}/\mathbb{C}^*$ be the projective space with standard coordinate charts U_i, $i = 0, ..., n$. The tautological line bundle is

$$\mathcal{O}_{\mathbb{P}^n}(-1) = \{(l, z) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid z \in l\},$$

with transition function

$$\varphi_{ij} = \frac{z_i}{z_j}, \quad \text{on } U_{ij}.$$

The Fubini-Study metric on $\mathbb{P}^n$ is given in each coordinate U_i by

$$\omega|_{U_i} = \frac{1}{2i\pi} \partial\overline{\partial} \log \left(\frac{1}{1 + \sum_{i=1}^n |z_i|^2} \right).$$

More generally, given a holomorphic vector bundle E of rank $r+1$, define the projective bundle over X associated to E:

$$\mathbb{P}(E) := (E \setminus \sigma_0)/\mathbb{C}^*,$$

where $\sigma_0 \subset E$ is the zero section and $\mathbb{C}^*$ acts on each fiber of E. $\mathbb{P}(E)$ inherits trivializations and transition maps from E via the quotient $\mathbb{C}^{r+1} \setminus \{0\} \to \mathbb{P}^r$. Now we have

$$\begin{array}{ccc} \pi^* E & & E \\ \downarrow q & & \downarrow p \\ \mathbb{P}(E) & \xrightarrow{\pi} & X \end{array}$$

and define the tautological bundle $\mathcal{O}_{\mathbb{P}(E)}(-1)$ over $\mathbb{P}(E)$ to be the line subbundle of $\pi^* E$ whose fiber at $(x, l \subset E_x)$ is the line $l \subset E_x$. Then $\mathcal{O}_{\mathbb{P}(E)}(1)$ is the dual of $\mathcal{O}_{\mathbb{P}(E)}(-1)$. The restriction of $\mathcal{O}_{\mathbb{P}(E)}(1)$ to each fiber $\pi^{-1}(x) \cong \mathbb{P}^r$ is naturally isomorphic to $\mathcal{O}_{\mathbb{P}^r}(1)$.

Proposition 5.12. *If X is compact Kähler and $E \to X$ is holomorphic vector bundle. Then $\mathbb{P}(E)$ is compact Kähler.*

Proposition 5.13 (Global Sections of $\mathcal{O}(k)$)**.** *For each $k \geq 0$, $H^0(\mathbb{P}^n, \mathcal{O}(k)) \cong \mathbb{C}[x_0, ..., x_n]_k$, the space of homogeneously polynomials of degree k. These vector space isomorphisms combine to give a ring isomorphism*

$$\bigoplus_{k \geq 0} H^0(\mathbb{P}^n, \mathcal{O}(k)) \cong \mathbb{C}[x_0, ..., x_n].$$

Corollary 5.14. *For $k < 0$, $H^0(\mathbb{P}^n, \mathcal{O}(k)) = 0$.*

Proposition 5.15. *The canonical bundle of the projective space is $K_{\mathbb{P}^n} \cong \mathcal{O}(-n-1)$.*

Proposition 5.16. *Let $Y \subset \mathbb{P}^n$ be a smooth hypersurface of degree k, i.e. defined by a section $0 \neq s \in H^0(\mathbb{P}^n, \mathcal{O}(k))$. Then $K_Y \cong \mathcal{O}(k-n-1)|_Y$.*

Proposition 5.17 (Euler Sequence)**.** *On $\mathbb{P}^n$ there is a natural short exact sequence of holomorphic vector bundles*

$$0 \to \mathcal{O} \to \oplus_{j=0}^n \mathcal{O}(1) \to T_{\mathbb{P}^n} \to 0.$$

Equivalently,

$$0 \to \Omega_{\mathbb{P}^n} \to \oplus_{j=0}^n \mathcal{O}(-1) \to \mathcal{O} \to 0,$$

or twisted by $\mathcal{O}(1)$:

$$0 \to \Omega_{\mathbb{P}^n}(1) \to \oplus_{j=0}^n \mathcal{O} \to \mathcal{O}(1) \to 0.$$

Here the map $\mathcal{O} \to \oplus_{j=0}^n \mathcal{O}(1)$ is induced by the standard basis $\{x_0, ..., x_n\}$ of $H^0(\mathbb{P}^n, \mathcal{O}(1)) \cong \mathbb{C}[x_0, ..., x_n]_1$. The map $\oplus_{j=0}^n \mathcal{O}(1) \to T_{\mathbb{P}^n}$ sends a section $\sigma = (\sigma_0, ..., \sigma_n)$ to

$$\pi_* \left(\sum_i \sigma_i \frac{\partial}{\partial X_i} \right),$$

where $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$, and σ_i, which sends each point $l \in \mathbb{P}^n$ to a linear functional on $l \subset \mathbb{C}^{n+1}$, extends naturally to a linear function $X \in \mathbb{C}^{n+1} \setminus \{0\} \mapsto \sigma_i(\pi(X))(X) \in \mathbb{C}$. In particular, $x_i(X) = X_i$.

More generally, given holomorphic vector bundle $E \to X$, there is relative Euler sequence of holomorphic vector bundles over $\mathbb{P}(E)$:

$$0 \to \Omega_\pi \otimes \mathcal{O}_{\mathbb{P}(E)}(1) \to \pi^* E^* \to \mathcal{O}_{\mathbb{P}(E)}(1) \to 0.$$

Here $\Omega_\pi = T_\pi^*$ is the relative cotangent bundle, defined such that T_π is the kernel of the vector bundle morphism $T_{\mathbb{P}(E)} \to \pi^* T_X$, i.e. directions along fibers $\pi^{-1}(x)$.

Let $Y \subset X$ be a complex submanifold of codimension k. Locally there are functions $f_1, ..., f_k \in \mathcal{O}_X(U)$ with independent differentials such that $Y \cap U = Z(f_1, ..., f_k)$. Define the local blow-up along Y on U to be

$$\tilde{U}_Y = \{(Z, z) \in \mathbb{P}^{k-1} \times U \mid Z_i f_j(z) = Z_j f_i(z) \quad \forall i, j \leq k\}.$$

The local blow-ups glue together to give a complex manifold $\tilde{X}_Y$, called the blow-up of X along Y.

Proposition 5.18. *The following holds for a blow-up along $Y \subset X$:*

- *The blow-up map $\pi : \tilde{X}_Y \to X$ is holomorphic, and biholomorphic above $X - Y$.*
- *The exceptional divisor $D := \pi^{-1}(Y) \cong \mathbb{P}(N_{Y|X})$.*
- $\mathcal{O}_{\tilde{X}_Y}(-D)\big|_D \cong \mathcal{O}_{\mathbb{P}\left(N_{Y|X}\right)}(1)$.
- *If X is Kähler and Y is compact complex submanifold of X, then $\tilde{X}_Y$ is Kähler. Moreover, $\tilde{X}_Y$ is compact if X is.*

If $Y = \{x\}$ is a point, we have

- $K_{\tilde{X}_x} \cong \pi^* K_X \otimes \mathcal{O}_{\tilde{X}_x}((n-1)D)$.
- *On the blow-up neighborhood U of $x \in X$, we have $\mathcal{O}_{\tilde{X}_x}(D)\big|_{\pi^{-1}(U)} \cong p^* \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$, where $p : \pi^{-1}(U) \subset \mathbb{P}^{n-1} \times U \to \mathbb{P}^{n-1}$ is the projection.*
- *$\tilde{X}_x$ is diffeomorphic as an oriented differentiable manifold to the connected sum $X \# \overline{\mathbb{P}^n}$.*

5.3 Holomorphic Vector Bundles

Lemma 5.19. *Let $f : E \to F$ be a holomorphic vector bundle homomorphism over a complex manifold X, in particular, f is of constant rank. Then $\mathrm{Im}(f)$ is a holomorphic vector subbundle of F, and we have the short exact sequence of holomorphic vector bundles*

$$0 \to \ker(f) \to E \to \mathrm{Im}(f) \to 0,$$

$$0 \to \mathrm{Im}(f) \to F \to \mathrm{coker}(f) \to 0.$$

Lemma 5.20. *Let X be a complex manifold. Suppose there is a short exact sequence of holomorphic vector bundles*

$$0 \to E \xrightarrow{e} F \xrightarrow{f} G \to 0.$$

Then:

1. *$F \cong E \oplus G$ as complex (smooth) vector bundles.*
2. *$F \cong E \oplus G$ as holomorphic vector bundles if and only if either of the following holds:*
   (a) *There exists a holomorphic vector bundle homomorphism $\varphi : G \to F$ such that $f \circ \varphi = \mathrm{Id}_G$.*
   (b) *There exists a holomorphic vector bundle homomorphism $\varepsilon : F \to E$ such that $\varepsilon \circ e = \mathrm{Id}_E$.*

6 Hodge Theory

6.1 Laplacians

Let X^n be a compact oriented differentiable manifold with metric g. The induced metric on tensor bundles and the identification of $\mathbb{R}$ with $\wedge^n\Omega_{X,x}$ via the volume form defines the Hodge $*$-operator

$$*: \Omega^k_{X,\mathbb{R}} \cong \Omega^{n-k}_{X,\mathbb{R}}$$

as an isomorphism of vector bundles (or sheaves). We use the same notation for the isomorphism on sections:

$$*: \mathcal{A}^k(X) \cong \mathcal{A}^{n-k}(X).$$

Lemma 6.1. *1. For $\alpha, \beta \in \mathcal{A}^k(X)$, we have*

$$(\alpha, \beta) = \int_X (\alpha, \beta) \operatorname{Vol} = \int_X \alpha \wedge *\beta.$$

2. *$*^2 = (-1)^{k(n-k)}$ on $\mathcal{A}^k(X)$.*

3. *The operator $d^*: \mathcal{A}^k(X) \to \mathcal{A}^{k-1}(X)$ defined by $d^* := (-1)^k *^{-1} d* = (-1)^{n(k+1)+1} * d*$ is the formal adjoint of d:*

$$(\alpha, d^*\beta) = (d\alpha, \beta), \quad \forall \alpha \in \mathcal{A}^{k-1}(X), \beta \in \mathcal{A}^k(X).$$

4. *The Laplacian $\Delta_d := dd^* + d^*d$ is self-adjoint, and*

$$(\alpha, \Delta_d \alpha) = \|d\alpha\|^2 + \|d^*\alpha\|^2.$$

From now assume X is compact complex manifold with $\dim_{\mathbb{C}} X = n$, equipped with metric g compatible with the almost complex structure I. The associated Hermitian metric on $(T_{X,\mathbb{R}}, I)$ extends to L^2 Hermitian metrics on the complexified tensor bundles $\Omega^k_{X,\mathbb{C}}$. Extend Hodge $*$-operator $\mathbb{C}$-linearly to $\Omega^k_{X,\mathbb{C}}$ as well and we have

$$*: \Omega^{p,q}_X \cong \Omega^{n-q,n-p}_X,$$

$$(\alpha, \beta) = \int_X (\alpha, \beta) \operatorname{Vol} = \int_X \alpha \wedge *\overline{\beta}, \quad \forall \alpha, \beta \in \mathcal{A}^{p,q}(X).$$

Lemma 6.2. *The operators*

$$\partial^* := - * \overline{\partial} * : \mathcal{A}^{p+1,q}(X) \to \mathcal{A}^{p,q}(X)$$

$$\overline{\partial}^* := - * \partial * : \mathcal{A}^{p,q+1}(X) \to \mathcal{A}^{p,q}(X)$$

are formal adjoints of ∂ and $\overline{\partial}$ respectively:

$$(\alpha, \partial^*\beta) = (\partial\alpha, \beta), \quad \forall \alpha \in \mathcal{A}^{p,q}(X), \beta \in \mathcal{A}^{p+1,q}(X),$$

$$\left(\alpha, \overline{\partial}^*\beta\right) = \left(\overline{\partial}\alpha, \beta\right), \quad \forall \alpha \in \mathcal{A}^{p,q}(X), \beta \in \mathcal{A}^{p,q+1}(X).$$

Hence the Laplacians

$$\Delta_\partial = \partial\partial^* + \partial^*\partial,$$

$$\Delta_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial},$$

are self-adjoint. Also, $\ker \Delta_\partial = \ker \partial \cap \ker \partial^*$, *and* $\ker \Delta_{\overline{\partial}} = \ker \overline{\partial} \cap \ker \overline{\partial}^*$.

