Notes on Calabi-Yau Manifolds
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1 Calabi Conjecture and Kähler-Einstein Metrics

Definition 1.1. A Kähler manifold X is called Calabi-Yau (CY) if its first Chern class vanishes in $H^2(X, \mathbb{R})$:

$$c_1(X) = \left[\frac{\operatorname{Ric}(\omega)}{2\pi}\right] = 0 \in H^2(X, \mathbb{R}).$$

Definition 1.2. A Kähler manifold (X, ω) is called Kähler-Einstein (KE) if there exists a real number $\lambda \in \mathbb{R}$ such that

$$\operatorname{Ric}(\omega) = \lambda\omega.$$

Example 1.3. Ricci-flat Kähler manifolds are trivially KE, e.g. ω_{Euc} on complex torus.

$\mathbb{P}^n$ is KE as $\operatorname{Ric}(\omega_{FS}) = (n+1)\omega_{FS}$.

$\mathbb{B}^n$ is KE with Poincaré metric ω_P satisfying $\operatorname{Ric}(\omega_P) = (-n-1)\omega_P$.

We can always assume, by scaling the Kähler metric, that $\lambda = 0, 1, -1$. This follows immediately from the local definition of Ricci curvature:

$$\operatorname{Ric}(\omega) := -i\partial\overline{\partial} \log \det\Big(g_{j\overline{k}}\Big).$$

Thus if $\tilde{\omega} = \mu\omega$, and $\operatorname{Ric}(\omega) = \lambda\omega$, then $\operatorname{Ric}(\tilde{\omega}) = \frac{\lambda}{\mu}\tilde{\omega}$.

Question 1.4. *Which compact Kähler manifolds admit Kähler-Einstein metrics?*

We approach this question from the definition. Suppose $\operatorname{Ric}(\omega) = \lambda\omega$, consider the three cases:

i) $\lambda = 0$. By Yau's theorem, this happens if and only if X is Calabi-Yau.

ii) $\lambda = 1$. In this case

$$2\pi c_1(X) = [\operatorname{Ric}(\omega)] = [\omega],$$

so it is necessary that X is Fano.

iii) $\lambda = -1$. As above, X must be canonically polarized, i.e. $c_1(X) < 0$.

In case iii) we have the following result.

Theorem 1.5 (Aubin-Yau '76)**.** *Let X be a compact Kähler manifold that is canonically polarized. Then there exists a unique Kähler metric ω on X with* $\operatorname{Ric}(\omega) = -\omega$.

In contrast, not all Fano manifolds admit KE metrics. There is an if and only if characterization of which Fano manifolds admit KE metrics, using algebraic geometry and theorem by Chen-Donaldson-Sun '12.

We can prove Theorem 1.5 in tandem with Yau's theorem solving the Calabi conjecture. Recall Yau's theorem:

Theorem 1.6 (Yau '76). *Let (X^n, ω) be a compact Kähler manifold. Given any closed real (1, 1)-form ψ with*

$$[\psi] = 2\pi c_1(X) = [\mathrm{Ric}(\omega)] \in H^2(X, \mathbb{R}),$$

there exists a unique Kähler metric $\tilde{\omega}$ such that

$$\begin{cases} [\tilde{\omega}] = [\omega] \in H^2(X, \mathbb{R}), \\ \mathrm{Ric}(\tilde{\omega}) = \psi. \end{cases}$$

Start proof of Yau's Theorem. We first show that the assertion $\mathrm{Ric}(\tilde{\omega}) = \psi$ we want is equivalent to a "prescribed volume form" problem. By assumption, $\mathrm{Ric}(\omega) - \psi$ is d-exact real (1, 1)-form. Hence by $\partial\overline{\partial}$-lemma, there exists $F \in C^\infty(X, \mathbb{R})$, unique up to adding a constant, such that

$$\mathrm{Ric}(\omega) - \psi = i\partial\overline{\partial}F.$$

We pick the unique constant added to F such that

$$\int_X e^F \omega^n = \int_X \omega^n.$$

Here we use compactness of X. Now F is uniquely determined.

Similarly, since we want to find $\tilde{\omega}$ with $[\tilde{\omega}] = [\omega] \in H^2(X, \mathbb{R})$, by $\partial\overline{\partial}$-lemma there exists some unique $\varphi \in C^\infty(X, \mathbb{R})$ such that

$$\tilde{\omega} = \omega + i\partial\overline{\partial}\varphi \quad \text{and} \quad \int_X \varphi\omega^n = 0.$$

Then we compute the Ricci curvature:

$$\begin{aligned} \mathrm{Ric}(\tilde{\omega}) &= \mathrm{Ric}(\omega) - i\partial\overline{\partial}\log\left(\frac{\tilde{\omega}^n}{\omega^n}\right) \\ &= \psi - i\partial\overline{\partial}\left[\log\left(\frac{\tilde{\omega}^n}{\omega^n}\right) - F\right]. \end{aligned}$$

Thus $\mathrm{Ric}(\tilde{\omega}) = \psi$ if and only if the real function $\log\left(\frac{\tilde{\omega}^n}{\omega^n}\right) - F$ is a constant. Taking exponential and using

$$\int_X e^F \omega^n = \int_X \omega^n \overset{\text{Stokes}}{=} \int_X \tilde{\omega}^n,$$

we see that this holds if and only if

$$\log\left(\frac{\tilde{\omega}^n}{\omega^n}\right) - F = 0 \Leftrightarrow \tilde{\omega}^n = e^F \omega^n,$$

which is a prescribed volume form problem.

□

To conclude what we compute so far, Yau's theorem is equivalent to

Theorem 1.7 (Yau '76). *Let (X^n, ω) be a compact Kähler manifold. Given $F \in C^\infty(X, \mathbb{R})$ with*

$$\int_X e^F \omega^n = \int_X \omega^n,$$

there exists a unique $\varphi \in C^\infty(X, \mathbb{R})$ such that

$$\begin{cases} \tilde{\omega} := \omega + i\partial\overline{\partial}\varphi > 0 \\ \int_X \varphi\omega^n = 0. \\ \tilde{\omega}^n = \left(\omega + i\partial\overline{\partial}\varphi\right)^n = e^F \omega^n. \end{cases}$$

The last equation is a 2nd order scalar PDE for φ, of complex Monge-Ampère type. In local coordinates:

$$\begin{cases} \det\left(g_{j\bar{k}} + \frac{\partial^2\varphi}{\partial z_j \partial \bar{z}_k}\right) = e^F \det\left(g_{j\bar{k}}\right) \\ \left(g_{j\bar{k}} + \frac{\partial^2\varphi}{\partial z_j \partial \bar{z}_k}\right) > 0 & \text{on } X. \end{cases}$$

The equation is non-linear for $n \geq 2$. For $n = 1$, this is trivial Poisson equation we have discussed.

We analyze Aubin-Yau (Theorem 1.5) similarly. Suppose we find KE metric $\tilde{\omega}$ such that $\mathrm{Ric}(\tilde{\omega}) = \lambda\tilde{\omega}$, $\lambda = \pm 1$. Then

$$2\pi c_1(X) = [\mathrm{Ric}(\tilde{\omega})] \Rightarrow [\tilde{\omega}] = \lambda \cdot 2\pi c_1(X),$$

which means the class $\lambda \cdot 2\pi c_1(X)$ contains some Kähler metric. Fix a Kähler metric ω in $\lambda \cdot 2\pi c_1(X)$. So by $\partial\overline{\partial}$-lemma, if KE metric $\tilde{\omega}$ exists, it must be of form

$$\tilde{\omega} = \omega + i\partial\overline{\partial}\varphi > 0,$$

where $\varphi \in C^\infty(X, \mathbb{R})$ is unique up to adding a constant. On the other hand, since $[\mathrm{Ric}(\omega)] = 2\pi c_1(X)$, we have

$$\mathrm{Ric}(\omega) - \lambda\omega = i\partial\overline{\partial}F$$

for some $F \in C^\infty(X, \mathbb{R})$. Then

$$\begin{aligned} \mathrm{Ric}(\tilde{\omega}) - \lambda\tilde{\omega} &= \mathrm{Ric}(\omega) - i\partial\overline{\partial}\log\left(\frac{\tilde{\omega}^n}{\omega^n}\right) - \lambda\omega - \lambda i\partial\overline{\partial}\varphi \\ &= i\partial\overline{\partial}\left[F - \lambda\varphi - \log\left(\frac{\tilde{\omega}^n}{\omega^n}\right)\right] \end{aligned}$$

Thus $\tilde{\omega}$ is KE metric if and only if the real function $F - \lambda\varphi - \log\left(\frac{\tilde{\omega}^n}{\omega^n}\right)$ is a constant. We can shift φ by this constant such that the condition is equivalent to

$$F - \lambda\varphi - \log\left(\frac{\tilde{\omega}^n}{\omega^n}\right) = 0 \Leftrightarrow \tilde{\omega}^n = e^{F-\lambda\varphi}\omega^n.$$

Now the Aubin-Yau theorem reduces to

Theorem 1.8. *Let (X^n, ω) be a compact Kähler manifold, and $F \in C^\infty(X, \mathbb{R})$. Then there exists a unique $\varphi \in C^\infty(X, \mathbb{R})$ such that*

$$\begin{cases} \tilde{\omega} := \omega + i\partial\overline{\partial}\varphi > 0 \\ \tilde{\omega}^n = (\omega + i\partial\overline{\partial}\varphi)^n = e^{F+\varphi}\omega^n \end{cases}$$

This implies that if X is canonically polarized, we can find $\omega \in -2\pi c_1(X)$ to start with, and find KE metric $\tilde{\omega}$ as defined above. The uniqueness of $\tilde{\omega}$ follows from the analysis above.

Proof of Uniqueness in Theorem 1.7. This immediately follows from Calabi's uniqueness. □

Proof of Uniqueness in Theorem 1.8. Let $\omega_i = \omega + i\partial\overline{\partial}\varphi_i > 0$ solving $\omega_i^n = e^{F+\varphi_i}\omega^n$ for $i = 1, 2$. Let $u := \varphi_2 - \varphi_1$. Then

$$(\omega_1 + i\partial\overline{\partial}u)^n = \omega_2^n = e^{F+\varphi_2}\omega^n = e^u\omega_1^n.$$

We want to show that $u \equiv 0$.

By compactness of X, we can pick a point $x \in X$ where u attains maximum. Then

$$i\partial\overline{\partial}u(x) \leq 0,$$

i.e. the matrix $\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(x)\right)$ is negative semi-definite. We can check this using the 2nd derivative test on the real coordinates and translate to complex coordinates. Hence

$$0 < \omega_2(x) = (\omega_1 + i\partial\bar{\partial}u)(x) \leq \omega_1(x).$$

Taking n-th power wedge, which takes determinant on the matrices in coordinates, we get

$$e^{u(x)}\omega_1^n(x) = \omega_2^n(x) \leq \omega_1^n(x) > 0,$$

so that

$$e^{u(x)} \leq 1 \Rightarrow u \leq 0 \text{ on } X.$$

Similarly, consider a point $x \in X$ where u attains minimum. Then

$$\omega_2(x) \geq \omega_1(x) > 0 \Rightarrow e^{u(x)}\omega_1^n(x) = \omega_2^n(x) \geq \omega_1^n(x) > 0,$$

so that $u \geq 0$ on X. In conclusion, $u \equiv 0$. □

Proof of existence in Theorem 1.7 and 1.8. We use the method of continuity. More precisely, we deform our PDE in a continuous way to another one that we can solve explicitly, and show that solvability persist through the deformation.

Let $t \in [0,1]$ be the deformation parameter. Let $\mu = 0$ and $+1$ for Theorem 1.7 and 1.8 respectively. Our PDE is

$$\begin{cases} \text{look for } \varphi_t \in C^\infty(X, \mathbb{R}) \text{ s.t.} \\ \omega + i\partial\bar{\partial}\varphi_t > 0 \\ \int_X \varphi_t \omega^n = 0 & \text{if } \mu = 0 \\ (\omega + i\partial\bar{\partial}\varphi_t)^n = c_t e^{tF + \mu\varphi_t}\omega^n & \text{on } X \\ 0 < c_t = \begin{cases} 1 & \mu = 1 \\ \frac{\int_X \omega^n}{\int_X e^{tF}\omega^n} & \mu = 0 \end{cases} \end{cases} \tag{$*_t$}$$

The choice of c_t when $\mu = 0$ ensures that

$$\int_X (\omega + i\partial\bar{\partial}\varphi_t)^n = \int_X \omega^n$$

for all $t \in [0,1]$. Indeed, problem $(*_1)$ is our desired PDE.

Let's define the set

$$I := \{t \in [0,1] \mid \text{PDE } (*_t) \text{ has a } C^\infty \text{ solution } \varphi_t\}.$$

I is non-empty since $0 \in I$ with trivial solution $\varphi_0 = 0$. We next show that I is open and closed in $[0,1]$. Openness means that for each solution φ_t of $(*_t)$, we can deform it slightly to get a solution of $(*_s)$ for s sufficiently close to t. Closedness means that if φ_{t_i} solves $(*_{t_i})$ and $t_i \to t_0$, then $\varphi_{t_i} \to \varphi_{t_0}$ solving $(*_{t_0})$.

First consider c_t as a function of t. For $\mu = 0$, we have $c_0 = c_1 = 1$. We claim that

$$e^{-\|F\|_{L^\infty(X)}} \leq c_t \leq e^{\|F\|_{L^\infty(X)}}. \tag{1.1}$$

Indeed,

$$\int_X e^{tF}\omega^n \leq e^{t\|F\|_{L^\infty(X)}} \int_X \omega^n \leq e^{\|F\|_{L^\infty(X)}} \int_X \omega^n,$$
$$\int_X e^{tF}\omega^n \geq e^{-t\|F\|_{L^\infty(X)}} \int_X \omega^n \geq e^{-\|F\|_{L^\infty(X)}} \int_X \omega^n.$$

In particular, c_t does not approach 0 or ∞ as t varies.

Recall the Hölder space $C^{k,\alpha}(U)$ defined by Hölder norm

$$\begin{aligned}\|u\|_{C^{k,\alpha}(U)} &= \|u\|_{C^k(U)} + [D^k u]_{C^\alpha(U)} \\ &= \sum_{i=0}^{k} \left\|D^i u\right\|_{L^\infty(U)} + \sup_{|I|=k} \sup_{x\neq y\in U} \frac{\left|D^I u(x) - D^I u(y)\right|}{|x-y|^\alpha}.\end{aligned}$$

We know that $C^{k,\alpha}(U)$ is Banach space containing $C^\infty(\overline{U})$, and $C^\infty(U)$ is not dense in $C^{k,\alpha}(U)$. We now move from local to global. Fix (X^n,ω) a compact Kähler manifold, or more generally a closed Riemannian manifold. Fix an atlas $\{(Uj,\varphi_j : U_j \cong V_j \subset \mathbb{C}^n)\}_{j=1}^N$, and $\{\rho_j\}$ a partition of unity subordinate to $\{U_j\}$.

Define for a k times differentiable function $u : X \to \mathbb{R}$ the $C^{k,\alpha}(X)$ norm, depending on the choices of atlas and POU above:

$$\|u\|_{C^{k,\alpha}(X)} := \sum_{j=1}^{N} \left\|(\rho_j u) \circ \varphi_j^{-1}\right\|_{C^{k,\alpha}(V_j)}$$

Then the Hölder space

$$C^{k,\alpha}(X) := \{u : X \to \mathbb{R} \mid u \text{ is k times differentiable and } \|u\|_{C^{k,\alpha}(X)} < \infty\}$$

is a Banach space containing $C^\infty(X)$, and $C^\infty(X)$ is not dense in $C^{k,\alpha}(X)$.

We fix $k = 3$ and any $\alpha \in (0,1)$ to prove openness of I. First consider $\mu = 0$. Define

$$\mathcal{U} = \left\{u \in C^{3,\alpha}(X) \mid \omega + i\partial\overline{\partial}u > 0 \text{ on } X \text{ and } \int_X u\omega^n = 0\right\}.$$

$$\mathcal{V} = \left\{v \in C^{1,\alpha}(X) \mid \int_X v\omega^n = \int_X \omega^n\right\}.$$

$\mathcal{U}$ is an open subset of the Banach space of the kernel of the bounded linear operator $\int_X (\cdot)\, \omega^n : C^{3,\alpha}(X) \to \mathbb{R}$. Similarly, V is an affine linear closed subspace of $C^{1,\alpha}$. Then define an opeartor

$$\mathcal{E} : \mathcal{U} \to \mathcal{V}, \quad \mathcal{E}(u) := \frac{\left(\omega + i\partial\overline{\partial}u\right)^n}{\omega^n}.$$

We can immediately see that $\mathcal{E}$ maps into $\mathcal{V}$. Functional analysis theory yields $\mathcal{E}$ is Fréchet differentiable as a map between Banach spaces.

Now we prove openness of I under $\mu = 0$. Suppose φ_t solves PDE $(*_t)$ for some $t \in I$. By definition of $\mathcal{E}$,

$$\mathcal{E}(\varphi_t) = c_t e^{tF}.$$

We try to find $\varphi_s \in \mathcal{U}$ (for now) solving $\mathcal{E}(\varphi_s) = c_s e^{sF}$ for all $s \in [0,1]$ sufficiently close to t. The key point is that if s is sufficiently close to t, then $c_s e^{sF}$ is as close as I want to $c_t e^{tF}$ in $\|\cdot\|_{C^{1,\alpha}(X)}$. To show this, we want to apply the **Inverse Function Theorem for Banach spaces**: if $D_{\varphi_t}\mathcal{E}$ is an isomorphism between (tangent) Banach spaces, then $\mathcal{E}$ is locally a bijection near φ_t and $\mathcal{E}(\varphi_t)$.

The tangent space to φ_t in $\mathcal{U}$ is

$$T_{\varphi_t}U = \left\{\psi \in C^{3,\alpha}(X) \mid \int_X \psi\omega^n = 0\right\}.$$

The tangent space to $\mathcal{E}(\varphi_t)$ in $\mathcal{V}$ is

$$T_{\mathcal{E}(\varphi_t)}V = \left\{\eta \in C^{1,\alpha}(X) \mid \int_X \eta\omega^n = 0\right\}.$$

We compute the Gateaux derivative $D_{\varphi_t}\mathcal{E} : T_{\varphi_t}U \to T_{\mathcal{E}(\varphi_t)}V$:

$$\begin{aligned} D_{\varphi_t}\mathcal{E}(\psi) &= \frac{d}{ds}\bigg|_{s=0} \mathcal{E}(\varphi_t + s\psi) \\ &= \frac{d}{ds}\bigg|_{s=0} \frac{\left(\omega + i\partial\overline{\partial}\left(\varphi_t + s\psi\right)\right)^n}{\omega^n} \\ &= \frac{n\left(\omega + i\partial\overline{\partial}\varphi_t\right)^{n-1} \wedge i\partial\overline{\partial}\psi}{\omega^n} \\ &= \frac{n\left(\omega + i\partial\overline{\partial}\varphi_t\right)^{n-1} \wedge i\partial\overline{\partial}\psi}{\left(\omega + i\partial\overline{\partial}\varphi_t\right)^n} \cdot \frac{\left(\omega + i\partial\overline{\partial}\varphi_t\right)^n}{\omega^n} \\ &= \left(\Delta_{\omega + i\partial\overline{\partial}\varphi_t}\psi\right) \cdot \mathcal{E}(\varphi_t) \end{aligned}$$

We then use the following linear PDE theory on manifolds.

Theorem 1.9 (Poisson equation in Hölder spaces)**.** *Let (X^n, ω) be a compact Kähler manifold. For any $k \in \mathbb{N}$, $\alpha \in (0,1)$, there exists $C > 0$ such that*

1) given any $f \in C^{k,\alpha}(X)$ with $\int_X f\omega^n = 0$, there exists a unique $u \in C^{k+2,\alpha}(X)$ solving

$$\begin{cases} \Delta_g u = f & \text{on } X \\ \int_X u\omega^n = 0 \end{cases}$$

and we have

$$\|u\|_{C^{k+2,\alpha}(X)} \leq C\|f\|_{C^{k,\alpha}(X)} \leq C'\|u\|_{C^{k+2,\alpha}(X)},$$

where the first inequality is ***global Schauder estimate*** *and the second inequality is trivial by $\Delta_g u = f$. Hence the map*

$$\Delta_g : \left\{u \in C^{k+2,\alpha}(X) \mid \int_X u\omega^n = 0\right\} \to \left\{f \in C^{k,\alpha}(X) \mid \int_X f\omega^n = 0\right\}$$

is a Banch space isomorphism.

2) given any $\lambda > 0$, and any $f \in C^{k,\alpha}(X)$, there exists a unique $u \in C^{k+2,\alpha}(X)$ solving the Helmholtz equation or eigenvalue equation

$$\Delta_g u = \lambda u + f \quad \text{on } X,$$

and we have the same Schauder estimate (second inequality is again trivial by $f = \Delta_g u - \lambda u$)

$$\|u\|_{C^{k+2,\alpha}(X)} \leq C\|f\|_{C^{k,\alpha}(X)} \leq C'\|u\|_{C^{k+2,\alpha}(X)}.$$

Hence the map

$$\Delta_g - \lambda\,\mathrm{Id} : C^{k+2,\alpha}(X) \to C^{k,\alpha}(X)$$

is a Banach space isomorphism.

Back to our proof. Let $\omega_t := \omega + i\partial\overline{\partial}\varphi_t$, a C^∞ Kähler metric. By computation above,

$$D_{\varphi_t}\mathcal{E}(\psi) = (\Delta_{\omega_t}\psi) \cdot \frac{\omega_t^n}{\omega^n}.$$

Then applying Theorem 1.9 1) to (X, ω_t), we see that $D_{\varphi_t}\mathcal{E} : T_{\varphi_t}U \to T_{\mathcal{E}(\varphi_t)}V$ is a Banach space isomorphism using trivial isomorphisms between kernel of $\int_X (\cdot)\omega^n$ and $\int_X (\cdot)\omega_t^n$. Therefore, by Inverse Function Theorem, there exist open neighborhoods

$$\varphi_t \in U \subset \mathcal{U}, \quad c_t e^{tF} \in V \subset \mathcal{V},$$

such that $\mathcal{E}: U \to V$ is bijection. Hence for all s sufficiently close to t, $c_s e^{sF} \in V$, and we can solve for φ_s using $(\mathcal{E}|_U)^{-1}$. We now have

$$\begin{cases} \varphi_s \in C^{3,\alpha}(X) \\ \omega + i\partial\overline{\partial}\varphi_s > 0 & \text{on } X \\ \int_X \varphi_s \omega^n = 0 \\ (\omega + i\partial\overline{\partial}\varphi_s)^n = c_s e^{sF}\omega^n \end{cases}$$

The last question to ask is whether $\varphi_s \in C^\infty(X, \mathbb{R})$. This is true by the following regularity theorem.

Theorem 1.10 (Regularity). *Let (X, ω) be a compact Kähler manifold. Suppose $\varphi \in C^{3,\alpha}(X)$ for some $\alpha \in (0,1)$ solves*

$$\omega + i\partial\overline{\partial}\varphi > 0, \quad (\omega + i\partial\overline{\partial}\varphi)^n = e^{F+\mu\varphi}\omega^n$$

for some $F \in C^\infty(X)$, $\mu \in \mathbb{R}$, then $\varphi \in C^\infty(X)$.

The same statement holds if we only assume $\varphi \in C^2(X)$. The proof is harder.

The proof of Theorem 1.10 uses local Schauder theory:

Theorem 1.11 (Schauder estimate). *Let g be any Kähler metric on the unit ball $B_1 = B_1(0) \subset \mathbb{C}^n$, and $g_{\mathbb{C}^n}$ the Euclidean metric. Fix $k \in \mathbb{N}$, $\alpha \in (0,1)$. Suppose*

$$\begin{cases} A^{-1} g_{\mathbb{C}^n} \le g \le A g_{\mathbb{C}^n} \\ \|g\|_{C^{k,\alpha}(B_1)} \le A \end{cases}$$

for some $A > 0$. Let $f \in C^{k,\alpha}(B_1)$, $u \in C^{2,\alpha}(B_1)$ solve

$$\Delta_g u = f \quad \text{on } B_1.$$

Then for any $\varepsilon > 0$, there exists some constant $C = C(n, A, k, \alpha, \varepsilon)$ such that $u \in C^{k+2,\alpha}(B_{1-\varepsilon})$, and

$$\|u\|_{C^{k+2,\alpha}(B_{1-\varepsilon})} \le C\left(\|f\|_{C^{k,\alpha}(B_1)} + \|u\|_{L^\infty(B_1)}\right).$$

In particular, on $B_{1/2}$, we have $u \in C^{k+2,\alpha}(B_{1/2})$ and there exists some constant $C = C(n, A, k, \alpha)$ such that

$$\|u\|_{C^{k+2,\alpha}(B_{1/2})} \le C\left(\|f\|_{C^{k,\alpha}(B_1)} + \|u\|_{L^\infty(B_1)}\right).$$

Proof of Regularity Theorem 1.10 assuming Schauder Theorem 1.11. The regularity is a local statement, so we can work in a chart isomorphic to $B_1(0) \subset \mathbb{C}^n$. Let our Kähler metric g given by metric g on B_1. In this coordinate, the Monge-Ampère equation that φ solves is

$$\det\left(g_{i\bar{j}} + \partial_i\partial_{\bar{j}}\varphi\right) = e^{F+\mu\varphi}\det\left(g_{i\bar{j}}\right).$$

Taking log,

$$\log\det\left(g_{i\bar{j}} + \partial_i\partial_{\bar{j}}\varphi\right) = F + \mu\varphi + \log\det\left(g_{i\bar{j}}\right).$$

Taking $\frac{\partial}{\partial z_k}$, as $\varphi \in C^{3,\alpha}(B_1)$, we get

$$\begin{aligned} \Delta_{\tilde{g}}(\partial_k\varphi) &= \tilde{g}^{i\bar{j}}\partial_k\partial_i\partial_{\bar{j}}\varphi \\ &= \tilde{g}^{i\bar{j}}\partial_k\left(\tilde{g}_{i\bar{j}} - g_{i\bar{j}}\right) \\ &= -\tilde{g}^{i\bar{j}}\partial_k g_{i\bar{j}} + \partial_k F + \mu\partial_k\varphi + g^{i\bar{j}}\partial_k g_{i\bar{j}} \end{aligned}$$

where

$$\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + \partial_i\partial_{\bar{j}}\varphi, \quad \left(\tilde{g}^{i\bar{j}}\right) = \left(\tilde{g}_{i\bar{j}}\right)^{-t}.$$

Note that $-\tilde{g}^{i\bar{j}}\partial_k g_{i\bar{j}} \in C^{1,\alpha}(B_1)$, $\partial_k\varphi \in C^{2,\alpha}(B_1)$, $\partial_k F + g^{i\bar{j}}\partial_k g_{i\bar{j}} \in C^\infty(B_1)$. To see the first one $\tilde{g}^{i\bar{j}} \in C^{1,\alpha}$, use that $C^{1,\alpha}$ is closed under product and division by function nowhere vanishing (**work on a slightly larger ball and use compactness in division**). By definition, $\tilde{g}$ has $C^{1,\alpha}(B_1)$ coefficients, and is comparable to the Euclidean metric on B_1. Thus we can apply Schauder Theorem 1.11 with $k=1$ to conclude that $\partial_k\varphi \in C^{3,\alpha}(B_{1/2})$. Repeat the argument above with $\partial_{\bar{k}}$ to get $\partial_{\bar{k}}\varphi \in C^{3,\alpha}(B_{1/2})$. Thus $\varphi \in C^{4,\alpha}(B_{1/2})$.

We can repeat the argument above with $k=2$ now, since we already have $\varphi \in C^{4,\alpha}(B_{1/2})$. This yields $\varphi \in C^{5,\alpha}(B_{1/4})$. Repeat this argument to see that φ is smooth at 0. This completes the proof.

□

Therefore, the solution φ_s we get from $\mathcal{E}^{-1}$ is smooth. This concludes openness of I when $\mu = 0$.

Next we show openness of I when $\mu = 1$. The proof can be adapted from above slightly. Suppose φ_t solves PDE $(*_t)$ for some $t \in I$. Define

$$\mathcal{F} : \mathcal{W} \to C^{1,\alpha}(X), \quad \mathcal{F}(w) := \log \frac{\left(\omega + i\partial\bar{\partial}w\right)^n}{\omega^n} - w$$

where

$$\mathcal{W} := \{w \in C^{3,\alpha}(X) \mid \omega + i\partial\bar{\partial}w > 0\}$$

is an open subset of the Banach space $C^{3,\alpha}(X)$. Indeed $\mathcal{F}$ maps $\mathcal{W}$ into $C^{1,\alpha}$: logarithm of a positive $C^{1,\alpha}(X)$ function is still $C^{1,\alpha}(X)$ by compactness of X. Then

$$\mathcal{F}(\varphi_t) = tF,$$

so for all $s \in [0,1]$ sufficiently close to t, the function sF is close to tF in $C^{1,\alpha}(X)$. As above, we compute the Gateaux derivative of $\mathcal{F}$ at φ_t to apply Inverse Function Theorem in Banach spaces.

$$D_{\varphi_t}\mathcal{F} : T_{\varphi_t}\mathcal{W} = C^{3,\alpha}(X) \to C^{1,\alpha},$$

and follow the calculaations above for $\mathcal{E}$ to get

$$\begin{aligned} D_{\varphi_t}\mathcal{F}(\psi) &= \frac{d}{ds}\bigg|_{s=0} \mathcal{F}(\varphi_t + s\psi) \\ &= \Delta_{\omega_t := \omega + i\partial\bar{\partial}\varphi_t}\psi - \psi \\ \Rightarrow D_{\varphi_t}\mathcal{F} &= \Delta_{\omega_t} - \mathrm{Id}\,. \end{aligned}$$

By Theorem 1.9 2), $D_{\varphi_t}\mathcal{F}$ is a Banach space isomorphism. Thus we solve for $\varphi_s \in C^{3,\alpha}(X)$ using local inverse $\mathcal{F}^{-1}(sF)$. By Regularity Theorem 1.10, $\varphi_s \in C^\infty(X)$. This concludes openness of I.

We are left to show that $I \subset [0,1]$ is closed. The main claim is the following:

Theorem 1.12 (Yau's a priori estimates). *Let (X^n, ω) be a compact Kähler manifold. Let $F \in C^\infty(X, \mathbb{R})$, $\mu = 0$ or 1. Suppose $\varphi \in C^\infty(X, \mathbb{R})$ solves*

$$\begin{cases} \omega + i\partial\bar{\partial}\varphi > 0 & \\ \int_X \varphi\omega^n = 0 & \text{if } \mu = 0 \\ (\omega + i\partial\bar{\partial}\varphi_t)^n = e^{F+\mu\varphi}\omega^n & \text{on } X. \end{cases}$$

Then given any $\alpha \in (0,1)$, there exists a constant $C = C((X,\omega), \|F\|_{C^{3,\alpha}(X)}, \alpha)$ such that

$$\begin{cases} \omega + i\partial\bar{\partial}\varphi \geq C^{-1}\omega \\ \|\varphi\|_{C^{2,\alpha}(X)} \leq C. \end{cases}$$

This means that if at $x \in X$ we pick coordinates that simultaneously diagonalize ω and $\omega + i\partial\overline{\partial}\varphi$ at x:

$$\begin{cases} g_{i\overline{j}}(x) = \delta_{ij} \\ \left(g_{i\overline{j}} + \partial_i \partial_{\overline{j}}\varphi\right)(x) = \lambda_j \delta_{ij}, \end{cases}$$

then the first a priori estimate says $\lambda_j \geq C^{-1}$ for each $j = 1, ..., n$; the second estimate says

$$\left\|\partial_i \partial_{\overline{j}}\varphi\right\|_{C^{0,\alpha}(X)} \leq C \Rightarrow \lambda_j \leq C$$

Let us assume Yau's a priori estimates first and finish the proof of closedness of I. See proof of Theorem 1.12 below.

Assume there is a sequence $t_i \in I$ such that $t_i \to \overline{t} \in [0,1]$, we want to show that $\overline{t} \in I$. By assumption, let $\varphi_{t_i} \in C^\infty(X, \mathbb{R})$ be solutions to the PDE $(*_{t_i})$. Let $\tilde{\omega}_i := \omega + i\partial\overline{\partial}\varphi_{t_i} > 0$, such that $\tilde{\omega}_i^n = c_{t_i} e^{t_i F + \mu\varphi_{t_i}} \omega^n$.