Suppose $(E, h) \to X$ is a holomorphic vector bundle equipped with a Hermitian metric h. We can then use g and h to define the $\mathbb{C}$-antilinear isomorphism

$$\overline{*}_E : \Omega_X^{p,q} \otimes E \cong \Omega_X^{n-p,n-q} \otimes E^*,$$

$$\alpha \otimes \sigma \mapsto (*\overline{\alpha}) \otimes (\langle \cdot, \sigma \rangle) .$$

We thus have L^2 Hermitian metric on $\mathcal{A}_E^{p,q}$:

$$(\alpha, \beta) = \int_X (\alpha, \beta) \operatorname{Vol} = \int_X \alpha \wedge \overline{*}_E \beta, \quad \forall \alpha, \beta \in \mathcal{A}_E^{p,q}(X).$$

Lemma 6.3. *The operator*

$$\overline{\partial}_E^* := -\overline{*}_{E^*} \circ \overline{\partial}_{E^*} \circ \overline{*}_E : \mathcal{A}_E^{p,q} \to \mathcal{A}_E^{p,q-1}$$

is formal adjoint of $\overline{\partial}_E$*:*

$$\left(\alpha, \overline{\partial}_E^* \beta\right) = \left(\overline{\partial}_E \alpha, \beta\right), \quad \forall \alpha \in \mathcal{A}_E^{p,q}(X), \beta \in \mathcal{A}_E^{p,q+1}(X).$$

Hence the Laplacian

$$\Delta_E = \overline{\partial}_E^* \overline{\partial}_E + \overline{\partial}_E \overline{\partial}_E^*$$

is self-adjoint and satisfies analogous properties as other Laplacians above.

6.2 Hodge Decomposition for Elliptic Differential Operators

Let E, F be two real differentiable vector bundles over differentiable manifold M. Let $\mathcal{A}^0_E$ denote the sheaf of smooth sections of E. For each differential operator

$$P : \mathcal{A}^0_E \to \mathcal{A}^0_F$$

of order k, we associate a global section σ_P of $\mathrm{Hom}(E, F) \otimes \mathrm{Sym}^k T_X$, called the symbol of the operator P, which depends only on the k-th order partial derivatives in P. P is **elliptic** if for every $m \in M$, and $0 \neq \alpha_m \in \Omega_{M,m}$, the homomorphism $\sigma_{P,m}(\alpha) : E_m \to F_m$ is injective.

Lemma 6.4. *Let X be a compact complex manifold with Hermitian metric g and Hermitian holomorphic vector bundle (E, h). The Laplacians $\Delta, \Delta_\partial, \Delta_{\overline{\partial}}, \Delta_E$ are elliptic operators. All symbols $\sigma_{P,x}$ have the form*

$$\alpha_x \in \Omega_{X,x} \mapsto -\|\alpha_x\|^2 \,\mathrm{Id}$$

except for a constant $\frac{1}{2}$ for Δ_∂ and $\Delta_{\overline{\partial}}$.

Theorem 6.5 (Demailly)**.** *Let (X, g) be a compact oriented manifold. Let $P : \mathcal{A}^0_E \to \mathcal{A}^0_F$ be an elliptic differential operator between vector bundles E and F of equal rank and equipped with metrics. Let P^* denote the formal adjoint of P. Then*

1. *$P^* : \mathcal{A}^0_F \to \mathcal{A}^0_E$ is a differential operator of the same order as P.*
2. *$\ker P \subset \mathcal{A}^0_E(X)$ is finite-dimensional.*
3. *$P(\mathcal{A}^0_E(X)) \subset \mathcal{A}^0_F(X)$ is closed and of finite codimension.*
4. *There is orthogonal decomposition with respect to the L^2 metric on $\mathcal{A}^0_E(X)$:*

$$\mathcal{A}^0_E(X) = \ker P \oplus P^* \left(\mathcal{A}^0_F(X)\right)$$

The main step in the proof is to prove the regularity: if $P^*\alpha = \beta$ in the sense of distributions and β is smooth, then α is smooth.

Theorem 6.6 (Hodge Decomposition: Riemannian Manifold Case)**.** *Let (X, g) be a compact oriented manifold. Let $\mathcal{H}^k(X, g)$ denote the vector space of Δ_d-harmonic real differential forms of degree k. There is orthogonal decomposition*

$$\mathcal{A}^k(X) = \mathcal{H}^k(X, g) \oplus d\left(\mathcal{A}^{k-1}(X)\right) \oplus d^*\left(\mathcal{A}^{k+1}(X)\right).$$

Then we have an isomorphism

$$\mathcal{H}^k(X, g) \cong H^k(X, \mathbb{R})$$

induced by each sending harmonic form to its class in the de Rham cohomology. Similarly, the complex harmonic forms $\mathcal{H}^k_{\mathbb{C}}(X, g) \cong H^k(X, \mathbb{C})$. $\mathcal{H}^k(X, g)$ is finite dimensional.

Note that in this case $*\Delta_d = \Delta_d *$, so there is isomorphism

$$* : \mathcal{H}^k(X, g) \cong \mathcal{H}^{n-k}(X, g).$$

Theorem 6.7 (Hodge Decomposition: Hermitian Manifold Case)**.** *Let X be a compact complex manifold equipped with a Hermitian metric g. Let $\mathcal{H}^{p,q}_\partial(X, g)$, $\mathcal{H}^{p,q}_{\overline{\partial}}(X, g)$ denote the vector space of Δ_∂- and $\Delta_{\overline{\partial}}$-harmonic forms in $\mathcal{A}^{p,q}(X)$, respectively. There are orthogonal decompositions*

$$\mathcal{A}^{p,q}(X) = \mathcal{H}^{p,q}_{\overline{\partial}}(X, g) \oplus \overline{\partial}\left(\mathcal{A}^{p,q-1}(X)\right) \oplus \overline{\partial}^*\left(\mathcal{A}^{p,q+1}(X)\right),$$

$$\mathcal{A}^{p,q}(X) = \mathcal{H}^{p,q}_{\partial}(X,g) \oplus \partial\left(\mathcal{A}^{p-1,q}(X)\right) \oplus \partial^*\left(\mathcal{A}^{p+1,q}(X)\right).$$

Then we have an isomorphism

$$\mathcal{H}^{p,q}_{\overline{\partial}}(X,g) \cong H^{p,q}(X)$$

induced by sending each harmonic form to its class in the Dolbeault cohomology. $\mathcal{H}^{p,q}_{\overline{\partial}}(X,g)$ and $\mathcal{H}^{p,q}_{\partial}(X,g)$ are finite dimensional.

Note that in this case $*\Delta_{\overline{\partial}} = \Delta_{\partial}*$, so there is isomorphism

$$* : \mathcal{H}^{p,q}_{\overline{\partial}}(X,g) \cong \mathcal{H}^{n-q,n-p}_{\partial}(X,g).$$

Theorem 6.8 (Hodge Decomposition: Holomorphic Vector Bundle Case)**.** *Let (E,h) be a Hermitian holomorphic vector bundle over a compact complex manifold X equipped with a Hermitian metric g. Let $\mathcal{H}^{p,q}(X,E)$ denote the vector space of Δ_E-harmonic sections in $\mathcal{A}^{p,q}_E(X)$. There is orthogonal decomposition*

$$\mathcal{A}^{p,q}_E(X) = \mathcal{H}^{p,q}(X,E) \oplus \overline{\partial}_E\left(\mathcal{A}^{p,q-1}_E(X)\right) \oplus \overline{\partial}^*_E\left(\mathcal{A}^{p,q+1}_E(X)\right).$$

Then we have an isomorphism

$$\mathcal{H}^{p,q}(X,E) \cong H^{p,q}(X,E)$$

induced by sending each harmonic section to its class in the Dolbeault cohomology. Both spaces are finite dimensional.

6.3 Duality Theorems

Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of A-modules over a topological space X, where A is a commutative ring with unit. There is natural A-module morphism

$$H^p(X,\mathcal{F}) \otimes_A H^q(X,\mathcal{G}) \to H^{p+q}(X,\mathcal{F}\otimes_A \mathcal{G}).$$

When $\mathcal{F} = \mathcal{G} = \mathcal{A}$ is a sheaf of rings and $A = \mathbb{Z}$, the map given by the product

$$\mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$$

induces homomorphisms $H^k(\mathcal{A}\otimes\mathcal{A}) \to H^k(\mathcal{A})$, which, composed with the map above, give the **cup product**

$$H^p(X,\mathcal{A}) \otimes H^q(X,\mathcal{A}) \to H^{p+q}(X,\mathcal{A}),$$

$$(\alpha,\beta) \mapsto \alpha \cup \beta.$$

If (X, g) is an n-dimensional connected compact oriented manifold, the cup product

$$H^p(X,\mathbb{R}) \otimes H^{n-p}(X,\mathbb{R}) \to H^n(X,\mathbb{R}) \cong \mathbb{R}$$

is induced by the bilinear map

$$(\alpha,\beta) \mapsto \int_M \alpha\wedge\beta$$

for closed p-form α and closed $(n-p)$-form β, once we identify $H^k(X,\mathbb{R}) \cong H^k_{DR}(X,\mathbb{R})$. This bilinear pairing

$$H^p(X,\mathbb{R}) \times H^{n-p}(X,\mathbb{R}) \to \mathbb{R}$$

is non-degenerate. The key observation is that $*$ commutes with Δ_g, and hence

$$* : \mathcal{H}^p(X,g) \cong \mathcal{H}^{n-p}(X,g).$$

Therefore we have a natural isomorphism

$$H_p(X,\mathbb{R})^* \cong H^p(X,\mathbb{R}) \cong H^{n-p}(X,\mathbb{R})^*.$$

In fact, the Poincaré duality gives a canonical isomorphism

$$H_p(X,\mathbb{Z}) \cong H^{n-p}(X,\mathbb{Z}).$$

Theorem 6.9 (Serre Duality)**.** *Let (E, h) be a Hermitian holomorphic vector bundle over a compact complex manifold X equipped with a Hermitian metric g. The bilinear pairing*

$$H^{p,q}(X,E) \times H^{n-p,n-q}(X,E^*) \to \mathbb{C}$$

induced by

$$(\alpha,\beta) \mapsto \int_X \alpha\wedge\beta, \quad \alpha \in \mathcal{A}^{p,q}_E(X) \cap \ker\overline{\partial}_E, \quad \beta \in \mathcal{A}^{n-p,n-q}_{E^*}(X) \cap \ker\overline{\partial}_{E^*}$$

is non-degenerate. Moreover, we have $\mathbb{C}$-antilinear isomorphism

$$\overline{*}_E : \mathcal{H}^{p,q}(X,E) \cong \mathcal{H}^{n-p,n-q}(X,E^*),$$

and natural $\mathbb{C}$-linear isomorphism

$$H^{p,q}(X,E) \cong H^{n-p,n-q}(X,E^*)^*.$$

6.4 Hodge Theory on Kähler Manifolds

In this section let (X, ω) be a Kähler manifold with complex dimension n.