Fix any $\alpha \in (0,1)$. To apply a priori estimates, we want $\tilde{F} := \log c_{t_i} + t_i F$, such that $e^{\tilde{F}} = c_{t_i} e^{t_i F}$. Recall from (1.1) that $|\log c_{t_i}| \leq \|F\|_{L^\infty(X)}$, so there exists some constant C independent of i such that

$$\|\log c_{t_i} + t_i F\|_{C^{3,\alpha}(X)} \leq C.$$

Thus Theorem 1.12 does apply, and there exists some constant C such that for all i,

$$\begin{cases} \tilde{\omega}_i \geq C^{-1}\omega \\ \|\varphi_{t_i}\|_{C^{2,\alpha}(X)} \leq C. \end{cases}$$

In local coordinates on B_1, write $(\tilde{g}_i)_{i\overline{j}} = g_{i\overline{j}} + \partial_i \partial_{\overline{j}}\varphi_{t_i}$. Then the matrices $\tilde{g}_i$ satisfy

$$\begin{cases} \|\tilde{g}_i\|_{C^{0,\alpha}(X)} \leq C \\ C^{-1}g \leq \tilde{g}_i \leq Cg \end{cases}$$

for some constant C independent of i. Then

$$\Delta_{\tilde{g}_i}(\partial_k \varphi_{t_i}) = -\tilde{g}_i^{p\overline{q}} \partial_k g_{p\overline{q}} + \partial_k \tilde{F} + \mu \partial_k \varphi_{t_i} + g^{p\overline{q}} \partial_k g_{p\overline{q}}.$$

The RHS has uniform $C^{0,\alpha}(B_1)$-norm bound independent of i. The only non-trivial part is the first term, where we use $\tilde{g}_i^{-1} = \frac{1}{\det(\tilde{g}_i)} \mathrm{Adj}(\tilde{g}_i)$, and compare $\det(\tilde{g}_i) \geq C \det(g)$ using the results above.

By Schauder Theorem 1.11, there exists some uniform constant C_ε such that

$$\|\partial_k \varphi_{t_i}\|_{C^{2,\alpha}(B_{1-\varepsilon})} \leq C_\varepsilon + C_\varepsilon \|\partial_k \varphi_{t_i}\|_{L^\infty(B_1)} \leq C,$$

where the last inequality follows from above: $\|\varphi_{t_i}\|_{C^{2,\alpha}(X)} \leq C$.

Similarly, $\left\|\partial_{\overline{k}}\varphi_{t_i}\right\|_{C^{2,\alpha}(B_{1-\varepsilon})} \leq C$ for some uniform constant C. Hence $\|\varphi_{t_i}\|_{C^{3,\alpha}(B_{1-\varepsilon})} \leq C$ for some constant C independent of i. Now we use compactness of X to pick $\varepsilon > 0$ small, and pick my charts "dense" such that the union of balls $B_{1-\varepsilon} \subset B_1$ still covers all of X. We can therefore conclude that

$$\|\varphi_{t_i}\|_{C^{3,\alpha}(X)} \leq C$$

where C is independent of i.

Fix $0 < \alpha' < \alpha < 1$. We have compact embedding $C^{3,\alpha}(X) \hookrightarrow C^{3,\alpha'}(X)$. Thus there exists a subsequence $t_{i_j} \to \overline{t}$ such that $\varphi_{t_{i_j}}$ converges in $C^{3,\alpha'}(X)$, say to some $\varphi_{\overline{t}} \in C^{3,\alpha'}(X)$. We want to verify that $\varphi_{\overline{t}}$ solves the PDE $(*_{\overline{t}})$. First, for positivity, we have

$$0 < C^{-1}\omega \leq \tilde{\omega}_{i_j} \to \omega + i\partial\overline{\partial}\varphi_{\overline{t}} \text{ as } j \to \infty.$$

Hence $\omega + i\partial\overline{\partial}\varphi_{\overline{t}} \geq C^{-1}\omega$ is a Kähler metric with $C^{1,\alpha'}$-coefficients.

Also, passing to limit in the PDE, as $j \to \infty$,

$$c_{\overline{t}} e^{\overline{t}F + \mu\varphi_{\overline{t}}}\omega^n \leftarrow c_{t_{i_j}} e^{t_{i_j}F + \mu\varphi_{t_{i_j}}}\omega^n = \tilde{\omega}^n_{i_j} \to \left(\omega + i\partial\overline{\partial}\varphi_{\overline{t}}\right)^n,$$

so that

$$\left(\omega + i\partial\overline{\partial}\varphi_{\overline{t}}\right)^n = c_{\overline{t}} e^{\overline{t}F + \mu\varphi_{\overline{t}}}\omega^n.$$

If $\mu = 0$, we check in addition

$$0 = \int_X \varphi_{t_{i_j}}\omega^n \to \int_X \varphi_{\overline{t}}\omega^n = 0.$$

Finally, by Regularity Theorem 1.10, $\varphi_{\overline{t}}$ is in fact $C^\infty(X, \mathbb{R})$. Therefore, $\varphi_{\overline{t}}$ solves the PDE $(*_{\overline{t}})$. This completes the proof of closedness of I.

Therefore, $I = [0, 1]$, and in particular our desired PDE $(*_1)$ has a solution.

□

We are now left only with proving Yau's a priori estimates. We first need the following.

Proposition 1.13 (Compact embeddings of Hölder spaces)**.** *Let (X^n, ω) be a compact Kähler manifold. Let $k, l \in \mathbb{N}$, $\alpha, \beta \in (0, 1)$ such that*

$$l + \beta > k + \alpha.$$

Hence $l \geq k$, and $\beta > \alpha$ if $l = k$. Then the natural inclusion $C^{l,\beta}(X) \hookrightarrow C^{k,\alpha}(X)$ is compact operator (mapping bounded subsets to precompact subsets).

Proof. The map $C^{l,\beta}(X) \hookrightarrow C^{k,\alpha}$ is clearly a bounded linear operator.

To show compactness, it suffices to consider $k = l = 0$. The rest of the cases follow by induction. Now $\beta > \alpha > 0$. Suppose u_i is a bounded sequence of functions in $C^{0,\beta}(X) = C^\beta(X)$. We want to show that u_i has a subsequence convergent in $C^\alpha(X)$. We have

$$\|u_i\|_{C^\beta(X)} := \|u\|_{L^\infty(X)} + [u_i]_{C^\beta(X)} \leq C$$

$$[u_i]_{C^\beta(X)} := \sup_{x \neq y \in X} \frac{|u_i(x) - u_i(y)|}{d(x, y)^\beta}.$$

$\|u\|_{L^\infty(X)} \leq C$ implies that u_i are uniformly bounded in $C(X)$, and $[u_i]_{C^\beta(X)} \leq C$ implies that u_i are equicontinuous in $C(X)$. Thus by Arzela-Ascoli, there exists a subsequence u_{i_j} that converges in $C(X)$, say to $u \in C(X)$. By pointwise convergence in particular, for any $x \neq y \in X$,

$$\frac{|u(x) - u(y)|}{d(x, y)^\beta} = \lim_j \frac{\left|u_{i_j}(x) - u_{i_j}(y)\right|}{d(x, y)^\beta} \leq C,$$

$$\Rightarrow [u]_{C^\beta(X)} \leq C.$$

Thus $u \in C^\beta(X)$.

It remains to show that $u_{i_j} \to u$ in $C^\alpha(X)$ as $j \to \infty$. Convergence in $C(X)$ is known already, so we want to show that

$$[u_{i_j} - u]_{C^\alpha(X)} \to 0 \text{ as } j \to \infty.$$

For $x \neq y \in X$, the value we consider is

$$\frac{\left|u_{i_j}(x) - u(x) - u_{i_j}(y) + u(y)\right|}{d(x, y)^\beta} \cdot \frac{d(x, y)^\beta}{d(x, y)^\alpha}.$$

When $d(x, y)$ is small, the second term is small, and the first term is uniformly bounded by $2C$. When $d(x, y)$ is not small, $d(x, y)^{-\alpha}$ is bounded, and $\left|u_{i_j}(x) - u(x) - u_{i_j}(y) + u(y)\right|$ is small for all j large and for all $x \neq y$. This proves that $[u_{i_j} - u]_{C^\alpha(X)} \to 0$ as $j \to \infty$, and completes the proof.

□

Proof of Theorem 1.12. We shall prove these in 3 steps.

Step 1. Prove the uniform bound

$$\|\varphi\|_{L^\infty(X)} \leq C = C((X,\omega), \|F\|_{L^\infty(X)}).$$

We first consider the easier case $\mu = 1$:

$$\begin{cases} \left(\omega + i\partial\overline{\partial}\varphi\right)^n = e^{F+\varphi}\omega^n \\ \omega + i\partial\overline{\partial}\varphi > 0. \end{cases}$$

Let $x \in X$ be a point where φ attains its maximum on X. By 2nd derivatives test, $i\partial\overline{\partial}\varphi \leq 0$, so

$$0 < (\omega + i\partial\overline{\partial}\varphi)(x) \leq \omega(x).$$

Taking n-th wedge product, we see $e^{F+\varphi}(x) \leq 1$, so $F(x) + \varphi(x) \leq 0$, $\varphi(x) \leq \|F\|_{L^\infty(X)}$. Similarly, considering any point $y \in X$ where φ attains its minimum, we get $-\varphi(y) \leq F(y) \leq \|F\|_{L^\infty(X)}$, so that

$$\|\varphi\|_{L^\infty(X)} \leq \|F\|_{L^\infty(X)}.$$

The case $\mu = 0$ is more delicate. Recall first the Euclidean Sobolev inequality.

Theorem 1.14 (Sobolev inequality in $\mathbb{R}^{n\geq 2}$)**.** *Given $1 \leq p < n$, and $q \in \mathbb{R}$ such that*

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{n},$$

then for all $f \in C_c^\infty(\mathbb{R}^n)$, we have

$$\|f\|_{L^q(\mathbb{R}^n)} \leq \frac{q(n-1)}{2n}\|\nabla f\|_{L^p(\mathbb{R}^n)}$$

The proof of this is standard real analysis. See, e.g. Evans §5.

We now bring Sobolev inequality onto compact Kähler manifolds.

Theorem 1.15 (Sobolev inequality on compact manifolds)**.** *Let $(X^{n\geq 1}, \omega)$ be a compact Kähler manifold. Given $1 \leq p < 2n$, let $q \in \mathbb{R}$ such that*

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{2n}.$$

Then there exists some constant $C = C((X,\omega), p)$ such that for all $f \in C^\infty(X, \mathbb{R})$, we have

$$\|f\|_{L^q(X)} \leq C\left(\|f\|_{L^p(X)} + \|\partial f\|_{L^p(X)}\right),$$

where

$$\|f\|_{L^p(X)} := \left(\int_X |f|^p\omega^n\right)^{\frac{1}{p}},$$

$$\|\partial f\|_{L^p(X)} := \left(\int_X \left(|\partial f|_g^2\right)^{\frac{p}{2}}\omega^n\right)^{\frac{1}{p}},$$

and recall that in local coordinates,

$$|\partial f|_g^2 = g^{i\bar{j}}\frac{\partial f}{\partial z_i}\overline{\frac{\partial f}{\partial z_j}}.$$

Proof. First cover X by a finite atlas $\{U_j, \varphi_j : U_j \cong V_j \subset \mathbb{C}^n\}_{j=1}^N$, such that in local coordinates on U_j, we have

$$\frac{1}{2}\,\mathrm{Id} \leq (g_{i\bar{j}}) \leq 2\,\mathrm{Id}\,.$$

Hence if we denote the Lebesgue measure by dx, we have

$$dx \leq \omega^n \leq 2^{2n} dx$$

on each U_j.

This atlas can be attained, because for each $x \in X$ we can pick local coordinates such that $g_{i\bar{j}}(x) = \mathrm{Id}$, so that the condition above is satisfied in a neighborhood of x.

Fix a partition of unity $\{\rho_j\}$ subordinate to $\{U_j\}$. Then

$$\|f\|_{L^q(X)} = \left\| \sum_{j=1}^N \rho_j f \right\|_{L^q(X)} \leq \sum_{j=1}^N \|\rho_j f\|_{L^q(U_j)}.$$

Apply Euclidean Sobolev inequality to $(\rho_j f) \circ \varphi_j^{-1} \in C_c^\infty(\mathbb{R}^{2n})$:

$$\begin{aligned}
\|\rho_j f\|_{L^q(U_j)} &= \left(\int_{U_j} |\rho_j f|^q \omega^n \right)^{\frac{1}{q}} \\
&\leq C \left(\int_{\varphi_j(U_j)} \left|(\rho_j f) \circ \varphi_j^{-1}\right|^q dx \right)^{\frac{1}{q}} && \text{by choice of coordinates above} \\
&\leq C \left(\int_{\varphi_j(U_j)} \left|D\left((\rho_j f) \circ \varphi_j^{-1}\right)\right|^p dx \right)^{\frac{1}{p}} && \text{Sobolev inequality} \\
&\leq C \left(\int_{U_j} |\partial(\rho_j f)|_g^p \omega^n \right)^{\frac{1}{p}} && \text{by choice of coordinates} \\
&\leq C \left(\int_{U_j} \left(|\partial \rho_j|_g^p \cdot |f|^p + |\rho_j|^p \cdot |\partial f|_g^p \right) \omega^n \right)^{\frac{1}{p}} \\
&\leq C \left(\int_{U_j} |f|^p \omega^n \right)^{\frac{1}{p}} + C \left(\int_{U_j} |\partial f|_g^p \omega^n \right)^{\frac{1}{p}} \\
&\leq C \left(\|f\|_{L^p(X)} + \|\partial f\|_{L^p(X)} \right)
\end{aligned}$$

where in the final steps we use the equivalence of all L^p norms on a finite dimensional space. Summing over j, we have the desired result. Indeed the constant C depends on partition of unity $\{\rho_j\}$, and hence on (X, ω), but not on f.

□

We next recall Poincaré inequality on compact Kähler manifolds.

Theorem 1.16 (Poincaré inequality on compact manifolds)**.** *Let (X^n, ω) be a compact Kähler manifold (or closed Riemannian manifold). Fix any $1 \leq p < \infty$. Then there exists some constant $C = C(p) > 0$ such that for all $f \in C^\infty(X, \mathbb{R})$, we have*

$$\int_X |f - a_f|^p \leq C \int_X |\nabla f|_g^p,$$

where

$$a_f := \frac{\int_X f}{\mathrm{Vol}(X,g)}$$

denotes the average of f.

There is a more general version of Poincaré inequality for all $1 \leq p < \infty$ and the constant C depends only on (X,g) and p. The proof uses Rellich-Kondrachov $W^{1,p}(X) \Subset L^p(X)$ and argue by contradiction.

Formally speaking, Poincaré inequality says that the operator $-\Delta_g$ acting on $C^\infty(X,\mathbb{R})$ with L^2 inner product has the zero eigenvalue first (one-dimensional space of constant functions), and then the next eigenvalue is positive ($1/C$ from above). Indeed eigenvalues of $-\Delta_g$ have zero average.

Let's now continue the proof of Yau's a priori estimate. Consider case $\mu = 0$ in Step 1. Notice the following lemma:

Lemma 1.17. *For each continuous function $f : X \to \mathbb{R}$ on a compact space X, we have*

$$\lim_{p\to\infty} \|f\|_{L^p(X)} = \|f\|_{L^\infty(X)}.$$

Proof. Notice that it suffices to consider the case $\|f\|_{L^\infty(X)} = 1$, as we can scale any non-zero function by its $L^\infty(X)$-norm.

Clearly, $\|f\|_{L^p(X)} \leq \|f\|_{L^\infty(X)} = 1$ for each $p \geq 1$.