Definition 6.10. The **Lefschetz operator** $L : \mathcal{A}^k \mapsto \mathcal{A}^{k+2}$ is defined such that $L\alpha = \alpha \wedge \omega$.

Lemma 6.11. *The* ***dual Lefschetz operator***

$$\Lambda : \mathcal{A}^k \mapsto \mathcal{A}^{k-2}$$

$$\Lambda = (-1)^k * L* = *^{-1} L*$$

is the formal adjoint of L with respect to the Hermitian metric on each $\Omega^*_{X,\mathbb{C},x}$ *induced by* ω.

Proposition 6.12 (Kähler Identities)**.** *Let* (X, ω) *be a Kähler manifold. Then*

$$\begin{aligned}
&[\overline{\partial}, L] = [\partial, L] = 0 \\
&[\overline{\partial}^*, \Lambda] = [\partial^*, \Lambda] = 0 \\
&[\overline{\partial}^*, L] = i\partial, \quad [\partial^*, L] = -i\overline{\partial} \\
&[\overline{\partial}, \Lambda] = i\partial^*, \quad [\partial, \Lambda] = -i\overline{\partial}^* \\
&\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d
\end{aligned}$$

and Δ_d *commutes with* $*, \partial, \overline{\partial}, \partial^*, \overline{\partial}^*, L$, *and* Λ.

Proof Idea. Compute in normal coordinate at a point. For line 3 and 4, it is equivalent to proving the equality of the symbols of the corresponding 1-st order differential operators, since they have no zero-order terms. □

Corollary 6.13. *Let* (X, ω) *be a Kähler manifold. Then*

$$\mathcal{H}^k(X, g) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X, g)$$

given by the bidegree decomposition.

Theorem 6.14 (Hodge Decomposition: Kähler Case)**.** *Let* (X, ω) *be a compact Kähler manifold. The decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

induced by the isomorphisms given above:

$$H^k(X, \mathbb{C}) \cong \mathcal{H}^k(X, g)$$

$$H^{p,q}(X) \cong \mathcal{H}^{p,q}(X, g)$$

$$\mathcal{H}^k(X, g) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X, g)$$

does not depend on the choice of Kähler metric ω.

The Hodge decomposition of cohomology depends essentially on the $\partial\overline{\partial}$-Lemma, which follows from the Hodge decomposition for forms.

Lemma 6.15 ($\partial\overline{\partial}$-Lemma)**.** *Let* (X, ω) *be a compact Kähler manifold. Let* $\alpha \in \mathcal{A}^{p,q}(X)$ *be d-closed. TFAE:*

1. α *is d-exact, i.e.* $\alpha = d\beta$ *for some* $\beta \in \mathcal{A}^{p+q-1}(X)$.

2. *α is ∂-exact, i.e. $\alpha = \partial\beta$ for some $\beta \in \mathcal{A}^{p-1,q}(X)$.*

3. *α is $\overline{\partial}$-exact, i.e. $\alpha = \overline{\partial}\beta$ for some $\beta \in \mathcal{A}^{p,q-1}(X)$.*

4. *α is $\partial\overline{\partial}$-exact, i.e. $\alpha = \partial\overline{\partial}\,\beta$ for some $\beta \in \mathcal{A}^{p-1,q-1}(X)$.*

By Hodge decomposition, the complex conjugation of forms induces natural maps on $H^k(X,\mathbb{C})$ such that $\overline{H^{p,q}(X)} = H^{q,p}(X)$. We can represent each class $[\alpha] \in H^{p,q}(X)$ by a d-closed (p,q)-form α, and then $\overline{[\alpha]} = [\overline{\alpha}] \in H^{q,p}(X)$.

Proposition 6.16. *Let X be a compact Kähler manifold. Then*

$$H^{1,1}(X,\mathbb{R}) \cong \frac{\{\text{d-closed real (1, 1)-forms}\}}{i\,\partial\overline{\partial}\,C^\infty(X,\mathbb{R})}.$$

6.5 Lefschetz Theorems

Let (X, ω) be a Kähler manifold with complex dimension n. Let L, Λ denote the corresponding Lefschetz and dual Lefschetz operators on the real and complexified tensor bundles. L is real operator of bidegree $(1, 1)$ in the bidegree decomposition of complexified bundles.

Lemma 6.17. $[L, \Lambda] = (k - n)\,\mathrm{Id}$, *and* $[L^i, \Lambda] = i(k - n + i - 1)L^{i-1}$ *on* $\mathcal{A}^k$.

Lemma 6.18. *The vector bundle morphism*

$$L^{n-k} : \Omega^k_{X,\mathbb{R}} \to \Omega^{2n-k}_{X,\mathbb{R}}$$

or equivalently, the operator of order zero

$$L^{n-k} : \mathcal{A}^k \to \mathcal{A}^{2n-k}$$

is an isomorphism. In the complexified case, there is isomorphism

$$L^{n-k} : \mathcal{A}^{p,q} \to \mathcal{A}^{n-q,n-p}$$

for each $p + q = k \leq n$, *and hence isomorphism*

$$L^{n-k} : \mathcal{H}^{p,q}(X, g) \cong \mathcal{H}^{n-q,n-p}(X, g).$$

Definition 6.19. An element $\alpha \in \Omega^k_{X,\mathbb{R},x}$ (or $\Omega^k_{X,\mathbb{C},x}$) is primitive if $\Lambda\alpha = 0$. Define $\Omega^k_{X,\mathbb{R},x,p}$ to be the primitive elements in $\Omega^k_{X,\mathbb{R},x}$.

Proposition 6.20 (Lefschetz Decomposition on Forms)**.** *For each* k, *there is orthogonal decomposition*

$$\Omega^k_{X,\mathbb{R},x} = \bigoplus_{r \geq 0} L^r \Omega^{k-2r}_{X,\mathbb{R},x,p}$$

with respect to the induced metric on forms. Moreover,

1. $\Omega^k_{X,\mathbb{R},x,p} = 0$ *if* $k > n$.
2. $\Omega^k_{X,\mathbb{R},x,p} = \ker L^{n-k+1}$ *if* $k \leq n$.

Therefore, we have orthogonal decomposition

$$\mathcal{A}^k = \bigoplus_{r \geq 0} L^r \mathcal{A}^{k-2r}_p.$$

In the complexified case, we have orthogonal decomposition

$$\mathcal{A}^{p,q} = \bigoplus_{r \geq 0} L^r \mathcal{A}^{p-r,q-r}_p.$$

Assume now (X, ω) is comapct Kähler manifold of complex dimension n. Let

$$L : H^k(X, \mathbb{R}) \to H^{k+2}(X, \mathbb{R})$$
$$\Lambda : H^k(X, \mathbb{R}) \to H^{k-2}(X, \mathbb{R})$$

denote the Lefschetz and dual Lefschetz operators on cohomology. Then $L[\alpha] = [\alpha \wedge \omega]$ for each d-closed $\alpha \in \mathcal{A}^k(X)$. $\Lambda[\alpha] = [\Lambda\alpha]$ for each harmonic $\alpha \in \mathcal{H}^k(X, g)$. These operators depend only on the cohomology class $[\omega] \in H^2(X, \mathbb{R})$. By Hodge decomposition, in the complexified case,

$$L : H^{p,q}(X) \to H^{p+1,q+1}(X)$$
$$\Lambda : H^{p,q}(X) \to H^{p-1,q-1}(X)$$

Let $H^k(X, \mathbb{R})_p$, $H^k(X, \mathbb{C})_p$, $H^{p,q}(X)_p$, $H^{p,p}(X, \mathbb{R})_p$ denote the primitive classes, i.e. kernel of Λ. Then

$$H^k(X, \mathbb{C})_p = H^k(X, \mathbb{R})_p \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}(X)_p.$$

Theorem 6.21 (Hard Lefschetz)**.** *For each $k \leq n$,*

$$L^{n-k} : H^k(X, \mathbb{R}) \to H^{2n-k}(X, \mathbb{R})$$

is an isomorphism. For any k, there is decomposition

$$H^k(X, \mathbb{R}) = \bigoplus_{r \geq 0} L^r H^{k-2r}(X, \mathbb{R})_p$$

$$H^k(X, \mathbb{C}) = \bigoplus_{r \geq 0} L^r H^{k-2r}(X, \mathbb{C})_p$$

$$H^{p,q}(X) = \bigoplus_{r \geq 0} L^r H^{p-r,q-r}(X)_p$$

$$H^{p,p}(X, \mathbb{R}) = \bigoplus_{r \geq 0} L^r H^{p-r,p-r}(X, \mathbb{R})_p$$

Proposition 6.22. *For each primitive element $\alpha \in \Omega^{p,q}_{X,x,p} \subset \Omega^k_{X,\mathbb{C},x}$, we have*

$$*\alpha = (-1)^{\frac{k(k+1)}{2}} i^{p-q} \frac{1}{(n-k)!} L^{n-k} \alpha.$$

Define the intersection form Q on $H^k(X, \mathbb{R})$ for each $k \leq n$:

$$Q(\alpha, \beta) = \langle L^{n-k}\alpha, \beta \rangle = \int_X \omega^{n-k} \wedge \alpha \wedge \beta.$$

Then $H_k(\alpha, \beta) = i^k Q(\alpha, \overline{\beta})$ is a Hermitian form on $H^k(X, \mathbb{C})$.

Proposition 6.23. *The Hard Lefschetz decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{r \geq 0} L^r H^{k-2r}(X, \mathbb{C})_p$$

is orthogonal with respect to H_k for each $k \leq n$.

Theorem 6.24 (Hodge-Riemann Bilinear Relation)**.** *The Hodge decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

is orthogonal with respect to H_k for each $k \leq n$. Moreover, for each $p + q = k \leq n$, the form

$$(-1)^{\frac{k(k-1)}{2}} i^{p-q-k} H_k$$

is positive definite on $H^{p,q}(X)_p$:

$$(-1)^{\frac{k(k-1)}{2}} i^{p-q} \int_X \omega^{n-k} \wedge \alpha \wedge \overline{\alpha} > 0, \quad \forall 0 \neq \alpha \in H^{p,q}(X)_p.$$

Proof Idea. A cohomology class $[\alpha]$ is primitive if and only its unique harmonic representative α is primitive form. Relate the form to $\|\alpha\|_2^2 = \int_X \alpha \wedge *\overline{\alpha}$. ☐

Corollary 6.25 (Hodge Index Theorem)**.** *Let X be a compact Kähler surface. The intersection form Q on $H^2(X, \mathbb{R})$:*

$$Q(\alpha, \beta) = \int_X \alpha \wedge \beta$$

has index $(2h^{2,0}(X) + 1, h^{1,1}(X) - 1)$. The restriction to $H^{1,1}(X, \mathbb{R})$ is of index $(1, h^{1,1}(X) - 1)$.

Corollary 6.26. *Let X be a compact Kähler manifold of even complex dimension n. The signature of the intersection form Q on $H^n(X,\mathbb{R})$ is*

$$\operatorname{sgn}(Q) = \sum_{p,q=0}^{n} (-1)^p h^{p,q}(X).$$

On a compact Kähler manifold (X^n,ω), some useful relations between (primitive) Hodge numbers include:

1. $h^{p,q} = h^{q,p}$ by complex conjugation,
2. $h^{p,q} = h^{n-p,n-q}$ by Serre duality,
3. $h^{p,q} = h^{n-q,n-p}$ by Hodge $*$-operator,
4. $h^{p,q} = \sum_{r\geq 0} h_p^{p-r,q-r}$ for $p+q \leq n$ by Hard Lefschetz.

Proposition 6.27. *Let X be a compact Kähler manifold. The two maps*

$$H^2(X,\mathbb{C}) \to H^{0,2}(X) = H^2(X,\mathcal{O}_X)$$

induced by

- *sheaf inclusion $\mathbb{C} \to \mathcal{O}_X$*
- *projection via Hodge decomposition*

coincide.

Proof Idea. The map on cohomology induced by morphism of sheaves can be described by any morphism of resolutions between these sheaves. Use de Rham resolution for $\mathbb{C}$ and Dolbeault resolution for $\mathcal{O}_X$ with morphism of resolution $\Pi^{0,k} : \mathcal{A}^k \to \mathcal{A}^{0,k}$. □

Theorem 6.28 (Lefschetz Theorem on $(1,1)$-Classes)**.** *Let X be a compact Kähler manifold. The image $\mathrm{NS}(X)$ (**Neron-Severi group**) of map $c_1 : \mathrm{Pic}(X) = H^1(X,\mathcal{O}_X^*) \to H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})$ induced by the short exact sequence*

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

equals $H^{1,1}(X,\mathbb{Z}) = H^{1,1}(X) \cap \operatorname{Im}\big(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})\big)$.

7 Connection and Curvature

7.1 Connection

Let (E, h) be a Hermitian vector bundle of (complex) rank r over a smooth manifold X. In local trivialization of E over $U \subset X$, h is given by a Hermitian matrix $H_U : U \to \mathrm{GL}(r, \mathbb{C})$. Let $\psi_{UV} : U \cap V \to \mathrm{GL}(r, \mathbb{C})$ denote the transition maps of E from V to U, then

$$H_V = \psi_{UV}^{\top} H_U \overline{\psi_{UV}}.$$

The following are basic constructions of Hermitian structures.

1. If E, F are Hermitian vector bundles, then there are natural Hermitian structures on $E \oplus F$, $E \otimes F$, $\mathrm{Hom}(E, F)$. In particular, (E^*, g) induced from (E, h) is given locally by Hermitian matrix

$$G_U = \overline{H_U}^{-1} = H_U^{-\top}.$$

   There is natural $\mathbb{C}$-antilinear isomorphism $E \cong E^*$ as real vector bundles.

2. If (X, g) is Hermitian manifold, then the Hermitian metric g on $(T_{X,\mathbb{R}}, I)$ induce natural Hermitian metrics on all complexified tensor bundles of X, e.g. $\bigwedge^{p,q} X$.

3. If (E, h) is Hermitian vector bundle and $F \subset E$ subbundle, then h restricts to a Hermitian metric on F and its orthogonal complement $F^{\perp}$ with respect to h. Since $E = F \oplus F^{\perp}$ and $F^{\perp} \cong E/F$, h induces Hermitian metric on E/F.

4. Let $f : X \to Y$ be a smooth map and $(E, h) \to Y$. Then $f^*E \to X$ has natural Hermitian structure f^*h given by $(f^*h)_x = h_{f(x)}$ on $(f^*E)_x = E_{f(x)}$. In local trivialization, the Hermitian matrix is $f^*H_U : f^{-1}(U) \to \mathrm{GL}(r, \mathbb{C})$ for each $U \subset Y$.

5. Let $L \to X$ be a holomorphic line bundle. Suppose L is globally generated by sections $s_1, ..., s_N \in H^0(X, L)$, then these sections induce a Hermitian structure on L, given in each local trivialization $\psi_U : L_U \to U \times \mathbb{C}$ by

$$H_U = \frac{1}{\sum_i |\psi_U(s_i)|^2}.$$

   An important example is $\mathcal{O}_{\mathbb{P}^n}(1)$ globally generated by $z_0, ..., z_n \in H^0(\mathbb{P}^n, \mathcal{O}(1))$.

Assume $\pi : E \to X$ is a complex vector bundle over a real manifold X.

Definition 7.1. A connection on E is a $\mathbb{C}$-linear sheaf morphism

$$\nabla : \mathcal{A}_E^0 \to \mathcal{A}_E^1$$

satisfying the Leibniz rule:

$$\nabla(f\sigma) = df \otimes \sigma + f \nabla \sigma$$

for all $U \subset X$ open, $\sigma \in \mathcal{A}_E^0(U)$, $f \in \mathcal{A}^0(U)$. A section σ of E is parallel if $\nabla(\sigma) = 0$.

As a sheaf morphism, $\nabla(\sigma)(x) \in \Omega_x^1 \otimes E_x$ depends only on the germ of σ at x.

Proposition 7.2. *The set of all connections on a vector bundle E is naturally an affine space over the complex vector space $\mathcal{A}^1_{\mathrm{End}(E)}(X)$.*

Proof Idea. The difference between any two connections is $\mathcal{A}^0$-linear. Use local frame for E to find the representation of $\nabla - \nabla'$ in $\mathcal{A}^1_{\mathrm{End}(E)}(X)$. □

Locally choose frame $\sigma_1, ..., \sigma_r$ for E on $U \subset X$. Then we can write $\nabla = d + A$ with respect to this frame, i.e.

$$\nabla \left(f^i \sigma_i\right) = df^i \otimes \sigma_i + f^i A_i^j \sigma_j, \quad A_i^j \in \mathcal{A}^1(U), \quad \nabla \sigma_i = A_i^j \otimes \sigma_j.$$

Some contructions of connections:

1. Given (E_i, ∇_i) for $i = 1, 2$, define connection on $E_1 \oplus E_2$ by

$$\nabla(s_1 \oplus s_2) = \nabla_1(s_1) \oplus \nabla_2(s_2).$$

2. As above, define connection on $E_1 \otimes E_2$ by

$$\nabla(s_1 \otimes s_2) = \nabla_1(s_1) \otimes s_2 + s_1 \otimes \nabla_2(s_2).$$

3. As above, define connection on $E_1^* \otimes E_2$ by

$$\nabla(f)(s_1) = \nabla_2(f(s_1)) - f\nabla_1(s_1).$$

   This follows from below.

4. Given (E, ∇), define connection ∇^* on E^* by

$$\nabla^*(\omega)(\sigma) = d(\omega(\sigma)) - \omega(\nabla(\sigma)), \quad \omega \in \mathcal{A}^0_{E^*}(U), \quad \sigma \in \mathcal{A}^0_E(U).$$

   If ω^i is the local dual frame to σ_i for E^*, then

$$\nabla^* \omega^i = -A_j^i \otimes \omega^j, \quad \nabla \sigma_i = A_i^j \otimes \sigma_j.$$

5. Given $f : X \to Y$ and $(E, \nabla) \to Y$, define the pullback connection $(f^*E, f^*\nabla) \to X$ as follows. Locally choose frame $\sigma_1, ...\sigma_r$ for E on $U \subset Y$, with pullback frame $f^*\sigma_i$ for f^*E on $f^{-1}(U)$. Then let

$$(f^*\nabla)(f^*\sigma_i) = f^*A_i^j f^*\sigma_j.$$

   Since $f^* \circ d = d \circ f^*$ on forms, this is well-defined and glue to a sheaf morphism $f^*\nabla$.

6. Given $(E = E_1 \oplus E_2, \nabla)$, define connections ∇_i on E_i by

$$\nabla_i(s_i) = \pi_i(\nabla(s_i)),$$

   where $\pi_i : E \to E_i$ is the projection.

Lemma 7.3. *Let E, F, G be vector bundles with connections. Then for any open set $U \subset X$, $\sigma \in \mathcal{A}^0_{E^* \otimes F}(U)$, $\tau \in \mathcal{A}^0_{E \otimes G}(U)$, the induced connections satsify*

$$\nabla\langle\sigma, \tau\rangle = \langle\nabla\sigma, \tau\rangle + \langle\sigma, \nabla\tau\rangle,$$

where $\langle\cdot, \cdot\rangle$ denotes the tensor of sections followed by contraction of E^* with E.

Definition 7.4. The second fundamental form of vector bundle pair $E_1 \subset E$ with respect to a connection ∇ on E is the section $\mathrm{II} \in \mathcal{A}^1(X, \mathrm{Hom}(E_1, E/E_1))$ defined for any local section $s \in \mathcal{A}^0_{E_1}(U)$ by

$$\mathrm{II}(s) = \pi_{E/E_1}\left(\nabla(s)\right)$$

If the vector bundle splits as $E = E_1 \oplus E_2$ with $E_2 \cong E/E_1$, then $\mathrm{II} = \nabla - \nabla_1$. We can always fix a Hermitian metric on E to produce splitting $E = E_1 \oplus E_1^{\perp}$ as Hermitian vector bundles.

Definition 7.5. Let (E, h) be a Hermitian vector bundle. A connection ∇ on E is compactible with the Hermitian structure h if

$$d\left(h(s_1, s_2)\right) = h(\nabla(s_1), s_2) + h(s_1, \nabla(s_2))$$

for every open set $U \subset X$, for all $s_1, s_2 \in \mathcal{A}^0_E(U)$.

Lemma 7.6. *Let (E, h) be a Hermitian vector bundle over X. In an orthonormal frame $\{\sigma_i\}$, the Hermitian connection matrix $\nabla(\sigma_i) = A_i^j \otimes \sigma_j$ is skew-Hermitian: $A^* = -A$. Moreover, for each $x_0 \in X$, there exists a local orthonormal frame $\{\sigma_i\}$ such that the Hermitian connection matrix vanishes at x_0: $A(x_0) = 0$.*

Proposition 7.7. *Let (E, h) be a Hermitian vector bundle. Let $\mathfrak{u}(E, h)$ denote the real vector subbundle of* $\mathrm{End}(E)$*:*

$$a \in \mathfrak{u}(E, h)_x \iff h_x(au, v) + h_x(u, av) = 0 \quad \forall u, v \in E_x.$$

Then the set of connections on E compatible with h is naturally an affine space over the real vector space $\mathcal{A}^1_{\mathfrak{u}(E,h)}(X)$.