For the other direction, split X according to the value of $|f|$. We claim that for each $0 < \delta < 1$, we can pick $\varepsilon(\delta) > 0$ and p_0 large such that $\forall p \geq p_0$,

$$\begin{aligned} \|f\|_{L^p(X)} &= \left(\int_{\{|f|>1-\varepsilon\}} |f|^p + \int_{\{|f|\leq 1-\varepsilon\}} |f|^p \right)^{\frac{1}{p}} \\ &\geq (1-\varepsilon) \cdot M(\{|f| > 1-\varepsilon\})^{1/p} \\ &> 1-\delta. \end{aligned}$$

Indeed this can be achieved. For example, pick $\varepsilon < \delta/2$, and since $M(\{|f| > 1-\varepsilon\}) > 0$ due to $\|f\|_{L^\infty(X)} = 1$, we can find p_0 large enough such that $M(\{|f| > 1-\varepsilon\})^{1/p} > 1-\varepsilon$ for all $p \geq p_0$. Therefore,

$$\lim_{p\to\infty} \|f\|_{L^p(X)} = 1.$$

□

Thus to bound $\|\varphi\|_{L^\infty(X)}$ uniformly, it suffices to bound $\|\varphi\|_{L^p(X)}$ uniformly. Observe that though $|t|^p$ is not differentiable at $t = 0$, the function $t|t|^\alpha$ is differentiable on $\mathbb{R}$ for any $\alpha \geq 0$, with derivative $(\alpha+1)|t|^\alpha$. Hence for $p \geq 2$, compute

$$\begin{aligned} \int_X \varphi|\varphi|^{p-2}(\omega^n - \tilde{\omega}^n) &= \int_X \varphi|\varphi|^{p-2}(1-e^F)\omega^n && \text{by assumption PDE} \\ &\leq \int_X |\varphi|^{p-1}(1+e^F)\omega^n \\ &\leq C \int_X |\varphi|^{p-1}\omega^n \end{aligned}$$

where C depends on $\left\|e^F\right\|_{L^\infty(X)}$. On the other hand, as in Calabi's uniqueness argument,

$$\begin{aligned}
\int_X \varphi|\varphi|^{p-2}(\omega^n-\tilde\omega^n) &= \int_X \varphi|\varphi|^{p-2}(\omega-\tilde\omega)\wedge\sum_{j=0}^{n-1}\left(\omega^j\wedge\tilde\omega^{n-1-j}\right)\\
&= -\int_X \varphi|\varphi|^{p-2} i\partial\overline{\partial}\varphi\wedge\sum_{j=0}^{n-1}\left(\omega^j\wedge\tilde\omega^{n-1-j}\right)\\
&= \int_X d\left(\varphi|\varphi|^{p-2}\right)\wedge i\overline{\partial}\varphi\wedge\sum_{j=0}^{n-1}\left(\omega^j\wedge\tilde\omega^{n-1-j}\right) && \text{Stokes and closedness of } \omega,\tilde\omega\\
&= (p-1)\int_X |\varphi|^{p-2} i d\varphi\wedge\overline{\partial}\varphi\wedge\sum_{j=0}^{n-1}\left(\omega^j\wedge\tilde\omega^{n-1-j}\right)\\
&= (p-1)\int_X |\varphi|^{p-2} i \partial\varphi\wedge\overline{\partial\varphi}\wedge\sum_{j=0}^{n-1}\left(\omega^j\wedge\tilde\omega^{n-1-j}\right)\\
&\geq (p-1)\int_X |\varphi|^{p-2} i \partial\varphi\wedge\overline{\partial\varphi}\wedge\omega^{n-1} && \text{proved in Calabi uniqueness}\\
&= \frac{p-1}{n}\int_X |\varphi|^{p-2}|\partial\varphi|_g^2\omega^n && n\alpha\wedge\omega^{n-1} \overset{\alpha\text{ real }(1,\,1)}{=} (\operatorname{tr}_\omega\alpha)\cdot\omega^n\\
&= \frac{4(p-1)}{np^2}\int_X \left|\partial\left(\varphi|\varphi|^{\frac{p-2}{2}}\right)\right|_g^2\omega^n
\end{aligned}$$

Remark 1.18. The wedge product of n positive real (1, 1)-forms on (X^n,ω) is positive multiple of the volume form.

In conclusion, we now have some kind of "reverse Sobolev inequality".

$$\int_X \left|\partial\left(\varphi|\varphi|^{\frac{p-2}{2}}\right)\right|_g^2\omega^n \leq C\frac{np^2}{4(p-1)}\int_X|\varphi|^{p-1}\omega^n \leq Cp\int_X|\varphi|^{p-1}\omega^n \tag{1.2}$$

for all $p\geq 2$ and C uniform independent of p. In particular, for $p=2$, we have

$$\int_X |\partial\varphi|_g^2\omega^n \leq C\int_X|\varphi|\omega^n. \tag{1.3}$$

Let's now assume $n\geq 2$ and combine the inequality above with Sobolev inequality. Let $\beta := \frac{n}{n-1}$. Applying Sobolev inequality to $f:=\varphi|\varphi|^{\frac{p-2}{2}}$, we have

$$\left(\int_X |f|^{2\beta}\omega^n\right)^{\frac{1}{\beta}} \leq C\left(\int_X|\partial f|_g^2\omega^n+\int_X f^2\omega^n\right).$$

Then using the reverse Sobolev inequality above for the second term on RHS,

$$\begin{aligned}
\left(\int_X |\varphi|^{p\beta}\omega^n\right)^{\frac{1}{\beta}} &\leq C\left(p\int_X|\varphi|^{p-1}\omega^n+\int_X|\varphi|^p\omega^n\right) && (1.4)\\
&\leq C\left(p\left(\int_X|\varphi|^p\omega^n\right)^{\frac{p-1}{p}}+\int_X|\varphi|^p\omega^n\right) \qquad \text{Hölder.} && (1.5)
\end{aligned}$$

Note that

$$\left(\int_X|\varphi|^p\omega^n\right)^{\frac{p-1}{p}} \leq \max\left(1,\int_X|\varphi|^p\omega^n\right),$$

so

$$\left(\int_X |\varphi|^{p\beta}\omega^n\right)^{\frac{1}{\beta}} \leq Cp\max\left(1, \int_X |\varphi|^p\omega^n\right).$$

Clearly, we also have $1 \leq$ RHS, so

$$\max\left(1, \|\varphi\|_{L^{p\beta}(X)}\right) \leq C^{\frac{1}{p}} p^{\frac{1}{p}} \max\left(1, \|\varphi\|_{L^p(X)}\right) \tag{1.6}$$

for all $p \geq 2$ and some uniform constant C independent of p. This looks like a "reverse Hölder inequality".

We are now ready to apply the Moser iteration technique. Since $\beta = \frac{n}{n-1} > 1$, iterate reverse Hölder inequality (1.6) to get

$$\begin{aligned}\max\left(1, \|\varphi\|_{L^{p\beta^2}(X)}\right) &\leq C^{\frac{1}{p\beta}} (p\beta)^{\frac{1}{p\beta}} \max\left(1, \|\varphi\|_{L^{p\beta}(X)}\right) \\ &\leq C^{\frac{1}{p\beta}} (p\beta)^{\frac{1}{p\beta}} C^{\frac{1}{p}} p^{\frac{1}{p}} \max\left(1, \|\varphi\|_{L^p(X)}\right).\end{aligned}$$

The k-th iteration of (1.6) becomes

$$\max\left(1, \|\varphi\|_{L^{p\beta^k}(X)}\right) \leq C^{\frac{1}{p}\cdot\sum_{i=0}^{k-1}\frac{1}{\beta^i}} \cdot p^{\frac{1}{p}\cdot\sum_{i=0}^{k-1}\frac{1}{\beta^i}} \cdot \beta^{\frac{1}{p}\cdot\sum_{i=1}^{k-1}\frac{i}{\beta^i}} \max\left(1, \|\varphi\|_{L^p(X)}\right).$$

Using $\beta > 1$ and Lemma 1.17, letting $k \to \infty$, and $p = 2$, we get

$$\max\left(1, \|\varphi\|_{L^\infty(X)}\right) \leq C\max\left(1, \|\varphi\|_{L^2(X)}\right). \tag{1.7}$$

Here we use the covergence $\sum_{i=1}^\infty \frac{i}{\beta^i} < \infty$.

We can further deal with $\|\varphi\|_{L^2(X)}$ using Poincaré inequality, Theorem 1.16. Indeed,

$$\begin{aligned}C^{-1}\int_X \varphi^2\omega^n &\leq \int_X |\partial\varphi|_g^2\omega^n && \int_X \varphi\omega^n = 0 \text{ and Theorem 1.16} \\ &\leq C\int_X |\varphi|\omega^n && \text{reverse Sobolev inequality (1.3)} \\ &\leq C\left(\int_X \varphi^2\omega^n\right)^{\frac{1}{2}}, && \text{Hölder}\end{aligned}$$

so that $\|\varphi\|_{L^2(X)} \leq C$ for some uniform constant C. Combined with inequality (1.7), we have uniform bound $\|\varphi\|_{L^\infty(X)} \leq C$, as desired.

Remark 1.19. Here the constant $C = C\left((X,\omega), \left\|e^F\right\|_{L^\infty(X)}\right)$. In fact, we can modify this argument to get $\|\varphi\|_{L^\infty(X)} \leq C = C\left((X,\omega), \left\|e^F\right\|_{L^q(X)}, q\right)$ for any $q > n$. The same claim holds for all $q > 1$, but with a different proof given by Kołodziej.

Question 1.20. *What about the case $n = 1$? Let $\beta = 2$. Sobolev inequality with $p = 4/3$, $q = 2p/(2-p) = 4$ gives*

$$\left(\int_X |f|^4\omega\right)^{\frac{1}{4}} \leq C\left(\left(\int_X |f|^{\frac{4}{3}}\omega\right)^{\frac{3}{4}} + \left(\int_X |\partial f|_g^{\frac{4}{3}}\omega\right)^{\frac{3}{4}}\right).$$

Plug in $f = \varphi|\varphi|^{\frac{p-2}{2}}$ for $p \geq 2$,

$$\begin{aligned}\left(\int_X |\varphi|^{2p}\omega\right)^{\frac{1}{4}} &\leq C\left(\left(\int_X |\varphi|^{\frac{2p}{3}}\omega\right)^{\frac{3}{4}} + \left(\int_X \left|\partial\left(\varphi|\varphi|^{\frac{p-2}{2}}\right)\right|_g^{\frac{4}{3}}\omega\right)^{\frac{3}{4}}\right) \\ &\leq C\left(\left(\int_X |\varphi|^p\omega\right)^{\frac{1}{2}} + \left(\int_X \left|\partial\left(\varphi|\varphi|^{\frac{p-2}{2}}\right)\right|_g^2\omega\right)^{\frac{1}{2}}\right)\end{aligned}$$

by Hölder. Then as above,

$$\left(\int_X |\varphi|^{2p}\omega\right)^{\frac{1}{2}} \leq C\left(\int_X |\varphi|^p\omega + \int_X \left|\partial\left(\varphi|\varphi|^{\frac{p-2}{2}}\right)\right|_g^2\omega\right)$$
$$\leq C\left(\int_X |\varphi|^p\omega + p\int_X |\varphi|^{p-1}\omega\right) \qquad \text{by reverse Sobolev inequality (1.2)}$$

which is now the same as (1.4) *in the case* $n \geq 2$*. We then proceed exactly as before with* $\beta = 2$ *now.*

We have now finished **Step 1** in the proof of Theorem 1.12, giving a uniform bound on the L^∞-norm of all solutions φ.

Step 2. We next show that there exists some constant $C = C\left((X,\omega), \|F\|_{C^2(X)}\right)$ such that

$$C^{-1}\omega \leq \tilde{\omega} := \omega + i\partial\overline{\partial}\varphi \leq C\omega \tag{1.8}$$

Locally, inequality (1.8) means that

$$C^{-1}(g_{i\bar{j}}) \leq (\tilde{g}_{i\bar{j}}) \leq C(g_{i\bar{j}}).$$

If we simultanesouly diagonalize g and $\tilde{g}$ at $x \in X$ such that $(g_{i\bar{j}})(x) = \delta_{ij}$, $(\tilde{g}_{i\bar{j}})(x) = \lambda_j\delta_{ij}$, $\lambda_j > 0$ indeed. Then $C^{-1} \leq \lambda_j \leq C$ for all $j = 1, ..., n$.

We first reduce the inequality between matrices/metrics/tensors to an inequality between functions. Recall the trace of a real (1, 1)-form defined by

$$\operatorname{tr}_\omega \tilde{\omega} = g^{i\bar{j}}\tilde{g}_{i\bar{j}} \Leftrightarrow n\omega^{n-1} \wedge \tilde{\omega} = \operatorname{tr}_\omega \tilde{\omega} \cdot \omega^n.$$

Indeed $\operatorname{tr}_\omega \tilde{\omega} \in C^\infty(X, \mathbb{R}_+)$, for in the diagonalized local coordinates above, $\operatorname{tr}_\omega \tilde{\omega}(x) = \sum_j \lambda_j$.

Claim 1.21. *If* $\operatorname{tr}_\omega \tilde{\omega} \leq C$ *on* X *for some uniform constant* C*, then inequality* (1.8) *follows.*

Proof of Claim 1.21. In local coordinates above, we have

$$\lambda_j < \sum_j \lambda_j \leq C \Rightarrow \tilde{\omega}(x) \leq C\omega(x)$$

Thus the inequality $\tilde{\omega} \leq C\omega$ follows immediately.

The other side needs uniform lower bound on λ_j. We use the PDE:

$$\prod_{j=1}^n \lambda_j = \frac{\det\left(\tilde{g}_{i\bar{j}}\right)}{\det\left(g_{i\bar{j}}\right)} = \frac{\tilde{\omega}^n}{\omega^n}(x) = e^{F+\mu\varphi}(x).$$

$\|F\|_{L^\infty(X)} \leq \|F\|_{C^2(X)}$ trivially, and we proved in **Step 1** that $\|\varphi\|_{L^\infty(X)} \leq C = C((X,\omega), \|F\|_{L^\infty(X)})$. Thus

$$\prod_{j=1}^n \lambda_j \geq e^{-\|F+\mu\varphi\|_{L^\infty(X)}} \geq C^{-1}$$

for some constant $C = C\left((X,\omega), \|F\|_{C^2(X)}\right)$. Meanwhile, $\sum_j \lambda_j \leq C$, so that $\lambda_j \geq C^{-n}$ for all j. This proves the claim. ☐

We are left to show that

$$\operatorname{tr}_\omega \tilde{\omega} \leq C.$$

The trick is to use maximum principle. We choose $\Delta_{\tilde g}$ over Δ_g as the former is the **linearized opeartor of PDE**. We compute

$$\Delta_{\tilde g} \operatorname{tr}_\omega \tilde\omega = \tilde g^{k\bar l}\partial_k\partial_{\bar l}\left(g^{i\bar j}\tilde g_{i\bar j}\right).$$

To simplify, we choose a coordinate at x that is normal for g and diagonalizes $\tilde g$. That is,

$$\begin{cases} g_{i\bar j}(x) = \delta_{ij} \\ \partial_k g_{i\bar j}(x) = 0 \\ \tilde g_{i\bar j}(x) = \lambda_j \delta_{ij} \end{cases}$$

and hence

$$\partial_k g^{i\bar j}(x) = 0.$$

This can be achieved: **start from the simultaneously diagonalized coordinates as above, and then perturb the coordinate by terms of order 2 such that $dg_{i\bar j}$ vanishes at x; meanwhile it's easy to check that $\tilde g(x)$ remains the same**. See e.g. Huybrechts §1.3. Then at x, (**typo for line 2?**)

$$\begin{aligned}
\Delta_{\tilde g} \operatorname{tr}_\omega \tilde\omega &= \tilde g^{k\bar l}\partial_k\partial_{\bar l}\left(g^{i\bar j}\tilde g_{i\bar j}\right)\\
&= \sum_{i,j,k,l} \tilde g^{k\bar l}\tilde g_{i\bar j}R_{j\bar i k\bar l} + \tilde g^{k\bar l} g^{i\bar j}\partial_k\partial_{\bar l}\tilde g_{i\bar j}\\
&= \sum_{i,k}\tilde g^{k\bar k}\tilde g_{i\bar i}R_{i\bar i k\bar k} - \sum_{i,j,k,l}\tilde g^{k\bar l}g^{i\bar j}\tilde R_{i\bar j k\bar l} + \sum_{i,j,k,l,p,q}\tilde g^{k\bar l}g^{i\bar j}\tilde g^{p\bar q}\partial_{\bar l}\tilde g_{p\bar j}\partial_k\tilde g_{i\bar q}\\
&= \sum_{i,k}\tilde g^{k\bar k}\tilde g_{i\bar i}R_{i\bar i k\bar k} - \operatorname{tr}_\omega \operatorname{Ric}(\tilde\omega) + \sum_{i,j,k,l,p,q}\tilde g^{k\bar l}g^{i\bar j}\tilde g^{p\bar q}\partial_{\bar l}\tilde g_{p\bar j}\partial_k\tilde g_{i\bar q}
\end{aligned}$$

where we recall that

$$R_{i\bar j k\bar l} := g^{p\bar q}\partial_{\bar l}g_{p\bar j}\partial_k g_{i\bar q} - \partial_k\partial_{\bar l}g_{i\bar j}$$

$$R_{i\bar j} = g^{k\bar l}R_{i\bar j k\bar l} = -\partial_i\partial_{\bar j}\log\det(g_{p\bar q})$$

$$\operatorname{Ric}(\omega) = iR_{i\bar j}dz_i\wedge d\bar z_j$$

$$R = \operatorname{tr}_\omega\operatorname{Ric}(\omega) = g^{i\bar j}R_{i\bar j}.$$

By PDE, $\tilde\omega^n = e^{F+\mu\varphi}\omega^n$, so $\det(\tilde g) = e^{F+\mu\varphi}\det(g)$, and

$$\operatorname{Ric}(\tilde\omega) - \operatorname{Ric}(\omega) = -i\partial\bar\partial(F+\mu\varphi) = -i\partial\bar\partial F - \mu\tilde\omega + \mu\omega. \tag{1.9}$$

Since $\operatorname{tr}_\omega\tilde\omega > 0$, $\operatorname{tr}_\omega\omega = n$, it follows that

$$-\operatorname{tr}_\omega\operatorname{Ric}(\tilde\omega) = -R + \Delta_g F + \mu\operatorname{tr}_\omega\tilde\omega - \mu n \geq -C$$

for some uniform constant $C = C\left((X,\omega), \|F\|_{C^2(X)}\right)$ by compactness of X.