Definition 7.8. Let $E \to X$ be a holomorphic vector bundle over a complex manifold X. A connection ∇ on E is compatible with the holomorphic structure if $\nabla^{0,1} = \overline{\partial}_E$, where $\nabla = \nabla^{1,0} \oplus \nabla^{0,1}$ is the bidegree decomposition of $\mathcal{A}^1_E = \mathcal{A}^{1,0}_E \oplus \mathcal{A}^{0,1}_E$.

Proposition 7.9. *Let $E \to X$ be a holomorphic vector bundle. The set of connections on E compatible with the holomorphic structure is naturally an affine space over the complex vector space $\mathcal{A}^{1,0}_{\mathrm{End}(E)}(X)$.*

Theorem 7.10 (Chern Connection)**.** *Let (E, h) be a holomorphic Hermitian vector bundle. There exists a unique connection on E that is compatible with both the Hermitian structure h and the holomorphic structure.*

Proof Idea. Uniqueness: $\mathcal{A}^{1,0}_{\mathrm{End}(E)}(X) \cap \mathcal{A}^1_{\mathfrak{u}(E,h)}(X) = 0$. This is in fact true on each fiber of $\Omega^1_{X,\mathbb{C}} \otimes \mathrm{End}(E)$, so we have local uniqueness that is useful below.

Existence: Choose local holomorphic frame $\sigma_1, ..., \sigma_r$ for E on $U \subset X$. Write $\nabla(\sigma_i) = A_i^j \otimes \sigma_j$, and we must have

$$A_i^j = \partial h_{ik} \cdot h^{jk} \in \mathcal{A}^{1,0}(U).$$

This uniquely determines a local connection ∇_U. Then by uniqueness, these local definitions glue. That is, we can determine $\nabla(s)$ by $\nabla(s)|_U = \nabla_U(s|_U)$. ☐

Example 7.11. When (X, g) is Kähler, $E = T_X$ is the holomorphic tangent bundle with Hermitian metric induced from g, then in holomorphic coordinate (z_i) we can compute the Chern connection

$$\nabla(\partial_{z_j})(\partial_{z_i}) = \Gamma^k_{ij} \partial_{z_k},$$

$$\Gamma^k_{ij} = \partial_i g_{j\bar{l}} \cdot g^{k\bar{l}}.$$

Example 7.12. Let (E, h) be a Hermitian vector bundle, and assume $E = E_1 \oplus E_2$ is orthogonal decomposition. Then (E_i, h_i) for $i = 1, 2$ are Hermitian vector bundles induced from (E, h).

If ∇ is a connection on E compatible with h, then the induced connections ∇_i on E_i are compatible with h_i. The second fundamental forms II_i satisfies

$$h_1(s_1, \mathrm{II}_2(s_2)) + h_2(\mathrm{II}_1(s_1), s_2) = 0$$

for all sections $s_i \in \mathcal{A}^0_{E_i}(U)$ on any open set $U \subset X$.

Suppose $E = E_1 \oplus E_2$ as complex vector bundles, and $E_1 \subset E$ is a pair of holomorphic vector bundles with holomorphic inclusion. Then given a connection ∇ on E with $\nabla^{0,1} = \overline{\partial}_E$, we have $\nabla_1^{0,1} = \overline{\partial}_{E_1}$, and hence

$$\mathrm{II}_1 = \nabla - \nabla_1 \in \mathcal{A}^{1,0}_{\mathrm{Hom}(E_1,E_2)}(X).$$

Definition 7.13 (Holomorphic Connection)**.** Let $E \to X$ be a holomorhic vector bundle over a complex manifold X. Let Ω_X denote the holomorphic cotagent bundle. A holomorphic connection on E is a $\mathbb{C}$-linear sheaf morphism

$$D : E \to \Omega_X \otimes E$$

satisfying the Leibnize rule

$$D(f\sigma) = \partial f \otimes \sigma + fD(\sigma)$$

for all $U \subset X$ open, $\sigma \in E(U)$, $f \in \mathcal{O}_X(U)$.

As above, the difference between any two holomorphic connections is $\mathcal{O}_X$-linear, so we can work in local holomorphic frame to represent it by a global section of the holomorphic vector bundle $\Omega_X \otimes \mathrm{End}(E)$:

$$D - D' \in (\Omega_X \otimes \mathrm{End}(E))\,(X)$$

Each holomorphic connection D extends to a $\mathbb{C}$-linear sheaf morphism $D : \mathcal{A}^0_E \to \mathcal{A}^{1,0}_E$ satisfying $D(f\sigma) = \partial f \otimes \sigma + fD(\sigma)$ for all smooth function f and holomorphic section σ. Unlike connections or Hermitian connections which always exist by partition of unity, the existence of holomorphic connection depends on a topological invariant of $E \to X$.

Definition 7.14 (Atiyah Class)**.** Let $E \to X$ be a holomorhic vector bundle over a complex manifold X. Let $\mathcal{U} = \{U_i\}$ be any open covering of X that provides local trivializations of E:

$$\psi_i : E|_{U_i} \cong U_i \times \mathbb{C}^r, \quad \psi_{ij} = \psi_i \circ \psi_j^{-1} : U_{ij} \to \mathrm{GL}(r, \mathbb{C}).$$

The Atiyah class $A(E) \in H^1(X, \Omega_X \otimes \mathrm{End}(E))$ is given by the Čech cocycle

$$\{(U_{ij}), \psi_j^{-1} \circ \left(\psi_{ij}^{-1} d\psi_{ij}\right) \circ \psi_j \in (\Omega_X \otimes \mathrm{End}(E))\,(U_{ij})\}.$$

The Atiyah class is well-defined, independent of the choice of trivializations.

Locally, choose holomorphic frame σ_i for E and its dual frame ω^i on U_j. Let A denote the matrix of holomorphic functions ψ_{ij}. Then the cocyle

$$\psi_j^{-1} \circ \left(\psi_{ij}^{-1} d\psi_{ij}\right) \circ \psi_j = A^{\alpha k} \frac{\partial A_{k\beta}}{\partial z^l} dz^l \otimes \omega^\beta \otimes \sigma_\alpha.$$

If E is a holomorphic line bundle, the cocyle simplifies to

$$\{(U_{ij}), \partial \log(\psi_{ij}) \in \Omega_X(U_{ij})\} \in H^{1,1}(X)$$

Theorem 7.15 (Exitence of Holomorphic Connection)**.** *Let $E \to X$ be a holomorhic vector bundle over a complex manifold X. A holomorphic connection on E exists if and only if the Atiyah class $A(E) = 0 \in H^1(X, \Omega_X \otimes \mathrm{End}(E))$.*

Proof Idea. Holomorphic connections on U_i are parametrized exactly by sections $\alpha_i \in (\Omega_X \otimes \mathrm{End}(E))\,(U_i)$. They agree on overlaps U_{ij} and hence glue to a global holomorphic connection, if and only if $\delta(\alpha_i)$ equals the Čech cocyle (with respect to this choice of trivializations) defining $A(E)$, where δ is the chain map in the Čech resolution. ☐

7.2 Curvature

Let E be a complex vector bundle over a differentiable manifold M. Each connection $\nabla : \mathcal{A}^0_E \to \mathcal{A}^1_E$ has natural extension

$$\nabla : \mathcal{A}^k_E \to \mathcal{A}^{k+1}_E$$

defined by

$$\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^{|\alpha|}\alpha \wedge \nabla(s)$$

for each open set $U \subset X$ and $\alpha \in \mathcal{A}^k(U)$, $s \in \mathcal{A}^0_E(U)$. Then ∇ satisfies generalized Leibniz rule

$$\nabla(\alpha \wedge s) = d\alpha \wedge s + (-1)^{|\alpha|}\alpha \wedge \nabla(s)$$

for each open set $U \subset X$ and $\alpha \in \mathcal{A}^k(U)$, $s \in \mathcal{A}^l_E(U)$. The natural extension of $d : \mathcal{A}^0 \to \mathcal{A}^1$ is the exterior differential $d : \mathcal{A}^k \to \mathcal{A}^{k+1}$.

Definition 7.16. The curvature of a connection (E, ∇) is the sheaf morphism

$$F_\nabla := \nabla \circ \nabla : \mathcal{A}^0_E \to \mathcal{A}^2_E.$$

Since F_∇ is $\mathcal{A}^0$-linear, we can describe curvature by a global section

$$F_\nabla \in \mathcal{A}^2_{\mathrm{End}(E)}(X).$$

In local frame $\{\sigma_i\}$ for E, writing $\nabla(\sigma_i) = A^j_i \otimes \sigma_j$ for $A^j_i \in \mathcal{A}^1(U)$, we have

$$F_\nabla(\sigma_i) = \left(d(A^j_i) - A^k_i \wedge A^j_k\right)\sigma_j.$$

Use induced connections on E^* and $\mathrm{End}(E)$ to get the **Bianchi identity**

$$\nabla(F_\nabla) = 0 \in \mathcal{A}^3_{\mathrm{End}(E)}(X).$$

For each $\sigma \in \mathcal{A}^k_E(U)$, $\nabla^2(\sigma) = F_\nabla(\sigma) \in \mathcal{A}^{k+2}_E(U)$.

Lemma 7.17. *Given $(E, \nabla), (E_1, \nabla_1), (E_2, \nabla_2)$, the curvature on the following vector bundles are given by:*

1. *On $E_1 \oplus E_2$,*
$$F = F_{\nabla_1} \oplus F_{\nabla_2}$$
2. *On $E_1 \otimes E_2$,*
$$F = F_{\nabla_1} \otimes \mathrm{Id}_{E_2} + \mathrm{Id}_{E_1} \otimes F_{\nabla_2}$$
3. *On E^*,*
$$F_{\nabla^*} = -F^\top_\nabla, \quad F_{\nabla^*}(s^\alpha \otimes \sigma_\beta) = -F_\nabla(s^\beta \otimes \sigma_\alpha)$$
4. *On pullback f^*E,*
$$F_{f^*\nabla} = f^* F_\nabla$$

Proposition 7.18. *The curvature of a Hermitian connection ∇ on a Hermitian vector bundle (E, h) satisfies*

$$0 = h(F_\nabla(\sigma), \tau) + h(\sigma, F_\nabla(\tau))$$

for any local sections σ, τ of E. That is, $F_\nabla \in \mathcal{A}^2_{\mathrm{End}(E,h)}(X)$.