Also,

$$\sum_{i,k}\tilde g^{k\bar k}\tilde g_{i\bar i}R_{i\bar i k\bar k} \geq -C\sum_{i,k}\tilde g^{k\bar k}\tilde g_{i\bar i} = -C\sum_{i,k}\lambda_i\lambda_k^{-1} = -C\operatorname{tr}_\omega\tilde\omega\cdot\operatorname{tr}_{\tilde\omega}\omega$$

for some uniform constant $C = C(X,\omega)$. In summary, in this normal coordinate at x, we have

$$\Delta_{\tilde g}\operatorname{tr}_\omega\tilde\omega \geq -C\operatorname{tr}_\omega\tilde\omega\cdot\operatorname{tr}_{\tilde\omega}\omega - C + \sum_{i,j,k,l,p,q}\tilde g^{k\bar l}\tilde g^{p\bar q}g^{i\bar j}\nabla_{\bar l}\tilde g_{p\bar j}\nabla_k\tilde g_{i\bar q}. \tag{1.10}$$

Both LHS and RHS are **coordinate-free quantities (tensorial)**. To apply maximum principle and show uniform bound on $\operatorname{tr}_\omega \tilde{\omega}$, we need some correction function on the LHS under $\Delta_{\tilde{g}}$. Observe that

$$\Delta_{\tilde{g}}\varphi = \tilde{g}^{i\bar{j}}\partial_i\partial_{\bar{j}}\varphi = \tilde{g}^{i\bar{j}}\left(\tilde{g}_{i\bar{j}} - g_{i\bar{j}}\right) = n - \operatorname{tr}_{\tilde{\omega}}\omega,$$

and for $u \in C^\infty(X, \mathbb{R}_+)$,

$$\Delta_{\tilde{g}}\log u = \tilde{g}^{i\bar{j}}\left(\partial_i\partial_{\bar{j}}\log u\right) = \frac{\Delta_{\tilde{g}}u}{u} - \frac{|\partial u|^2_{\tilde{g}}}{u^2}.$$

It follows that

$$\begin{aligned}\Delta_{\tilde{g}}\log\operatorname{tr}_\omega\tilde{\omega} &= \frac{\Delta_{\tilde{g}}\operatorname{tr}_\omega\tilde{\omega}}{\operatorname{tr}_\omega\tilde{\omega}} - \frac{|\partial\operatorname{tr}_\omega\tilde{\omega}|^2_{\tilde{g}}}{\left(\operatorname{tr}_\omega\tilde{\omega}\right)^2}\\ &\geq -C\operatorname{tr}_{\tilde{\omega}}\omega - \frac{C}{\operatorname{tr}_\omega\tilde{\omega}} + \frac{\sum_{i,j,k,l,p,q}\tilde{g}^{k\bar{l}}\tilde{g}^{p\bar{q}}g^{i\bar{j}}\nabla_{\bar{l}}\tilde{g}_{p\bar{j}}\nabla_k\tilde{g}_{i\bar{q}}}{\operatorname{tr}_\omega\tilde{\omega}} - \frac{|\partial\operatorname{tr}_\omega\tilde{\omega}|^2_{\tilde{g}}}{\left(\operatorname{tr}_\omega\tilde{\omega}\right)^2}.\end{aligned}$$

Observe that

$$\operatorname{tr}_\omega\tilde{\omega}\cdot\operatorname{tr}_{\tilde{\omega}}\omega = \sum_{j,k}\lambda_j\lambda_k^{-1} \geq n,$$

so the second term

$$-\frac{C}{\operatorname{tr}_\omega\tilde{\omega}} \geq \frac{-C\operatorname{tr}_{\tilde{\omega}}\omega}{n}$$

and can be absorbed into the first term $-C\operatorname{tr}_{\tilde{\omega}}\omega$. Now we have

$$\Delta_{\tilde{g}}\log\operatorname{tr}_\omega\tilde{\omega} \geq -C\operatorname{tr}_{\tilde{\omega}}\omega + \frac{\sum_{i,j,k,l,p,q}\tilde{g}^{k\bar{l}}\tilde{g}^{p\bar{q}}g^{i\bar{j}}\nabla_{\bar{l}}\tilde{g}_{p\bar{j}}\nabla_k\tilde{g}_{i\bar{q}}}{\operatorname{tr}_\omega\tilde{\omega}} - \frac{|\partial\operatorname{tr}_\omega\tilde{\omega}|^2_{\tilde{g}}}{\left(\operatorname{tr}_\omega\tilde{\omega}\right)^2}.$$

Taking $A := C + 1$, and replacing C by the new constant An, we get

$$\Delta_{\tilde{g}}\left(\log\operatorname{tr}_\omega\tilde{\omega} - A\varphi\right) \geq \operatorname{tr}_{\tilde{\omega}}\omega - C + \frac{\sum_{i,j,k,l,p,q}\tilde{g}^{k\bar{l}}\tilde{g}^{p\bar{q}}g^{i\bar{j}}\nabla_{\bar{l}}\tilde{g}_{p\bar{j}}\nabla_k\tilde{g}_{i\bar{q}}}{\operatorname{tr}_\omega\tilde{\omega}} - \frac{|\partial\operatorname{tr}_\omega\tilde{\omega}|^2_{\tilde{g}}}{\left(\operatorname{tr}_\omega\tilde{\omega}\right)^2}$$

We claim that the error term is non-negative:

$$\frac{\sum_{i,j,k,l,p,q}\tilde{g}^{k\bar{l}}\tilde{g}^{p\bar{q}}g^{i\bar{j}}\nabla_{\bar{l}}\tilde{g}_{p\bar{j}}\nabla_k\tilde{g}_{i\bar{q}}}{\operatorname{tr}_\omega\tilde{\omega}} - \frac{|\partial\operatorname{tr}_\omega\tilde{\omega}|^2_{\tilde{g}}}{\left(\operatorname{tr}_\omega\tilde{\omega}\right)^2} \geq 0 \tag{1.11}$$

Assuming this claim first, we get

$$\Delta_{\tilde{g}}\left(\log\operatorname{tr}_\omega\tilde{\omega} - A\varphi\right) \geq \operatorname{tr}_{\tilde{\omega}}\omega - C. \tag{1.12}$$

We now apply maximum principle. Pick $x \in X$ where $\log\operatorname{tr}_\omega\tilde{\omega} - A\varphi$ attains its maximum. Then

$$0 \geq \Delta_{\tilde{g}}\left(\log\operatorname{tr}_\omega\tilde{\omega} - A\varphi\right)(x) \geq \operatorname{tr}_{\tilde{\omega}}\omega(x) - C \Rightarrow \operatorname{tr}_{\tilde{\omega}}\omega(x) \leq C.$$

To compare $\operatorname{tr}_{\tilde{\omega}}\omega$ and $\operatorname{tr}_\omega\tilde{\omega}$, note the elementary inequality

$$\operatorname{tr}_\omega\tilde{\omega} \leq \frac{1}{(n-1)!}\left(\operatorname{tr}_{\tilde{\omega}}\omega\right)^{n-1}\cdot\frac{\tilde{\omega}^n}{\omega^n}, \tag{1.13}$$

which can be easily checked at each point on X using the diagonalized coordinates above:

$$\begin{aligned}\operatorname{tr}_\omega \tilde{\omega} &= \sum_i \lambda_i \\ &\le \frac{1}{(n-1)!}\left(\sum_j \lambda_j^{-1}\right)^{n-1}\left(\prod_k \lambda_k\right) \\ &= \frac{1}{(n-1)!}\left(\operatorname{tr}_{\tilde{\omega}}\omega\right)^{n-1}\cdot\frac{\tilde{\omega}^n}{\omega^n}.\end{aligned}$$

Then by (1.13),

$$\operatorname{tr}_\omega \tilde{\omega}(x) \le C\frac{\tilde{\omega}^n}{\omega^n}(x) = Ce^{F+\mu\varphi}(x) \le C$$

using $\|\varphi\|_{L^\infty(X)} \le C$ from **Step 1**. Hence $\log \operatorname{tr}_\omega \tilde{\omega}(x) \le C$. Using $\|\varphi\|_{L^\infty(X)} \le C$ again, we get

$$\log \operatorname{tr}_\omega \tilde{\omega}(x) - A\varphi(x) \le C.$$

Then by maximality at x,

$$\log \operatorname{tr}_\omega \tilde{\omega} - A\varphi \le C \text{ on } X.$$

Again, since $\|\varphi\|_{L^\infty(X)} \le C$, we have the desired inequality

$$\operatorname{tr}_\omega \tilde{\omega} \le C = C\left((X,\omega), \|F\|_{C^2(X)}\right).$$

It remains to prove inequality (1.11). We claim that

$$\frac{\sum_{i,j,k,l,p,q} \tilde{g}^{k\bar{l}}\tilde{g}^{p\bar{q}}g^{i\bar{j}}\nabla_{\bar{l}}\tilde{g}_{p\bar{j}}\nabla_k\tilde{g}_{i\bar{q}}}{\operatorname{tr}_\omega\tilde{\omega}} - \frac{|\partial \operatorname{tr}_\omega \tilde{\omega}|^2_{\tilde{g}}}{(\operatorname{tr}_\omega\tilde{\omega})^2} = \frac{1}{\operatorname{tr}_\omega\tilde{\omega}}\tilde{g}^{k\bar{l}}\tilde{g}^{p\bar{q}}g^{i\bar{j}}B_{ki\bar{q}}\overline{B_{lj\bar{p}}} = \frac{|B|^2_{mixed}}{\operatorname{tr}_\omega\tilde{\omega}} \ge 0, \tag{1.14}$$

where

$$B_{ki\bar{q}} := \nabla_k\tilde{g}_{i\bar{q}} - \frac{\partial_k(\operatorname{tr}_\omega\tilde{\omega})}{\operatorname{tr}_\omega\tilde{\omega}}\tilde{g}_{i\bar{q}}.$$

are coordinates of the 3-tensor

$$B := \nabla\tilde{g} - \partial(\log \operatorname{tr}_\omega\tilde{\omega})\otimes\tilde{g},$$

and we define the mixed norm exactly as above:

$$|B|^2_{mixed} := \tilde{g}^{k\bar{l}}\tilde{g}^{p\bar{q}}g^{i\bar{j}}B_{ki\bar{q}}\overline{B_{lj\bar{p}}}.$$

Clearly $|B|^2_{mixed} \ge 0$, for if we choose coordinates near $x \in X$ normal to g and diagonal to $\tilde{g}$ as above, we get

$$|B|^2_{mixed} = \sum_{k,i,p}\lambda_k^{-1}\lambda_p^{-1}B_{ki\bar{p}}\overline{B_{ki\bar{p}}} \ge 0.$$

To see the first equality in (1.14), we compute by hand:

$$\begin{aligned}\tilde{g}^{k\bar{l}}\tilde{g}^{p\bar{q}}g^{i\bar{j}}B_{ki\bar{q}}\overline{B_{lj\bar{p}}} &= \tilde{g}^{k\bar{l}}\tilde{g}^{p\bar{q}}g^{i\bar{j}}\nabla_{\bar{l}}\tilde{g}_{p\bar{j}}\nabla_k\tilde{g}_{i\bar{q}} \\ &\quad - \frac{1}{\operatorname{tr}_\omega\tilde{\omega}}\tilde{g}^{k\bar{l}}g^{i\bar{j}}\nabla_k\tilde{g}_{i\bar{j}}\cdot\partial_{\bar{l}}(\operatorname{tr}_\omega\tilde{\omega}) \\ &\quad - \frac{1}{\operatorname{tr}_\omega\tilde{\omega}}\tilde{g}^{k\bar{l}}g^{i\bar{j}}\nabla_{\bar{l}}\tilde{g}_{i\bar{j}}\cdot\partial_k(\operatorname{tr}_\omega\tilde{\omega}) \\ &\quad + \frac{1}{(\operatorname{tr}_\omega\tilde{\omega})^2}\tilde{g}^{k\bar{l}}\tilde{g}^{p\bar{q}}g^{i\bar{j}}\partial_k(\operatorname{tr}_\omega\tilde{\omega})\,\partial_{\bar{l}}(\operatorname{tr}_\omega\tilde{\omega})\,\tilde{g}_{i\bar{q}}\tilde{g}_{p\bar{j}}.\end{aligned}$$

The first term is as desired. For the rest, compute in coordinates chosen as above at a fixed point $x \in X$.

$$g^{i\bar{j}}\partial_k\tilde{g}_{i\bar{j}} = \partial_k\left(g^{i\bar{j}}\tilde{g}_{i\bar{j}}\right) = \partial_k\left(\operatorname{tr}_\omega\tilde{\omega}\right) \Rightarrow \tilde{g}^{k\bar{l}}g^{i\bar{j}}\nabla_k\tilde{g}_{i\bar{j}}\cdot\partial_{\bar{l}}\left(\operatorname{tr}_\omega\tilde{\omega}\right) = \left|\partial\operatorname{tr}_\omega\tilde{\omega}\right|^2_{\tilde{g}},$$

and similarly,

$$\tilde{g}^{k\bar{l}}g^{i\bar{j}}\nabla_{\bar{l}}\tilde{g}_{i\bar{j}}\cdot\partial_k\left(\operatorname{tr}_\omega\tilde{\omega}\right) = \left|\partial\operatorname{tr}_\omega\tilde{\omega}\right|^2_{\tilde{g}}.$$

For the last term,

$$\tilde{g}^{k\bar{l}}\tilde{g}^{p\bar{q}}g^{i\bar{j}}\partial_k\left(\operatorname{tr}_\omega\tilde{\omega}\right)\partial_{\bar{l}}\left(\operatorname{tr}_\omega\tilde{\omega}\right)\tilde{g}_{i\bar{q}}\tilde{g}_{p\bar{j}} = g^{i\bar{j}}\tilde{g}_{i\bar{j}}\cdot\tilde{g}^{k\bar{l}}\partial_k\left(\operatorname{tr}_\omega\tilde{\omega}\right)\partial_{\bar{l}}\left(\operatorname{tr}_\omega\tilde{\omega}\right) = \operatorname{tr}_\omega\tilde{\omega}\cdot\left|\partial\operatorname{tr}_\omega\tilde{\omega}\right|^2_{\tilde{g}}.$$

Combined, this yields the first equality in (1.14). This completes **Step 2**.

Step 3. Last step in the proof of Yau's a priori estimates is the following.

Theorem 1.22 (Calabi-Yau-Nirenberg). *Let (X^n,ω) be a compact Kähler manifold. Let $F \in C^\infty(X,\mathbb{R})$, $\mu = 0$ or 1. Then there exists some constant $C = C\left((X,\omega), \|F\|_{C^3(X)}\right)$ such that for all $\varphi \in C^\infty(X,\mathbb{R})$ solving the problem*

$$\begin{cases} \tilde{\omega} := \omega + i\partial\bar{\partial}\varphi > 0 & \\ \int_X \varphi\omega^n = 0 & \text{if } \mu = 0 \\ \tilde{\omega}^n = e^{F+\mu\varphi}\omega^n & \text{on } X, \end{cases}$$

we have uniform bound

$$\|\tilde{\omega}\|_{C^1(X,g)} \leq C.$$

Let's first finish the proof of Theorem 1.12 assuming Theorem 1.22, whose proof is further below. Since ω is given, the estimate above implies

$$\left\|i\partial\bar{\partial}\varphi\right\|_{C^1(X,g)} \leq C$$
$$\Rightarrow \|\Delta_g\varphi\|_{C^1(X,g)} \leq C$$
$$\Rightarrow \|\Delta_g\varphi\|_{C^\alpha(X,g)} \leq C, \quad \forall\alpha\in(0,1), \quad C \textbf{ independent of } \alpha.$$

Our goal is to bound $\|\varphi\|_{C^2(X,g)}$ uniformly. Thus we apply global Schauder estimate, Theorem 1.9. Let

$$\tilde{\varphi} := \varphi - \frac{\int_X \varphi\omega^n}{\int_X \omega^n},$$

so that $\int_X \tilde{\varphi}\omega^n = 0$ and $\Delta_g\tilde{\varphi} = \Delta_g\varphi$. Then Theorem 1.9 1) gives

$$\|\tilde{\varphi}\|_{C^{2,\alpha}(X,g)} \leq C\|\Delta_g\varphi\|_{C^\alpha(X,g)}$$

$$\Rightarrow \|\varphi\|_{C^{2,\alpha}(X,g)} \leq \|\tilde{\varphi}\|_{C^{2,\alpha}(X,g)} + \left|\frac{\int_X \varphi\omega^n}{\int_X \omega^n}\right| \leq C\|\Delta_g\varphi\|_{C^\alpha(X,g)} + \|\varphi\|_{L^\infty(X)}.$$

Combining with the uniform bound $\|\varphi\|_{L^\infty(X)} \leq C$ proved in **Step 1**, we get

$$\|\varphi\|_{C^{2,\alpha}(X,g)} \leq C = C\left((X,\omega), \|F\|_{C^3(X)}, \alpha\right), \quad \forall\alpha\in(0,1).$$

Along with the metric comparison result in **Step 2**, this concludes the proof of Theorem 1.12.

Question 1.23. *The dependence of C on α comes solely from global Schauder Theorem 1.9.*

Also C only depends on $\|F\|_{C^3(X)}$, instead of $\|F\|_{C^{3,\alpha}(X)}$ as stated in Theorem 1.12? This is because $\|\Delta_g\varphi\|_{C^1(X,g)} \leq C$ implies $\|\Delta_g\varphi\|_{C^\alpha(X,g)} \leq C$ for another C independent of α?

□

Proof of Theorem 1.22. Recall from **Step 2** that

$$C^{-1}\omega \le \tilde{\omega} \le C\omega \tag{1.15}$$

for some uniform constant C. Thus

$$\|\tilde{\omega}\|_{C^1(X,g)} = \|\tilde{g}\|_{C^1(X,g)} := \|\tilde{g}\|_{C^0(X,g)} + \|\nabla\tilde{g}\|_{C^0(X,g)} \le C + \|\nabla\tilde{g}\|_{C^0(X,g)},$$

where $\nabla = \nabla^g$ is the **Chern connection with respect to metric** g.

We are left to show that

$$\|\nabla\tilde{g}\|_{C^0(X,g)} = \sup_X |\nabla\tilde{g}|_g \le C,$$

which by (1.15) is equivalent to

$$\sup_X |\nabla\tilde{g}|^2_{\tilde{g}} \le C.$$

As before, the idea for such uniform bound is to apply the maximum principle. We first claim that

$$|\nabla\tilde{g}|^2_{\tilde{g}} = |T|^2_{\tilde{g}} \tag{1.16}$$

where T is a 3-tensor defined by

$$T^k_{ij} := \tilde{\Gamma}^k_{ij} - \Gamma^k_{ij},$$

the difference of Christoffel symbols of $\nabla^{\tilde{g}}$ and ∇^g. Recall that in general local coordinates

$$\Gamma^k_{ij} := \sum_l g^{k\bar{l}}\partial_i g_{j\bar{l}}.$$

To see (1.16), compute $\nabla\tilde{g}$ in coordinates:

$$\nabla_i\tilde{g}_{j\bar{l}} = \partial_i\tilde{g}_{j\bar{l}} - \Gamma^p_{ij}\tilde{g}_{p\bar{l}}$$

This is in fact the coefficient for $dz_j \wedge d\bar{z}_l$ in $\nabla_i\tilde{g}$. Then observe that

$$\tilde{g}^{k\bar{l}}\nabla_i\tilde{g}_{j\bar{l}} = \tilde{g}^{k\bar{l}}\partial_i\tilde{g}_{j\bar{l}} - \Gamma^k_{ij} = T^k_{ij}.$$

Since raising the $\bar{l}$ index is an isometry for $\tilde{g}$, we get (1.16).

Remark 1.24. More details to expalin this. $S := \nabla\tilde{g}$ is a tensor of type $(0,3)$, i.e. tensor product of 3 covectors. On the other hand, we view T as a tensor of type $(1,2)$, which is tensor product of a vector with two covectors. Raising index is the sharp # isomorphism that relates S and T. In coordinates,

$$\begin{aligned}
|\nabla\tilde{g}|^2_{\tilde{g}} &= \left|\nabla_i\tilde{g}_{j\bar{l}}dz_i \otimes (dz_j \wedge d\bar{z}_l)\right|^2_{\tilde{g}} \\
&= \nabla_i\tilde{g}_{j\bar{l}}\overline{\nabla_\alpha\tilde{g}_{\beta\bar{\rho}}}\tilde{g}^{i\overline{\alpha}}\tilde{g}^{j\overline{\beta}}\tilde{g}^{\rho\bar{l}} \\
&= \nabla_i\tilde{g}_{j\bar{l}}\overline{\nabla_\alpha\tilde{g}_{\beta\bar{\rho}}}\tilde{g}^{i\overline{\alpha}}\tilde{g}^{j\overline{\beta}}\tilde{g}^{k\bar{l}}\overline{\tilde{g}^{\gamma\bar{\rho}}}\tilde{g}_{k\overline{\gamma}} && \tilde{g}^{i\bar{j}} = \overline{\tilde{g}^{j\bar{i}}} \\
&= T^k_{ij}\overline{T^\gamma_{\alpha\beta}}\tilde{g}^{i\overline{\alpha}}\tilde{g}^{j\overline{\beta}}\tilde{g}_{k\overline{\gamma}} \\
&= \left|T^k_{ij}dz_i \otimes dz_j \otimes \frac{\partial}{\partial z_k}\right|^2_{\tilde{g}} \\
&= |T|^2_{\tilde{g}}.
\end{aligned}$$

To apply the maximum principle to $|\nabla\tilde{g}|^2_{\tilde{g}}$, we must compute its Laplacian. We choose the Laplacian with respect to $\tilde{g}$. This is the **linearized operator of Monge-Ampère equation**. Compute

$$\begin{aligned}
\Delta_{\tilde{g}}|\nabla\tilde{g}|^2_{\tilde{g}} &= \Delta_{\tilde{g}}|T|^2_{\tilde{g}} \\
&= \tilde{g}^{p\overline{q}}\partial_p\partial_{\overline{q}}\left(T^k_{ij}\overline{T^\gamma_{\alpha\beta}}\tilde{g}^{i\overline{\alpha}}\tilde{g}^{j\overline{\beta}}\tilde{g}_{k\overline{\gamma}}\right) \\
&= \tilde{g}^{p\overline{q}}\tilde{\nabla}_p\tilde{\nabla}_{\overline{q}}\left(T^k_{ij}\overline{T^\gamma_{\alpha\beta}}\tilde{g}^{i\overline{\alpha}}\tilde{g}^{j\overline{\beta}}\tilde{g}_{k\overline{\gamma}}\right) && (*) \\
&= \tilde{g}^{p\overline{q}}\left(\tilde{\nabla}_pT^k_{ij}\right)\left(\tilde{\nabla}_{\overline{q}}\overline{T^\gamma_{\alpha\beta}}\right)\tilde{g}^{i\overline{\alpha}}\tilde{g}^{j\overline{\beta}}\tilde{g}_{k\overline{\gamma}} + \tilde{g}^{p\overline{q}}\left(\tilde{\nabla}_p\overline{T^\gamma_{\alpha\beta}}\right)\left(\tilde{\nabla}_{\overline{q}}T^k_{ij}\right)\tilde{g}^{i\overline{\alpha}}\tilde{g}^{j\overline{\beta}}\tilde{g}_{k\overline{\gamma}} \\
&\quad + \tilde{g}^{p\overline{q}}\tilde{\nabla}_p\tilde{\nabla}_{\overline{q}}\left(T^k_{ij}\right)\overline{T^\gamma_{\alpha\beta}}\tilde{g}^{i\overline{\alpha}}\tilde{g}^{j\overline{\beta}}\tilde{g}_{k\overline{\gamma}} + \tilde{g}^{p\overline{q}}T^k_{ij}\tilde{\nabla}_p\tilde{\nabla}_{\overline{q}}\left(\overline{T^\gamma_{\alpha\beta}}\right)\tilde{g}^{i\overline{\alpha}}\tilde{g}^{j\overline{\beta}}\tilde{g}_{k\overline{\gamma}} && (**) \\
&= \left|\tilde{\nabla}T\right|^2_{\tilde{g}} + \left|\overline{\tilde{\nabla}}T\right|^2_{\tilde{g}} + \tilde{g}^{p\overline{q}}\overline{T^\gamma_{\alpha\beta}}\tilde{\nabla}_p\tilde{\nabla}_{\overline{q}}\left(T^k_{ij}\right)\cdot\tilde{g}^{i\overline{\alpha}}\tilde{g}^{j\overline{\beta}}\tilde{g}_{k\overline{\gamma}} + \tilde{g}^{p\overline{q}}T^k_{ij}\tilde{\nabla}_p\tilde{\nabla}_{\overline{q}}\left(\overline{T^\gamma_{\alpha\beta}}\right)\cdot\tilde{g}^{i\overline{\alpha}}\tilde{g}^{j\overline{\beta}}\tilde{g}_{k\overline{\gamma}}.
\end{aligned}$$

Remark 1.25. For lines $(*)$ and $(**)$ above, the following facts about the Chern connection ∇ are used:

- For a function f one has $\nabla_i f = \partial_i f$.
- For a tensor, e.g. T represented by T^k_{ij} above, one has $\nabla_p T^k_{ij} = \partial_p T^k_{ij} - \Gamma^l_{pi}T^k_{lj} - \Gamma^l_{pj}T^k_{il} + \Gamma^k_{pl}T^l_{ij}$. **This is abuse of notation, and $\nabla_p T^k_{ij}$ actually denotes the coordinate entries of $\nabla_p T$.** In particular, $\nabla_p g_{i\overline{j}} = \partial_p g_{i\overline{j}} - \Gamma^l_{pi}g_{l\overline{j}} = \partial_p g_{i\overline{j}} - g^{l\overline{m}}\left(\partial_p g_{i\overline{m}}\right)g_{l\overline{j}} = 0$, which means $\nabla_p g = 0$. Similarly, $\nabla_p\left(g^{\#\#}\right) = 0$, as $\nabla_p g^{i\overline{j}} = \partial_p g^{i\overline{j}} + \Gamma^i_{pl}g^{l\overline{j}} = \partial_p g^{i\overline{j}} + g^{i\overline{m}}\left(\partial_p g_{l\overline{m}}\right)g^{l\overline{j}} = 0$.
- ∇_p satisfies the "Leibniz rule" for the natural pairing between vector fields and covector fields, **or more generally the natural pairing between a (k,l)-tensor field and a (m,n)-tensor field**.
$$\nabla\langle S,T\rangle_g = \langle\nabla S,T\rangle_g + \langle S,\overline{\nabla}T\rangle_g$$
In the computation above, the function $T^k_{ij}\overline{T^\gamma_{\alpha\beta}}\tilde{g}^{i\overline{\alpha}}\tilde{g}^{j\overline{\beta}}\tilde{g}_{k\overline{\gamma}}$ is a natural pairing of a $(2,4)$-tensor $T\otimes\overline{T}$ with the $(4,2)$-tensor $g\otimes g^{\#\#}\otimes g^{\#\#}$, where g is $(0,2)$-tensor.
- ∇_p also satisfies the "Leibniz rule" for the tensor product of tensor fields: $\nabla_p(F\otimes G) = \nabla_p F\otimes G + F\otimes\nabla_p G$.
- Another way to see this is to view $\|T\|^2_{\tilde{g}} = \langle T,T\rangle_{\tilde{g}}$ as an inner product on tensor fields extended from $\tilde{g}$, an inner product on vector fields. Thus we have **Cauchy-Schwarz** inequality which will be useful below.

The last two terms are almost complex conjugates of each other, except that $\tilde{\nabla}\overline{\tilde{\nabla}}$ are in wrong order. This motivates the use of curvature to relate them. Thus compute

$$\begin{aligned}
&\tilde{\nabla}_{\overline{p}}\tilde{\nabla}_q T^c_{ab} - \tilde{\nabla}_q\tilde{\nabla}_{\overline{p}}T^c_{ab} \\
&= \partial_{\overline{p}}\left(\partial_q T^c_{ab} - \tilde{\Gamma}^l_{qa}T^c_{lb} - \tilde{\Gamma}^l_{qb}T^c_{al} + \tilde{\Gamma}^c_{ql}T^l_{ab}\right) - \left(\partial_q\partial_{\overline{p}}T^c_{ab} - \tilde{\Gamma}^l_{qa}\partial_{\overline{p}}T^c_{lb} - \tilde{\Gamma}^l_{qb}\partial_{\overline{p}}T^c_{al} + \tilde{\Gamma}^c_{ql}\partial_{\overline{p}}T^l_{ab}\right) \\
&= -\partial_{\overline{p}}\tilde{\Gamma}^l_{qa}\cdot T^c_{lb} - \partial_{\overline{p}}\tilde{\Gamma}^l_{qb}\cdot T^c_{al} + \partial_{\overline{p}}\tilde{\Gamma}^c_{ql}\cdot T^l_{ab}.
\end{aligned}$$

Observe that

$$\begin{aligned}
\tilde{g}^{q\overline{p}}\left(-\partial_{\overline{p}}\tilde{\Gamma}^l_{qa}\right) &= -\tilde{g}^{q\overline{p}}\partial_{\overline{p}}\left(\tilde{g}^{l\overline{m}}\partial_q\tilde{g}_{a\overline{m}}\right) \\
&= -\tilde{g}^{q\overline{p}}\partial_{\overline{p}}\left(\tilde{g}^{l\overline{m}}\right)\partial_q\tilde{g}_{a\overline{m}} - \tilde{g}^{q\overline{p}}\tilde{g}^{l\overline{m}}\partial_{\overline{p}}\partial_q\tilde{g}_{a\overline{m}} \\
&= \tilde{g}^{q\overline{p}}\tilde{g}^{l\overline{\beta}}\tilde{g}^{\alpha\overline{m}}\partial_{\overline{p}}g_{\alpha\overline{\beta}}\partial_q\tilde{g}_{a\overline{m}} - \tilde{g}^{q\overline{p}}\tilde{g}^{l\overline{m}}\partial_{\overline{p}}\partial_q\tilde{g}_{a\overline{m}} \\
&= \tilde{g}^{l\overline{m}}\left(\tilde{g}^{q\overline{p}}\tilde{g}^{\alpha\overline{\beta}}\partial_{\overline{p}}g_{\alpha\overline{m}}\partial_q\tilde{g}_{a\overline{\beta}} - \tilde{g}^{q\overline{p}}\partial_{\overline{p}}\partial_q\tilde{g}_{a\overline{m}}\right) \\
&= \tilde{g}^{l\overline{m}}\tilde{R}_{a\overline{m}} \\
&= \tilde{R}^l_a,
\end{aligned}$$

and similarly,

$$\tilde{g}^{q\overline{p}}\left(-\partial_{\overline{p}}\tilde{\Gamma}^{l}_{qb}\right) = \tilde{R}^{l}_{b}, \quad \tilde{g}^{q\overline{p}}\left(\partial_{\overline{p}}\tilde{\Gamma}^{c}_{ql}\right) = -\tilde{R}^{c}_{l}.$$

Combining, we get

$$\tilde{g}^{q\overline{p}}\tilde{\nabla}_{\overline{p}}\tilde{\nabla}_{q}T^{c}_{ab} = \tilde{g}^{q\overline{p}}\tilde{\nabla}_{q}\tilde{\nabla}_{\overline{p}}T^{c}_{ab} + \tilde{R}^{l}_{a}T^{c}_{lb} + \tilde{R}^{l}_{b}T^{c}_{al} - \tilde{R}^{c}_{l}T^{l}_{ab}.$$

By the Monge-Ampère equation (assumption PDE), we have computed in equality (1.9) above that

$$\tilde{R}_{i\bar{j}} = R_{i\bar{j}} - \partial_i\partial_{\bar{j}}F - \mu\tilde{g}_{i\bar{j}} + \mu g_{i\bar{j}}. \tag{1.17}$$

We know from **Step 2** (1.8) that $\tilde{g}$ and g are uniformly equivalent. Hence

$$\left|\tilde{R}\right|^2_{\tilde{g}} = \tilde{g}^{i\bar{k}}\tilde{g}^{l\bar{j}}\tilde{R}_{i\bar{j}}\overline{\tilde{R}_{k\bar{l}}} \le C$$

for some uniform constant C. **This is easily seen at each point in the nice coordinate chosen in Step 2**. By the same reason,

$$\left|\tilde{R}^{l}_{a}T^{c}_{lb}\right|^2_{\tilde{g}} \le C|T|^2_{\tilde{g}}, \quad \left|\tilde{R}^{l}_{b}T^{c}_{al}\right|^2_{\tilde{g}} \le C|T|^2_{\tilde{g}}, \quad \left|\tilde{R}^{c}_{l}T^{l}_{ab}\right|^2_{\tilde{g}} \le C|T|^2_{\tilde{g}}.$$