Proof Idea. For Hermitian connection ∇, $\sigma \in \mathcal{A}^k_E(U)$, $\tau \in \mathcal{A}^l_E(U)$,

$$dh(\sigma, \tau) = h(\nabla(\sigma), \tau) + (-1)^k h(\sigma, \nabla(\tau)).$$

∎

Proposition 7.19. *The curvature of a connection ∇ on a holomorphic vector bundle E compatible with the holomorphic structure satisfies*

$$F_\nabla \in \left(\mathcal{A}^{2,0}_{\mathrm{End}(E)} \oplus \mathcal{A}^{1,1}_{\mathrm{End}(E)}\right)(X).$$

Combined, the Chern connection on a holomorphic Hermitian bundle (E, h) satisfies

$$F_\nabla \in \left(\mathcal{A}^{1,1}_{\mathrm{End}(E)} \cap \mathcal{A}^{2}_{\mathrm{End}(E,h)}\right)(X).$$

In local frame σ_i for E, write $h_{ij} = h(\sigma_i, \sigma_j)$ to get

$$F_\nabla = \overline{\partial}\left(\partial h_{ik} \cdot h^{jk}\right)\omega^i \otimes \sigma_j.$$

Thus $\overline{\partial}_{\mathrm{End}(E)}(F_\nabla) = 0$. If E is line bundle, then $F_\nabla = -\partial\overline{\partial}\log h$, which is a closed purely imaginary $(1,1)$-form.

Example 7.20. Let (X, g) be Kähler. The holomorphic tangent bundle T_X with Hermitian metric induced from g has Chern curvature

$$F_\nabla = R_{i\ k\bar{l}}^{\ j} dz^i \otimes \frac{\partial}{\partial z^j} \otimes (dz^k \wedge d\bar{z}^{\bar{l}}), \quad R_{i\ k\bar{l}}^{\ j} = -\partial_k \partial_{\bar{l}} h_{i\bar{\beta}} \cdot h^{j\bar{\beta}} + \partial_k h_{i\bar{\beta}} \cdot \partial_{\bar{l}} h_{p\bar{q}} \cdot h^{p\bar{\beta}} \cdot h^{j\bar{q}}.$$

Lowering the index, $R_{i\bar{j}k\bar{l}} = \mathrm{Rm}(\partial_i, \partial_{\bar{j}}, \partial_k, \partial_{\bar{l}})$, where Rm is the $\mathbb{C}$-linear extension of the Riemann curvature tensor on (X, g).

Proposition 7.21. *Let (E, h) be a holomorphic Hermitian vector bundle. The Chern connection ∇ on E has curvature representing the Atiyah class:*

$$[F_\nabla] = -A(E) \in H^1(X, \Omega_X \otimes \mathrm{End}(E)).$$

The idea is to compare the Dolbeault and Čech resolutions for $\Omega_X \otimes \mathrm{End}(E)$. More generally, given two resolutions $\mathcal{I}^\cdot$ and $\mathcal{C}^\cdot(\mathcal{U}, \mathcal{F})$ for a sheaf $\mathcal{F}$, we have the following diagram

$$\begin{array}{ccccccc}
\mathcal{F} & \longrightarrow & \mathcal{I}^0 & \xrightarrow{d} & \mathcal{I}^1 & \longrightarrow & \mathcal{I}^2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{C}^0(\mathcal{U},\mathcal{F}) & \longrightarrow & \mathcal{C}^0(\mathcal{U},\mathcal{I}^0) & \longrightarrow & \mathcal{C}^0(\mathcal{U},\mathcal{I}^1) & \longrightarrow & \mathcal{C}^0(\mathcal{U},\mathcal{I}^2) \\
\delta\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{C}^1(\mathcal{U},\mathcal{F}) & \longrightarrow & \mathcal{C}^1(\mathcal{U},\mathcal{I}^0) & \longrightarrow & \mathcal{C}^1(\mathcal{U},\mathcal{I}^1) & \longrightarrow & \mathcal{C}^1(\mathcal{U},\mathcal{I}^2) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{C}^2(\mathcal{U},\mathcal{F}) & \longrightarrow & \mathcal{C}^2(\mathcal{U},\mathcal{I}^0) & \longrightarrow & \mathcal{C}^2(\mathcal{U},\mathcal{I}^1) & \longrightarrow & \mathcal{C}^2(\mathcal{U},\mathcal{I}^2)
\end{array}$$

and the associated resolution which naturally contains both resolutions given

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0(\mathcal{U},\mathcal{I}^0) \xrightarrow{D} \mathcal{C}^0(\mathcal{U},\mathcal{I}^1) \oplus \mathcal{C}^1(\mathcal{U},\mathcal{I}^0) \longrightarrow \mathcal{C}^0(\mathcal{U},\mathcal{I}^2) \oplus \mathcal{C}^1(\mathcal{U},\mathcal{I}^1) \oplus \mathcal{C}^2(\mathcal{U},\mathcal{I}^0) \longrightarrow \dots$$

with $D = d + (-1)^p \delta$ on $\mathcal{K}^{p,q} = \mathcal{C}^q(\mathcal{U}, \mathcal{I}^p)$. It suffices to compare the representatives in this new resolution.

7.3 Chern Classes

In this section let $G := \mathrm{GL}(n,\mathbb{C})$, and $V = \mathfrak{gl}(n,\mathbb{C})$.

Definition 7.22. A polynomial function $P : V \to \mathbb{C}$ homogeneous of degree k in the entries is invariant if $P(A) = P(gAg^{-1})$ for all $A \in V$, $g \in G$.

The key example is the elementary symmetric polynomials $P^j(A)$ of the eigenvalues of $A \in V$:

$$\det(\mathrm{Id} + tA) = \sum_{k=0}^{n} P^k(A) \cdot t^k. \tag{7.1}$$

Other examples include:

$$\mathrm{tr}\big(e^{tA}\big) = \sum_{k=0}^{\infty} P^k(A) \cdot t^k, \tag{7.2}$$

$$\frac{\det(tA)}{\det(\mathrm{Id} - e^{-tA})} = \sum_{k=0}^{\infty} P^k(A) \cdot t^k. \tag{7.3}$$

Definition 7.23. A k-linear form $\tilde{P} : V \times \cdots \times V \to \mathbb{C}$ is invariant if

$$\tilde{P}(A_1, ..., A_k) = \tilde{P}(gA_1g^{-1}, \ldots, gA_kg^{-1})$$

for all $A_1, ..., A_k \in V$, $g \in G$.

Lemma 7.24. *There is one-to-one correspondence between symmetric invariant k-linear forms $\tilde{P} : V \times \cdots \times V \to \mathbb{C}$ and invariant homogeneous polynomial $P : V \to \mathbb{C}$ of degree k, given by diagonal and polarization:*

$$P(A) = \tilde{P}(A, ..., A).$$

For example, when $k = 2$, polarization writes $\tilde{P}$ by P via

$$\tilde{P}(A, B) = \frac{1}{2}\left(P(A + B) - P(A) - P(B)\right).$$

The invariance property enables us to go global from local. Let $E \to X$ be a complex vector bundle of rank n on a smooth manifold X, with a connection ∇ and curvature F_∇. Let $\{U_\alpha\}$ be an open covering of M by local trivializations $\varphi_\alpha : E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^n$.

Lemma 7.25. *Let $P : V \to \mathbb{C}$ be any invariant homogeneous polynomial of degree k.*

1. *There is a well-defined global $2k$-form $P(F_\nabla)$ on X, independent of the trivializations chosen. On each U_α, the $2k$-form can be given by $P(\Theta_\alpha)$, where Θ_α is the $n \times n$ curvature matrix of 2-forms.*
2. *$P(F_\nabla)$ is d-closed.*
3. *The cohomology class $[P(F_\nabla)] \in H^{2k}_{dR}(X, \mathbb{C})$ is independent of the choice of the connection ∇, hence a class determined only by E.*

Proof Idea. Use Bianchi identity and that

$$d\tilde{P}(\gamma_1, ..., \gamma_k) = \sum_{j=1}^{k} (-1)^{\sum_{l=1}^{j-1} i_l} P(\gamma_1, ..., \nabla(\gamma_j), ..., \gamma_k)$$

for any $\gamma_j \in \mathcal{A}^{i_j}_{\mathrm{End}(E)}(X)$. For $a \in \mathcal{A}^1_{\mathrm{End}(E)}(X)$,

$$\left.\frac{d}{dt}\right|_{t=0} P(F_{\nabla + ta}) = k\tilde{P}\left(\nabla(a), F_\nabla, ..., F_\nabla\right) = kd\tilde{P}(a, F_\nabla, ..., F_\nabla).$$

□

Definition 7.26. Let Φ denote the graded algebra of invariant homogeneous polynomials $V \to \mathbb{C}$. Fix a complex vector bundle $E \to X$. The Weil homomorphism is the $\mathbb{C}$-algebra homomorphism

$$\Phi \to H^{2*}_{dR}(X, \mathbb{C})$$

$$P \mapsto [P(F_\nabla)]$$

where F_∇ is the curvature with respect to any connection on E.

Definition 7.27. The Chern forms $c_i(F_\nabla)$ of the curvature F_∇ on E are

$$c_i(F_\nabla) := P^i\left(\frac{i}{2\pi}F_\nabla\right) \in \mathcal{A}^{2i}(X),$$

where P^i are invariant polynomials defined by (7.1). The Chern classes of E are

$$c_i(E) = \left[P^i\left(\frac{i}{2\pi}F_\nabla\right)\right] \in H^{2i}_{dR}(X, \mathbb{C}),$$

where $c_0(E) = 1 \in H^0_{dR}(X, \mathbb{C})$. The total Chern class is

$$c(E) = c_0(E) + c_1(E) + \cdots \in H^{2*}_{dR}(X, \mathbb{C}).$$

The Chern classes of a complex manifold are $c_i(T_X)$, where T_X is the holomorphic tangent bundle.

The Chern characters ch_i and Todd classes td_i are defined analogously with respect to invariant polynomials given in (7.2) and (7.3).

Lemma 7.28. *Let $f : X \to Y$ be any smooth map between smooth manifolds. Suppose $E \to Y$ is a complex vector bundle. Then*

$$f^*c_i(E) = c_i(f^*E).$$

Lemma 7.29. *Let $E \to X$ be a complex vector bundle. Then for each i,*

$$c_i(E^*) = (-1)^i c_i(E) \in H^{2i}_{dR}(X, \mathbb{C}).$$

Proposition 7.30 (Whitney Product Formula)**.** *For two complex vector bundles $E, F \to X$ we have*

$$c(E \oplus F) = c(E) \cdot c(F) \in H^{2*}_{dR}(X, \mathbb{C}).$$

Lemma 7.31. *For two complex vector bundles $E, F \to X$ we have*

$$\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F) \in H^{2*}_{dR}(X, \mathbb{C}),$$

$$\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \cdot \mathrm{ch}(F) \in H^{2*}_{dR}(X, \mathbb{C}).$$

Lemma 7.32. *The total Chern character and total Todd class can be expressed in terms of the total Chern class:*

$$\mathrm{ch}(E) = \mathrm{rank}(E) + c_1(E) + \frac{c_1^2(E) - 2c_2(E)}{2} + \frac{c_1^3(E) - 3c_1(E)c_2(E) + 3c_3(E)}{6} + \dots$$

$$\mathrm{td}(E) = 1 + \frac{c_1(E)}{2} + \frac{c_1^2(E) + c_2(E)}{12} + \frac{c_1(E)c_2(E)}{24} + \dots$$

Lemma 7.33. *The Chern classes, Chern characters, and Todd classes are all real, i.e. lies in $H^{2*}(X, \mathbb{R})$. Moreover, if X is compact Kähler and E is holomorphic vector bundle, then these characteristic classes lie in $H^{*,*}(X, \mathbb{R})$.*

Proof Idea. Choose Hermitian connection to see iF_∇ is Hermitian matrix under Hermitian trivialization. □

The splitting principle reduces the problem of computing Chern classes of a vector bundle E to the easier problem when E splits into line bundles $E = E_1 \oplus \cdots \oplus E_n$.