Combining the computation of $\Delta_{\tilde{g}}|T|^2_{\tilde{g}}$ and estimates above, and using Cauchy-Schwarz,

$$\Delta_{\tilde{g}}|T|^2_{\tilde{g}} \ge 2\operatorname{Re}\left\{\tilde{g}^{p\bar{q}}\overline{T^{\gamma}_{\alpha\beta}}\tilde{\nabla}_p\tilde{\nabla}_{\bar{q}}\left(T^{k}_{ij}\right)\cdot\tilde{g}^{i\bar{\alpha}}\tilde{g}^{j\bar{\beta}}\tilde{g}_{k\bar{\gamma}}\right\} - C|T|^2_{\tilde{g}}. \tag{1.18}$$

Now we use the definition of T to understand the first term on RHS. Compute

$$\begin{aligned}
\tilde{g}^{p\bar{q}}\tilde{\nabla}_p\tilde{\nabla}_{\bar{q}}\left(T^{k}_{ij}\right) &= \tilde{g}^{p\bar{q}}\tilde{\nabla}_p\partial_{\bar{q}}\left(T^{k}_{ij}\right)\\
&= \tilde{g}^{p\bar{q}}\tilde{\nabla}_p\partial_{\bar{q}}\left(\tilde{\Gamma}^{k}_{ij} - \Gamma^{k}_{ij}\right)\\
&= \tilde{g}^{p\bar{q}}\tilde{\nabla}_p\left(R^{k}_{ji\bar{q}} - \tilde{R}^{k}_{ji\bar{q}}\right)\\
&= \tilde{g}^{p\bar{q}}\tilde{\nabla}_p\left(R^{k}_{ij\bar{q}} - \tilde{R}^{k}_{ij\bar{q}}\right) && \text{Bianchi I: } R_{i\bar{j}k\bar{l}} = R_{k\bar{j}i\bar{l}}\\
&= \tilde{g}^{p\bar{q}}\left(\nabla_p R^{k}_{ij\bar{q}} - T^{r}_{pi}R^{k}_{rj\bar{q}} - T^{r}_{pj}R^{k}_{ir\bar{q}} + T^{k}_{pr}R^{r}_{ij\bar{q}}\right) - \tilde{g}^{p\bar{q}}\tilde{\nabla}_p\tilde{R}^{k}_{ij\bar{q}}.
\end{aligned}$$

By Bianchi identities, the last term above yields Ricci of $\tilde{g}$:

$$\begin{aligned}
\tilde{g}^{p\bar{q}}\tilde{\nabla}_p\tilde{R}^{k}_{ij\bar{q}} &= \tilde{g}^{p\bar{q}}\tilde{\nabla}_p\left(\tilde{R}_{i\bar{l}j\bar{q}}\tilde{g}^{k\bar{l}}\right)\\
&= \tilde{g}^{p\bar{q}}\tilde{\nabla}_p\left(\tilde{R}_{i\bar{l}j\bar{q}}\right)\tilde{g}^{k\bar{l}} && \tilde{\nabla}_p\tilde{g}^{k\bar{l}} = 0\\
&= \tilde{g}^{p\bar{q}}\tilde{\nabla}_j\left(\tilde{R}_{p\bar{l}i\bar{q}}\right)\tilde{g}^{k\bar{l}}\\
&= \tilde{\nabla}_j\left(\tilde{g}^{p\bar{q}}\tilde{R}_{p\bar{l}i\bar{q}}\tilde{g}^{k\bar{l}}\right)\\
&= \tilde{\nabla}_j\tilde{R}^{k}_{i}\\
&= \tilde{g}^{k\bar{l}}\tilde{\nabla}_j\tilde{R}_{i\bar{l}}\\
&= \tilde{g}^{k\bar{l}}\nabla_j\tilde{R}_{i\bar{l}} - \tilde{g}^{k\bar{l}}T^{p}_{ji}\tilde{R}_{p\bar{l}}.
\end{aligned}$$

Again, the Monge-Ampère equation implies (1.17) and furthermore,

$$\nabla_j\tilde{R}_{i\bar{l}} = \nabla_j R_{i\bar{l}} - \nabla_j\partial_i\partial_{\bar{l}}F - \mu\nabla_j\tilde{g}_{i\bar{l}}. \tag{1.19}$$

Combined, we estimate as above to get

$$\left|\tilde{g}^{p\overline{q}}\tilde{\nabla}_p\tilde{\nabla}_{\overline{q}}\left(T^k_{ij}\right)\right|^2_{\tilde{g}} \leq C + C|T|^2_{\tilde{g}} \Rightarrow \left|\tilde{g}^{p\overline{q}}\tilde{\nabla}_p\tilde{\nabla}_{\overline{q}}\left(T^k_{ij}\right)\right|_{\tilde{g}} \leq C + C|T|_{\tilde{g}}.$$

Now the term $\tilde{g}^{p\overline{q}}\overline{T^{\gamma}_{\alpha\beta}}\tilde{\nabla}_p\tilde{\nabla}_{\overline{q}}\left(T^k_{ij}\right)\cdot\tilde{g}^{i\overline{\alpha}}\tilde{g}^{j\overline{\beta}}\tilde{g}_{k\overline{\gamma}}$ in (1.18) can be viewed as the $\tilde{g}$-inner product of the tensor $\tilde{g}^{p\overline{q}}\tilde{\nabla}_p\tilde{\nabla}_{\overline{q}}\left(T^k_{ij}\right)$ (which is Laplcacian of tensor T) with the tensor T. Hence Cauchy-Schwarz yields

$$\Delta_{\tilde{g}}|T|^2_{\tilde{g}} \geq C|T|^2_{\tilde{g}} - C|T|_{\tilde{g}} \geq -C|T|^2_{\tilde{g}} - C. \tag{1.20}$$

Recall from **Step 2**, the inequality (1.10) and bounds on $\operatorname{tr}_\omega\tilde{\omega}$ and $\operatorname{tr}_{\tilde{\omega}}\omega$, that

$$\Delta_{\tilde{g}}\operatorname{tr}_\omega\tilde{\omega} \geq -C + \tilde{g}^{k\overline{l}}\tilde{g}^{p\overline{q}}g^{i\overline{j}}\nabla_{\overline{l}}\tilde{g}_{p\overline{j}}\nabla_k\tilde{g}_{i\overline{q}} \geq -C + C^{-1}|\nabla\tilde{g}|^2_{\tilde{g}}, \tag{1.21}$$

as we can replace $g^{i\overline{j}}$ by $C^{-1}\tilde{g}^{i\overline{j}}$ by uniform comparability of $\tilde{g}$ and g deduced in **Step 2**.

Combining (1.16), (1.20), and (1.21), we can pick a sufficiently large but uniform constant A such that

$$\Delta_{\tilde{g}}(|\nabla\tilde{g}|^2_{\tilde{g}} + A\operatorname{tr}_\omega\tilde{\omega}) \geq |\nabla\tilde{g}|^2_{\tilde{g}} - C. \tag{1.22}$$

We are now ready to apply the maximum principle. Consider some point $x \in X$ where $|\nabla\tilde{g}|^2_{\tilde{g}} + A\operatorname{tr}_\omega\tilde{\omega}$ attains its maximum. Then

$$|\nabla\tilde{g}|^2_{\tilde{g}}(x) - C \leq 0 \Rightarrow |\nabla\tilde{g}|^2_{\tilde{g}}(x) + A\operatorname{tr}_\omega\tilde{\omega}(x) \leq C,$$

as we derived the uniform bound on $\operatorname{tr}_\omega\tilde{\omega}$ in **Step 2**. Thus

$$|\nabla\tilde{g}|^2_{\tilde{g}} \leq |\nabla\tilde{g}|^2_{\tilde{g}} + A\operatorname{tr}_\omega\tilde{\omega} \leq C \quad \text{on } X$$

for some uniform constant $C = C\Big((X,\omega), \|F\|_{C^3(X)}\Big)$. This completes the proof of Theorem 1.22.

□

Remark 1.26. The uniform bound on the third derivative of F is used at (1.19).

The computation of $\Delta_{\tilde{g}}|T|^2_{\tilde{g}}$ above is the complex version of the **Bochner Formula**: in summary,

$$\Delta|T|^2 = |\nabla T|^2 + \left|\overline{\nabla}T\right|^2 + 2\operatorname{Re}\langle\Delta T, T\rangle + Q(T),$$

where $Q(T)$ is the error term above involving Ricci curvature. Compare with the real version on smooth manifolds:

$$\frac{1}{2}\Delta|T|^2 = |\nabla T|^2 + \langle\Delta T, T\rangle + Q(T), \quad T \text{ any tensor field.}$$

$$\frac{1}{2}\Delta|X|^2 = |\nabla X|^2 + \langle\Delta X, X\rangle + \operatorname{Ric}(X,X), \quad X \text{ any vector field.}$$

We have so far finished the proof of Yau's answer to the Calabi conjecture (Theorem 1.6) and Aubin-Yau's Theorem 1.5 on the existence of Kähler-Einstein metrics on canonically polarized compact Kähler manifolds.

We add another digression from the proof of Theorem 1.22, on the localized higher order estimates of a Kähler metric, for later use.

Theorem 1.27 (Local higher order estimates)**.** *Let $B_1 = B_1(0) \subset \mathbb{C}^n$ denote the unit ball. For each $A \geq 1$, $k \in \mathbb{N}$, there exists some constant $C = C(k,n,A)$ such that for any Ricci-flat Kähler metric ω on B_1 satisfying*

$$A^{-1}\omega_{\mathbb{C}^n} \leq \omega \leq A\omega_{\mathbb{C}^n} \quad \text{on } B_1,$$

we have

$$\|\omega\|_{C^k(B_{\frac{1}{2}}, g_{\mathbb{C}^n})} \leq C(k,n,A).$$

Here $\omega_{\mathbb{C}^n}$ denotes the Euclidean metric on B_1.

Proof. We start the same as in proof of Theorem 1.22. By uniform comparability of ω and $\omega_{\mathbb{C}^n}$, it suffices to bound $|\nabla g|_g^2$ on $B_{\frac{1}{2}}$. ∇ is the Chern connection with respect to $g_{\mathbb{C}^n}$, which is now trivial in the standard coordinate.

In particular, the Christoffel symbols with respect to $g_{\mathbb{C}^n}$ vanish, so letting T be the tensor with entries

$$T_{ij}^k := \Gamma_{ij}^k,$$

the Christoffel symbols of g, we have

$$|\nabla g|_g^2 = |T|_g^2.$$

Since ω is Ricci flat and $\omega_{\mathbb{C}^n}$ is flat, the Bochner formula we computed above simplifies to

$$\Delta_g |\nabla g|_g^2 = \Delta_g |T|_g^2 = |\nabla T|_g^2 + \left|\overline{\nabla} T\right|_g^2.$$

Let $\rho \in C_c^\infty(B_1)$ be a cutoff function that is 1 on $B_{1/2}$. As $\Delta_g = g^{p\overline{q}} \nabla_p \nabla_{\overline{q}}$, compute

$$\begin{aligned}
\Delta_g \left(\rho^2 |\nabla g|_g^2\right) &= \rho^2 \left(|\nabla T|_g^2 + \left|\overline{\nabla} T\right|_g^2\right) + |\nabla g|_g^2 \Delta_g(\rho^2) + 2\operatorname{Re}\langle \nabla(\rho^2), \nabla |T|_g^2\rangle_g \\
&\geq \rho^2 \left(|\nabla T|_g^2 + \left|\overline{\nabla} T\right|_g^2\right) - C|\nabla g|_g^2 - 4\rho \left|\langle \nabla \rho, \nabla |T|_g^2\rangle_g\right|.
\end{aligned}$$

Again by uniform comparability of ω and $\omega_{\mathbb{C}^n}$, we have $|\nabla \rho|_g \leq C$. Thus the last term above can be further estimated:

$$\begin{aligned}
4\rho \left|\langle \nabla \rho, \nabla |T|_g^2\rangle_g\right| &\leq C\rho \left|\nabla |T|_g^2\right|_g \\
&\leq C\rho \left|\langle \nabla T, T\rangle_g + \langle T, \overline{\nabla} T\rangle\right|_g \\
&\leq C\rho \left(|\nabla T|_g |T|_g + \left|\overline{\nabla} T\right|_g |T|_g\right) \\
&\leq \rho^2 \left(|\nabla T|_g^2 + \left|\overline{\nabla} T\right|_g^2\right) + C|T|_g^2.
\end{aligned}$$

Combine to get

$$\Delta_g \left(\rho^2 |\nabla g|_g^2\right) \geq -C|\nabla g|_g^2. \tag{1.23}$$

Compare this with (1.20). Thus as before the analogous of (1.21) now is

$$\Delta_g \operatorname{tr}_{\omega_{\mathbb{C}^n}} \omega = \delta_{ij} g^{k\bar{l}} g^{p\overline{q}} \partial_k g_{i\overline{q}} \partial_{\bar{l}} g_{p\bar{j}} = g_{\mathbb{C}^n}^{i\bar{j}} g^{k\bar{l}} g^{p\overline{q}} \nabla_k g_{i\overline{q}} \nabla_{\bar{l}} g_{p\bar{j}} \geq C^{-1} |\nabla g|_g^2, \tag{1.24}$$

using Ricci flatness of ω and comparability of ω and $\omega_{\mathbb{C}^n}$. We have checked the first equality of (1.24) in order to conclude (1.10). The second equality makes the quantity coordinate-free as **a mixed norm of ∇g.**

Combining (1.23) and (1.24), we can again pick constant A sufficiently large such that

$$\Delta_g \left(\rho^2 |\nabla g|_g^2 + A \operatorname{tr}_{\omega_{\mathbb{C}^n}} \omega\right) \geq 0 \quad \text{in } B_1.$$

By maximum principle, the maximum of function $\rho^2 |\nabla g|_g^2 + A \operatorname{tr}_{\omega_{\mathbb{C}^n}} \omega$ is attained on the boundary, say $x \in \partial B_1$. Since $\rho = 0$ on ∂B_1,

$$\rho^2 |\nabla g|_g^2 + A \operatorname{tr}_{\omega_{\mathbb{C}^n}} \omega \leq \left(\rho^2 |\nabla g|_g^2 + A \operatorname{tr}_{\omega_{\mathbb{C}^n}} \omega\right)(x) = A \operatorname{tr}_{\omega_{\mathbb{C}^n}} \omega(x) \leq C$$

as $\operatorname{tr}_{\omega_{\mathbb{C}^n}} \omega$ is uniformly bounded by comparability of ω and $\omega_{\mathbb{C}^n}$. Thus on $B_{1/2}$ where $\rho = 1$,

$$|\nabla g|_g^2 \leq \rho^2 |\nabla g|_g^2 + A \operatorname{tr}_{\omega_{\mathbb{C}^n}} \omega \leq C.$$

This is uniform bound on $|\nabla g|_g^2$, which completes the proof.

For the case $k \geq 2$, we use a standard bootstrap argument. The Ricci-flatness condition implies that the component functions $g_{i\bar{j}}$ of ω satisfies

$$\Delta_g g_{i\bar{j}} = g^{k\bar{l}} \partial_k \partial_{\bar{l}} g_{i\bar{j}} = g^{k\bar{l}} g^{p\bar{q}} \partial_k g_{i\bar{q}} \partial_{\bar{l}} g_{p\bar{j}} := Q_{i\bar{j}}.$$

For any nested balls $B \subset B' \subset B''$, we first have

$$\left\| Q_{i\bar{j}} \right\|_{L^p(B'', g_{\mathbb{C}^n})} \leq C_B$$

for all $p \geq 1$ by the result for $k = 1$ proved above. Then

$$\left\| g_{i\bar{j}} \right\|_{W^{2,p}(B', g_{\mathbb{C}^n})} \leq C_{B,p}$$

by L^p estimates (see e.g. Gilbarg-Trudinger §9.5) for $p > 1$. Then

$$\left\| g_{i\bar{j}} \right\|_{C^{1,\alpha}(B', g_{\mathbb{C}^n})} \leq C_{B,\alpha}$$

for all $\alpha \in (0, 1)$ by Morrey's inequality.

We can thus apply local Schauder estimates repeatedly to get uniform bound on $\|\omega\|_{C^k(B_{1/2}, g_{\mathbb{C}^n})}$ for shrinking balls and increasing $k = 2, 3, ...$ This completes the proof for all $k \geq 2$.

□

2 Intermezzo

Before seeing applications of Calabi-Yau Theorem, we first recall some important concepts and properties.