Theorem 7.34 (Splitting Principle)**.** *Let $E \to X$ be a complex vector bundle of rank n over a smooth manifold X. There exists a space $Y = Fl(E)$, called the flag bundle associated to E, and a map $p : Y \to X$, such that*

1. *The induced map on cohomology $p^* : H^*(X) \to H^*(Y)$ is injective.*
2. *The pullback bundle $p^*E \to Y$ splits as a direct sum of line bundles: $p^*E = L_1 \oplus \cdots \oplus L_n$.*

Example 7.35. For any complex vector bundle $E \to X$, one has $c_1(E) = c_1(\det E)$.

Proposition 7.36 (Chern Class of Holomorphic Line Bundle)**.** *Let L be a holomorphic line bundle on a complex manifold X. The first Chern class $c_1(L) \in H^2(X, \mathbb{C})$ coincides with the image of L under the map $H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C})$.*

Proof Idea. Compare the Čech resolution and de Rham resolution for $\mathbb{C}$ in the double-complex resolution as discussed above. L is represented by the cocycle of transition maps $(g_{ij}) \in \mathcal{C}^1(\mathcal{U}, \mathcal{O}_X^*)$, whose image in $H^2(X, \mathbb{C})$ is given by $(f_{ij} + f_{jk} + f_{ki}) \in \mathcal{C}^2(\mathcal{U}, \mathcal{A}^0)$ where $f_{ij} = \frac{1}{2\pi i} \log g_{ij}$. This is cohomologous to $c_1(L)$ represented by a Hermitian curvature form $-\frac{i}{2\pi} \partial\overline{\partial} \log h_i \in \mathcal{C}^0(\mathcal{U}, \mathcal{A}^2)$, because $h_i = |g_{ij}|^2 h_j$. □

Proposition 7.37. *Let X be compact Kähler manifold. For every closed real $(1,1)$-form $\omega \in c_1(L)$, there exists a Hermitian metric h on L whose Chern curvature form $-i\,\partial\overline{\partial} \log h = \omega$.*

Let X be a compact complex manifold. For any analytic subvariety $V \subset X$ of dimension k, the fundamental class $(V) \in H_{2k}(X, \mathbb{R})$ is defined as the linear functional

$$\varphi \mapsto \int_V \varphi$$

on $H^{2k}(X, \mathbb{R})$. Denote its Poincaré dual by $[V] \in H^{2n-2k}(X, \mathbb{R})$.

Lemma 7.38. *Let X be a compact Kähler manifold. For any analytic subvariety $V \subset X$ of dimension k, the fundmanetal class*

$$[V] \in H^{n-k,n-k}(X, \mathbb{Z}).$$

Proposition 7.39. *Let X be a compact complex manifold, and $D \in \mathrm{Div}(X)$. Then $c_1(\mathcal{O}(D)) = [D] \in H^2(X, \mathbb{R})$.*

8 Applications of Cohomology

8.1 Hirzebruch-Riemann-Roch

Definition 8.1. Let $E \to X$ be a holomorphic vector bundle over a compact complex manifold X. Define the **Euler-Poincaré characteristic**

$$\chi(X,E) := \sum_{i=0}^{\dim(X)} (-1)^i h^i(X,E).$$

Definition 8.2. Let C be a compact complex curve. The genus of C is

$$g(C) := \frac{2-\chi(C)}{2} = \frac{1}{2}\deg K_C + 1,$$

where $\chi(C)$ is the Euler characteristic of C, and the second equality follows from Gauss-Bonnet.

Lemma 8.3. *Let X be a compact complex surface, and $C \subset X$ a smooth irreducible curve (analytic subvariety of dimension 1). The genus of C is*

$$g(C) = \frac{K_X \cdot C + C \cdot C}{2} + 1$$

where the second equality is immediate consequence of adjunction formula $K_C \cong (K_X \otimes \mathcal{O}_X(C))|_C$.

We use the formula above to define the genus for any curve on a compact complex surface.

Proposition 8.4 (Riemann-Roch)**.** *Let $E \to C$ be a holomorphic vector bundle over a compact complex curve C. Then*

$$\chi(C,E) = \deg(E) + \mathrm{rank}(E) \cdot (1 - g(C)) \tag{8.1}$$

The Hirzebruch-Riemann-Roch (HRR) formula generalizes this. There are further generalizations, including the Grothendieck-Riemann-Roch formula and Atiyah-Singer index theorem.

Theorem 8.5 (Hirzebruch-Riemann-Roch)**.** *Let E be a holomorphic vector bundle on a compact complex manifold X. Then its Euler-Poincaré characteristic equals*

$$\chi(X,E) = \int_X \mathrm{ch}(E)\,\mathrm{td}(X) = \int_X \sum_{i=0}^{n} \mathrm{ch}_i(E)\,\mathrm{td}_{n-i}(X).$$

Example 8.6 (Line bundles on a curve)*.* Let C be a connected compact curve and $L \in \mathrm{Pic}(C)$. Then

$$\chi(C,L) = \int_C c_1(L) + \frac{c_1(C)}{2} = \deg(L) - \frac{\deg(\mathcal{K}_C)}{2} = \deg(L) + (1 - g(C)),$$

which recovers Riemann-Roch (8.1).

Example 8.7 (Line bundles on a surface)*.* Let X be a compact complex surface. Then

$$\chi(X,L) = \chi(X,\mathcal{O}_X) + \frac{L \cdot L - K_X \cdot L}{2}.$$

Definition 8.8. Let X be a compact complex manifold of dimension n. Define the **arithmetic genus** of X as

$$p_a(X) := (-1)^n(\chi(X,\mathcal{O}_X) - 1).$$

Definition 8.9. The **Hirzebruch χ_y-genus** of a compact complex manifold X of dimension n is the polynomial

$$\chi_y := \sum_{p=0}^{n} \chi(X, \Omega_X^p) y^p = \sum_{p,q=0}^{n} (-1)^q h^{p,q}(X) y^p.$$

We can calculate the Hirzebruch χ_y-genus using HRR and Chern roots:

Proposition 8.10. *Let γ_i denote the formal Chern roots of T_X. Then*

$$\chi_y = \int_X \prod_{i=1}^{n} (1 + y e^{-\gamma_i}) \frac{\gamma_i}{1 - e^{-\gamma_i}}.$$

The formal Chern roots only locally diagonalize the curvature matrix $\frac{i}{2\pi} F_\nabla$ for T_X. However, the symmetric polynomials in (γ_i) represent well-defined cohomology classes in $H^{2*}(X, \mathbb{R})$, and the RHS of the formula above makes sense in this way.

Some important special values of the Hirzebruch χ_y-genus:

1. $y = 0$: $\chi_{y=0} = \chi(X, \mathcal{O}_X) = \int_X \operatorname{td}(X)$ gives the arithmetic genus.
2. $y = 1$: $\chi_{y=1} = \sum_{p,q=0}^{n} (-1)^q h^{p,q}(X) = \operatorname{sgn}(X)$ if X is compact Kähler of even dimension n, by Hodge index theorem. Combining with the proposition above, we get **Hirzebruch signature theorem** for compact Kähler manifolds of even dimension:

$$\operatorname{sgn}(X) = \chi\left(\bigwedge \Omega_X\right) = \int_X L(X),$$

where $L(X)$ is the **L-genus** defined in terms of the Chern roots by

$$L(X) = \prod_{i=1}^{n} \gamma_i \frac{(1 + e^{-\gamma_i})}{1 - e^{-\gamma_i}} = \prod_{i=1}^{n} \gamma_i \cdot \coth\left(\frac{\gamma_i}{2}\right).$$

The same result holds for any compact complex manifold of even dimension.

3. $y = -1$: Suppose X is compact Kähler manifold of dimension n. Then

$$\chi_{y=-1} = \sum_{p,q=0}^{n} (-1)^{p+q} h^{p,q}(X) = \chi(X) = \int_X \prod_{i=1}^{n} \gamma_i = \int_X c_n(X).$$

This is the **Chern-Gauss-Bonnet formula**, which holds more generally for any compact complex manifolds.

Example 8.11. Consider a smooth hypersurface $Y \subset X$. The short exact sequence of sheaves on X

$$0 \to \mathcal{O}(-Y) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0,$$

gives

$$\chi(Y, \mathcal{O}_Y) = \chi(X, \mathcal{O}_Y) = \chi(X, \mathcal{O}_X) - \chi(X, \mathcal{O}(-Y)) = \int_X \left(1 - e^{-[Y]}\right) \operatorname{td}(X).$$

8.2 Kodaira Vanishing Theorem

Definition 8.12. Let X be a complex manifold. A holomorphic line bundle L on X is called a **positive line bundle** if its first Chern class $c_1(L) \in H^2(X, \mathbb{R})$ can be represented by a closed positive real (1, 1)-form.

Theorem 8.13 (Kodaira Vanishing Theorem)**.** *Let L be a positive holomorphic line bundle on a compact Kähler manifold X of dimension n. Then*

$$H^{p,q}(X, L) = H^q(X, \Omega_X^p \otimes L) = 0, \quad \textit{for all } p + q > n.$$

Let (E, h) be a holomorphic vector bundle over X with fixed Hermitian structure. Recall from §6 the operators $\overline{\partial}_E$ and $\overline{\partial}_E^*$ on $\mathcal{A}_E^{p,q}$. We also extend the Lefschetz operator L and dual Lefschetz operator Λ from $\mathcal{A}^{p,q}$ to $\mathcal{A}_E^{p,q}$ via $L = L \otimes \mathrm{Id}_E$, $\Lambda = \Lambda \otimes \mathrm{Id}_E$. The Kähler identity

$$[\Lambda, L] = (n - (p + q)) \cdot \mathrm{Id}$$

now holds on $\mathcal{A}_E^{p,q}$.

Lemma 8.14 (Nakano Identity)**.** *Let ∇ be the Chern connection on (E, h). Then*

$$[\Lambda, \overline{\partial}_E] = -i \left(\nabla_E^{1,0}\right)^* : \mathcal{A}_E^{p,q} \to \mathcal{A}_E^{p-1,q},$$

where

$$\left(\nabla_E^{1,0}\right)^* := -\overline{*}_{E^*} \circ \nabla_{E^*}^{1,0} \circ \overline{*}_E.$$

Nakano identity implies the Kähler identity

$$[\Lambda, \overline{\partial}] = -i\partial^*,$$

by letting $E = \mathcal{O}_X$, in which case $\nabla = d$.

Proof. Work in orthonormal trivialization such that the connection matrix A satisfies $A^* = -A$ and $A(x_0) = 0$ for a fixed point $x_0 \in X$. □

Lemma 8.15. *Let (E, h) be a Hermitian holomorphic vector bundle over compact Kähler manifold (X, g). Let ∇ be the Chern connection on E, and $\alpha \in \mathcal{H}^{p,q}(X, E)$ any harmonic form. Then*

$$\frac{i}{2\pi} (F_\nabla \Lambda(\alpha), \alpha) \leq 0, \quad \frac{i}{2\pi} (\Lambda F_\nabla(\alpha), \alpha) \geq 0.$$

Proof. Use $F_\nabla = \nabla^{1,0} \circ \overline{\partial}_E + \overline{\partial}_E \circ \nabla^{1,0} : \mathcal{A}_E^{p,q} \to \mathcal{A}_E^{p+1,q+1}$. □

Proof of Kodaira Vanishing Theorem. Fix a Kähler form $\omega \in c_1(L)$, which is the Chern curvature form of a Hermitian metric on L. For any harmonic form $\alpha \in \mathcal{H}^{p,q}(X, L)$,

$$(n - (p + q))\|\alpha\|^2 = ([\Lambda, L]\alpha, \alpha) = \frac{i}{2\pi} ([\Lambda, F_\nabla]\alpha, \alpha) \geq 0.$$

Therefore, $0 = \mathcal{H}^{p,q}(X, L) \cong H^{p,q}(X, L) = H^q(X, \Omega_X^p \otimes L)$ for any $p + q > n$. □

Theorem 8.16 (Weak Lefschetz Theorem)**.** *Let X be a compact Kähler manifold of dimension n, and let $Y \subset X$ be a smooth hypersurface such that the induced line bundle $\mathcal{O}(Y)$ is positive. Then the canonical restriction map*

$$i^* : H^k(X, \mathbb{C}) \to H^k(Y, \mathbb{C})$$

is bijective for $k \leq n - 2$ and injective for $k \leq n - 1$.

Lemma 8.17. *Let X be a compact Kähler manifold, and $Y \subset X$ smooth hypersurface. Let $i : Y \to X$ denote the inclusion. For any holomorphic vector bundle E on X, we have*

$$H^q(Y, i^*E) = H^q(Y, i^*E \otimes \mathcal{O}_Y) \cong H^q(X, E \otimes i_*\mathcal{O}_Y), \quad \forall q \geq 0.$$

We consider the structure sequence (and its twist by a holomorphic vector bundle on X)

$$0 \to \mathcal{O}_X(-Y) \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0$$

as instead

$$0 \to \mathcal{O}_X(-Y) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0,$$

since the induced long exact sequence of sheaf cohomologies conincide by the lemma above. Lemma of this type involves the tool of spectral sequence, which is beyond the scope of the current notes.

Proof. Use Hodge decomposition. Apply Kodaira vanishing theorem to variants of structure sequence and normal bundle sequence

$$0 \to \Omega_X^p(-Y) \to \Omega_X^p \to \Omega_X^p|_Y \to 0,$$
$$0 \to \Omega_Y^{p-1}(-Y) \to \Omega_X^p|_Y \to \Omega_Y^p \to 0.$$

□

Theorem 8.18 (Serre's Vanishing Theorem)**.** *Let L be a positive line bundle on a compact Kähler manifold X of dimension n. For any holomorphic vector bundle E on X there exists a constant m_0 such that*

$$H^q(X, E \otimes L^m) = 0$$

for all $m \geq m_0$, $q > 0$.

Proof Idea. Repeat the argument for Kodaira vanishing to $\mathcal{H}^{n,q}(E \otimes L^m)$. □

Corollary 8.19 (Grothendieck Lemma)**.** *Every holomorphic vector bundle E on $\mathbb{P}^1$ is isomorphic to a direct sum of line bundles*

$$\bigoplus_{i=1}^{r} \mathcal{O}(a_i),$$

where the integers $a_1 \geq \cdots \geq a_r$ are uniquely determined by E.

Proof Idea. Prove by induction on rank r. The first Chern class map $c_1 : H^1(\mathbb{P}^n, \mathcal{O}^*) \to H^2(\mathbb{P}^n, \mathbb{Z})$ is isomorphism. Use Serre's vanishing and Riemann-Roch to choose a_1 as the maximal integer a such that $H^0(\mathbb{P}^1, E(-a)) \neq 0$. A non-zero section $s \in H^0(\mathbb{P}^1, E(-a_1))$ induces a short exact sequence of holomorphic vector bundles

$$0 \to \mathcal{O}(a_1) \to E \to E_1 = E/\mathcal{O}(a_1) \to 0.$$

Show that $E \cong \mathcal{O}(a_1) \oplus E_1$ as holomorphic vector bundles, and use induction hypothesis on E_1. □

8.3 Kodaira Embedding Theorem

Theorem 8.20 (Kodaira Embedding Theorem)**.** *Let X be a compact complex manifold, and $L \to X$ a positive line bundle. Then there exists $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$, L^k is globally generated and the canonical map*

$$\varphi_{L^k} : X \to \mathbb{P}(H^0(X, L^k)^*)$$
$$x \mapsto \big(s \in H^0(X, L^k) \mapsto s(x)\big)$$

is a holomorphic embedding.

If we fix a basis $\{s_0, ..., s_N\}$ of $H^0(X, L^k)$ with dual basis for $H^0(X, L^k)^*$, then φ_{L^k} is given explicitly by

$$x \mapsto [s_0(x) : \cdots : s_N(x)] \in \mathbb{P}^N.$$

Lemma 8.21. *Let $L \to X$ be a holomorphic line bundle over a compact complex manifold X. Then L is globally generated (or base-point free) if $H^0(X, L) \to L_x$ is surjective for each $x \in X$. This map fits into the short exact sequence of sheaves*

$$0 \to L \otimes \mathcal{I}_{\{x\}} \to L \to L_x \to 0$$

Lemma 8.22. *Let $L \to X$ be a globally generated holomorphic line bundle over a compact complex manifold X. Then $\varphi_L : X \to \mathbb{P}^N$ is an embedding if both conditions below hold:*

1. *φ_L is injective. It is sufficient to check, for any $x_1 \neq x_2 \in X$, $H^1(X, L \otimes \mathcal{I}_{\{x_1,x_2\}}) = 0$, according to the short exact sequence of sheaves*

$$0 \to L \otimes \mathcal{I}_{\{x_1,x_2\}} \to L \to L_{x_1} \oplus L_{x_2} \to 0.$$

2. *φ_L is immersion. It is sufficient to check, for any $x \in X$, $H^1(X, L \otimes \mathcal{I}^2_{\{x\}}) = 0$, according to the short exact sequence of sheaves*

$$0 \to L \otimes \mathcal{I}^2_{\{x\}} \to L \otimes \mathcal{I}_{\{x\}} \xrightarrow{d_x} \Omega_{X,x} \otimes L_x \to 0.$$

In other words, $H^0(X, L)$ separates points and generates 1-jets at every point of X.

Lemma 8.23. *Let X be a complex manifold of dimension n and L a postive line bundle on X. Let $\sigma : \hat{X} \to X$ be the blow-up of X along a finite number of points $x_1, ..., x_l \in X$, and let $E_j := \sigma^{-1}(x_j) \cong \mathbb{P}(T_x X)$ be the exceptional divisors for each $j = 1, ..., l$. Then for any holomorphic line bundle M on X and integers $n_1, ..., n_k > 0$, the line bundle*

$$\sigma^*(L^k \otimes M) \otimes \mathcal{O}(-\sum_j n_j E_j)$$

on $\hat{X}$ is positive for $k \gg 0$.

Proof. Use partition of unity to define Hermitian metrics on $\mathcal{O}(-E_j)$ whose curvature is the Fubini-Study metric on $E_j \cong \mathbb{P}^{n-1}$ and vanish outside a small neighborhood of E_j. $\sigma^*(c_1(L))$ ensures positivity in the direction normal to E_j for points on E_j and positivity outside a neighborhood of E. □

Proof of Kodaira Embedding Theorem. Follow the outline in Lemma 8.22. Translate the problem to the blow-up of X along point(s) of interest, and use Kodaira vanishing to conclude the vanishing of the corresponding H^1 cohomologies. This provides k_0 but may depend on the choice of points we consider. Observe finally that the properties we require on φ_{L^k} are "open", which combined with compactness of X gives a uniform choice of k_0. □

Corollary 8.24. *Let X be a compact complex manifold. A line bundle $L \to X$ is positive if and only if it is ample.*

Definition 8.25. Let X be a compact Kähler manifold. The Kähler cone $C_X \subset H^{1,1}(X,\mathbb{R})$ consists of all classes which admits a Kähler form on X.

Lemma 8.26. *The Kähler cone $C_X \subset H^{1,1}(X,\mathbb{R})$ is an open convex cone and contains no lines $\{\alpha + t\beta \mid t \in \mathbb{R}\}$ for any $\beta \neq 0$.*

Corollary 8.27. *A compact Kähler manifold X is projective if and only if*

$$C_X \cap \operatorname{Im}(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})) \neq \emptyset.$$

Corollary 8.28. *Every compact Kähler manifold X with $h^{2,0}(X) = h^{0,2}(X) = 0$ is projective.*

Corollary 8.29. *Let $X := V/\Gamma$ be a complex torus. Then X is projective if and only if X admits a **Riemann form**, i.e. an alternating $\mathbb{R}$-bilinear form $\omega : V \times V \to \mathbb{R}$ such that*

i) $\omega(iu, iv) = \omega(u, v)$,

ii) $\omega(\cdot, i(\cdot))$ *is positive definite, and*

iii) $\omega(u, v) \in \mathbb{Z}$ *if* $u, v \in \Gamma$.

Proposition 8.30. *Let X be a projective manifold. Then the natural homomorphism*

$$\mathcal{O} : \operatorname{Div}(X) \to \operatorname{Pic}(X)$$

is surjective.

Proof. For each ample line bundle L and any line bundle M, $M \otimes L^k$ and L^k are both contained in the image of $\mathcal{O}$ for some sufficiently large k by Kodaira vanishing and Hirzebruch-Riemann-Roch. □

Proposition 8.31. *Let X be a projective manifold. Then the kernel of the map*

$$c_1 : \operatorname{Pic}(X) \xrightarrow{\delta} H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})$$

consists of numerically trivial line bundles $\operatorname{NT}(X)$ (line bundles of degree zero on any curve $C \subset X$). Therefore,

$$\operatorname{NS}(X) \cong \operatorname{Pic}(X)/\operatorname{NT}(X).$$

Proof. Let $\omega \in H^{1,1}(X,\mathbb{Z})$ denote the pullback of the Fubini-Study metric on $\mathbb{P}^N$. Apply Hard Lefschetz and Hodge-Riemann bilinear relations to $c_1(L)$ and consider the curves $[D] \wedge [\omega]^{n-2}$ for any divisor D. □

Conjecture 8.32 (Hodge Conjecture). *Let X be a projective manifold of dimension n. Then for any $0 \leq k \leq n$,*

$$H^{k,k}(X,\mathbb{Q}) := H^{k,k}(X) \cap H^{2k}(X,\mathbb{Q}) = \operatorname{span}_{\mathbb{Q}}\{[V] \in H^{k,k}(X,\mathbb{Z}) \mid V \textit{ analytic subvariety of dimension } n-k\}.$$