Consult, in addition to the lecture notes, Huybrechts or Griffiths-Harris, for basic definitions of holomorphic vector bundles, sections, hermitian metrics, curvature, and Chern classes. We emphasize some important points below.

Proposition 2.1. *The space $H^0(\mathbb{P}^n, \mathcal{O}(k))$ for each $k > 0$ is isomorphic to $\mathbb{C}[z_0, ..., z_n]_k$, the space of homogeneous polynomials of degree k, which is $\mathbb{C}$-vector space of dimension $\binom{n+k}{k}$.*

Corollary 2.2. *$H^0(\mathbb{P}^n, \mathcal{O}(k)) = 0$ for each $k < 0$.*

Theorem 2.3 (Birkar-Cascini-Hacon-McKernan, Siu '06)**.** *The canonical ring*

$$R(X, \mathcal{K}_X) = \bigoplus_{m \geq 0} H^0(X, \mathcal{K}_X^m)$$

of any compact Kähler manifold X is a finitely generated $\mathbb{C}$-algebra.

For an example of a compact complex non-Kähler manifold whose canonical ring is not finitely generated, see Example 6.4 of https://arxiv.org/pdf/1309.3015

Lemma 2.4. *For each complex manifold X, $c_1(X) = -c_1(\mathcal{K}_X) = c_1(\mathcal{K}_X^*)$. We can see this using the metric $h = (\det g)^{-1}$ on $\mathcal{K}_X$.*

Recall the holomorphic sectional curvature (HSC) for a (1, 0)-tangent vector $V = V^i \frac{\partial}{\partial z_i}$ with $|V|_g^2 = 1$:

$$\text{HSC}(V) := R_{i\bar{j}k\bar{l}} V^i \overline{V^j} V^k \overline{V^l},$$

which is real and coordinate-free.

The complex space forms are the three model spaces of constant HSC: $(\mathbb{C}^n, \omega_{Euc})$, $(\mathbb{P}^n, \omega_{FS})$, and $(\mathbb{B}^n, \omega_P)$. Recall the Poincaré metric on $\mathbb{B}^n$ has constant HSC $= -2$.

Theorem 2.5 (Hopf, see Kobayashi-Nomizu Vol.II §IX.7)**.** *Let (X^n, ω) be a Kähler manifold. Then*

1. *ω has constant HSC $= \lambda \in \mathbb{R}$ if and only if $R_{i\bar{j}k\bar{l}} = \frac{\lambda}{2}\left(g_{i\bar{j}} g_{k\bar{l}} + g_{i\bar{l}} g_{k\bar{j}}\right)$.*
2. *If we further assume X is compact, and ω has constant HSC $= \lambda \in \mathbb{R}$, then*
   - (a) *$\lambda = 0$ if and only if X has a finite covering space $\pi : \tilde{X} \to X$ such that $\tilde{X}$ is biholomorphic to a torus $\mathbb{C}^n/\Lambda$ and $\pi^*\omega$ is a Euclidean metric. This is true if and only if the universal covering $p : \hat{X} \to X$ is biholomorphic to $\mathbb{C}^n$ and $p^*\omega$ is a Euclidean metric.*
   - (b) *$\lambda > 0$ if and only if X is biholomorphic to $\mathbb{P}^n$ and ω is isometric to $\frac{2}{\lambda}\omega_{FS}$.*
   - (c) *$\lambda < 0$ if and only if X is biholomorphic to $\mathbb{B}^n/\Gamma$ for some discrete subgroup Γ acting on $\mathbb{B}^n$ by isometries of ω_P, and ω is isometric to $-\frac{2}{\lambda}\omega_P$. This is true if and only if the universal covering $p : \hat{X} \to X$ is biholomorphic to $\mathbb{B}^n$ and $p^*\omega = -\frac{2}{\lambda}\omega_P$.*

3 Applications of Calabi-Yau Theorem

3.1 Positivity of Chern Classes

We are now ready to state and prove two consequences of Theorem 1.6.

Theorem 3.1 (Positivity of 2nd Chern class). *Let $X^{n\geq 2}$ be a compact Calabi-Yau manifold, i.e. $c_1(X)=0$. Then for every Kähler metric ω on X, we have*

$$\int_X c_2(X)\wedge\omega^{n-2}\geq 0,$$

Moreover,

$$\int_X c_2(X)\wedge\omega^{n-2}=0 \text{ for some Kähler metric } \omega \iff X \text{ is finitely covered by } \mathbb{C}^n/\Lambda.$$

Corollary 3.2. *A compact Kähler manifold $X^{n\geq 2}$ is finitely covered by $\mathbb{C}^n/\Lambda$ if and only if $c_1(X)=0\in H^2(X,\mathbb{R})$ and $c_2(X)=0\in H^4(X,\mathbb{R})$.*

Proof. $\Leftarrow$. this is immediate consequence of Theorem 3.1.

$\Rightarrow$. Let $\pi:\mathbb{C}^n/\Lambda\to X$ be a (holomorphic) finite cover. Then $\pi^*:H^i(X,\mathbb{R})\to H^i(\mathbb{C}^n/\Lambda,\mathbb{R})$ is injective. We consider

$$\pi^*c_k(X)=c_k(\mathbb{C}^n/\Lambda)=0,\quad k=1,2$$

since $\mathbb{C}^n/\Lambda$ admits a flat metric. Thus $c_1(X)=0$ and $c_2(X)=0$. □

Question 3.3. *Following the same essential idea, we can also descend a Kähler metric $\tilde{\omega}$ on $\mathbb{C}^n/\Lambda$ to a Kähler metric on X via π by averaging over the fibers of each point:*

$$\omega(x)(u,v):=\frac{1}{|p^{-1}(x)|}\sum_{p(\tilde{x})=x}\tilde{\omega}(\tilde{x})(d\pi_{\tilde{x}}^{-1}(u),d\pi_{\tilde{x}}^{-1}(v)).$$

It is easy to check that ω is closed positive real (1, 1)-form, and it is flat.

Another consequence of Theorem 1.6 is:

Theorem 3.4 (Miyaoka-Yau inequality). *Let $X^{n\geq 2}$ be a compact Kähler manifold that is canonically polarized, i.e. $c_1(X)<0$. Then*

$$(-1)^n\int_X\left(\frac{2(n+1)}{n}c_2(X)-c_1^2(X)\right)\wedge c_1(X)^{n-2}\geq 0.$$

Moreover, the equality holds if and only if X is biholomorphic to $\mathbb{B}^n/\Gamma$.

Both Theorem 3.1 and Theorem 3.4 follow from Theorem 1.6 and Theorem 1.5, together with the following.

Theorem 3.5. *Let $(X^{n\geq 2},\omega)$ be a compact Kähler-Einstein manifold. Say $\mathrm{Ric}(\omega)=\lambda\omega$ for some $\lambda\in\mathbb{R}$. Then*

$$\int_X\left(\frac{2(n+1)}{n}c_2(X)-c_1^2(X)\right)\wedge\omega^{n-2}\geq 0.$$

Moreover, the equality holds if and only if ω has constant HSC $=\frac{2\lambda}{n+1}$.

Proof of Theorem 3.1. Let ω be any fixed Kähler metric. By Yau's Theorem 1.6, there exists some unique metric $\tilde{\omega} = \omega + i\partial\overline{\partial}\omega$ such that $\mathrm{Ric}(\tilde{\omega}) = 0 \in c_1(X)$. Thus Theorem 3.5 applies to $\tilde{\omega}$ to yield

$$\int_X c_2(X) \wedge \omega^{n-2} = \int_X c_2(X) \wedge \tilde{\omega}^{n-2} \geq 0.$$

Here we use Stokes' theorem and closedness of any representative of $c_2(X)$. Moreover, the equality holds if and only if $\tilde{\omega}$ has constant HSC $= 0$. By Hopf's Theorem 2.5, this holds if and only if X is finitely covered by a torus $\mathbb{C}^n/\Lambda$. □

Proof of Theorem 3.4. By Theorem 1.5, there exists a unique Kähler metric ω such that $\mathrm{Ric}(\omega) = -\omega$. Then

$$c_1(X) = \left[\frac{\mathrm{Ric}(\omega)}{2\pi}\right] = -\frac{1}{2\pi}[\omega] \in H^2(X, \mathbb{R}).$$

Thus we can pick the representative $\frac{-\omega}{2\pi}$ for $c_1(X)$, and it follows immediately from Theorem 3.5 that

$$(-1)^n \int_X \left(\frac{2(n+1)}{n} c_2(X) - c_1^2(X)\right) \wedge c_1(X)^{n-2} = \int_X \left(\frac{2(n+1)}{n} c_2(X) - c_1^2(X)\right) \wedge \omega^{n-2} \geq 0.$$

Moreover, the equality holds if and only if ω has constant HSC $= \frac{-2}{n+1}$. By Hopf's Theorem 2.5, this holds if and only if X is biholomorphic to $\mathbb{B}^n/\Gamma$. □

Proof of Theorem 3.5. By assumption, $R_{i\bar{j}} = \lambda g_{i\bar{j}}$, and hence $R = R_{i\bar{j}} g^{i\bar{j}} = n\lambda$.

Consider the 4-tensor R^0 defined by

$$R^0_{i\bar{j}k\bar{l}} := R_{i\bar{j}k\bar{l}} - \frac{\lambda}{n+1}\left(g_{i\bar{j}} g_{k\bar{l}} + g_{i\bar{l}} g_{k\bar{j}}\right).$$

Hence by Theorem 2.5, ω has constant HSC $= \frac{2\lambda}{n+1}$ if and only if $R^0 = 0$, i.e. $\left|R^0\right|_g^2 = 0$. This suggests we compute $\left|R^0\right|_g^2$.

$$\begin{aligned}
\left|R^0\right|_g^2 &= R^0_{i\bar{j}k\bar{l}} R^0_{p\bar{q}r\bar{s}} g^{i\bar{q}} g^{p\bar{j}} g^{k\bar{s}} g^{r\bar{l}} \qquad\qquad \overline{R_{i\bar{j}k\bar{l}}} = R_{j\bar{i}l\bar{k}} \\
&= |R|_g^2 + \frac{\lambda^2}{(n+1)^2}\left(g_{i\bar{j}} g_{k\bar{l}} + g_{i\bar{l}} g_{k\bar{j}}\right)\left(g_{p\bar{q}} g_{r\bar{s}} + g_{p\bar{s}} g_{r\bar{q}}\right) g^{i\bar{q}} g^{p\bar{j}} g^{k\bar{s}} g^{r\bar{l}} \\
&\quad - \frac{\lambda}{n+1} R_{i\bar{j}k\bar{l}}\left(g_{p\bar{q}} g_{r\bar{s}} + g_{p\bar{s}} g_{r\bar{q}}\right) g^{i\bar{q}} g^{p\bar{j}} g^{k\bar{s}} g^{r\bar{l}} \\
&\quad - \frac{\lambda}{n+1} R_{j\bar{i}l\bar{k}}\left(g_{q\bar{p}} g_{s\bar{r}} + g_{q\bar{r}} g_{s\bar{p}}\right) g^{q\bar{i}} g^{j\bar{p}} g^{s\bar{k}} g^{l\bar{r}} \\
&= |R|_g^2 + \frac{2\lambda^2}{(n+1)^2}(n^2+n) - \frac{2\lambda}{n+1}\mathrm{Re}\Big\{R_{i\bar{j}k\bar{l}}\left(g_{p\bar{q}} g_{r\bar{s}} + g_{p\bar{s}} g_{r\bar{q}}\right) g^{i\bar{q}} g^{p\bar{j}} g^{k\bar{s}} g^{r\bar{l}}\Big\} \\
&= |R|_g^2 + \frac{2n\lambda^2}{(n+1)} - \frac{4n\lambda^2}{n+1} \\
&= |R|_g^2 - \frac{2n\lambda^2}{(n+1)}.
\end{aligned}$$

Recall that

$$\frac{1}{4\pi^2}\mathrm{tr}(\Omega \wedge \Omega) = \frac{1}{4\pi^2} R^k_{ip\bar{q}} R^i_{kr\bar{s}} \left(i dz_p \wedge d\bar{z}_q\right) \wedge \left(i dz_r \wedge d\bar{z}_s\right)$$

is a real (2, 2)-form representing the class $c_1^2(X) - 2c_2(X) \in H^4(X, \mathbb{R})$, where $\Omega \in A^2(X, \mathrm{End}(T^{1,0}X))$ is the curvature form. We consider the integral of $\frac{1}{4\pi^2}\operatorname{tr}(\Omega \wedge \Omega) \wedge \omega^{n-2}$.

For each $x \in X$, we compute in a coordinate $\{z_i\}$ such that $g_{i\bar{j}}(x) = \delta_{ij}$. Let $A_i = i dz_i \wedge d\bar{z}_i$. Then

$$\omega(x) = \sum_j A_j,$$

$$\omega^n(x) = n! A_1 \wedge \cdots \wedge A_n,$$
$$\omega^{n-2}(x) = (n-2)! \sum_{i<j} A_1 \wedge \cdots \wedge \widehat{A_i} \wedge \cdots \wedge \widehat{A_j} \wedge \cdots \wedge A_n.$$

Then at x,

$$\begin{aligned}(n(n-1))\operatorname{tr}(\Omega \wedge \Omega) \wedge \omega^{n-2}(x) &= \omega^n(x) \sum_{i,k} \sum_{p \neq q} \left(R^k_{ip\bar{p}} R^i_{kq\bar{q}} - R^k_{ip\bar{q}} R^i_{kq\bar{p}} \right) \\ &= \omega^n(x) \sum_{i,k} \sum_{p,q} \left(R^k_{ip\bar{p}} R^i_{kq\bar{q}} - R^k_{ip\bar{q}} R^i_{kq\bar{p}} \right),\end{aligned}$$

where the minus sign comes from

$$dz_p \wedge d\bar{z}_q \wedge dz_q \wedge d\bar{z}_p = -dz_p \wedge \bar{z}_p \wedge dz_q \wedge d\bar{z}_q.$$

Now since $g_{i\bar{j}}(x) = \delta_{ij}$, we have at x,

$$\sum_p R^k_{ip\bar{p}} = \sum_p R_{i\bar{k}p\bar{p}} = R_{i\bar{k}p\bar{q}} g^{p\bar{q}} = R_{i\bar{k}} = R^k_i.$$

Thus we continue to compute

$$\begin{aligned}(n(n-1))\operatorname{tr}(\Omega \wedge \Omega) \wedge \omega^{n-2}(x) &= \omega^n(x) \left(R^k_i R^i_k - R^k_{ip\bar{q}} R^i_{kq\bar{p}} \right) \\ &= \omega^n(x) \left(|\mathrm{Ric}(\omega)|_g^2 - |R|_g^2 \right) \\ &= \omega^n(x) \left(\lambda^2 n - |R|_g^2 \right). \qquad \mathrm{Ric}(\omega) = \lambda\omega\end{aligned}$$

Combined with the computation of $\left|R^0\right|_g^2$ above, we get

$$\frac{\left|R^0\right|_g^2 \omega^n}{n(n-1)} = -\operatorname{tr}(\Omega \wedge \Omega) \wedge \omega^{n-2}(x) + \frac{\lambda^2 \omega^n}{n-1} - \frac{2\lambda^2 \omega^n}{(n+1)(n-1)} = -\operatorname{tr}(\Omega \wedge \Omega) \wedge \omega^{n-2}(x) + \frac{\lambda^2}{n+1}\omega^n.$$

Finally, integrate over X:

$$\begin{aligned}0 &\leq \int_X \frac{\left|R^0\right|_g^2 \omega^n}{4\pi^2 n(n-1)} \\ &= \int_X \left(2c_2(X) - c_1^2(X)\right) \wedge \omega^{n-2} + \frac{1}{n+1} \int_X \left(\frac{\lambda\omega}{2\pi}\right)^2 \wedge \omega^{n-2} \\ &= \int_X \left(2c_2(X) - c_1^2(X)\right) \wedge \omega^{n-2} + \frac{1}{n+1} \int_X c_1^2(X) \wedge \omega^{n-2} \qquad \mathrm{Ric}(\omega) = \lambda\omega \\ &= \int_X \left(2c_2(X) - \frac{n}{n+1} c_1^2(X)\right) \wedge \omega^{n-2}.\end{aligned}$$

This completes the proof.

□

3.2 Applications in Classification

The Miyaoka-Yau inequality (Theorem 3.4) further implies the following:

Theorem 3.6 (Conjecture of Severi, Proof by Hirzebruch-Kodaira '57, Yau '76)**.** *Let X^2 be a compact complex surface. If X is homotopy equivalent to $\mathbb{P}^2$, then X is biholomorphic to $\mathbb{P}^2$.*

The most important ingredient of the proof is the following theorem of Hirzebruch and Kodaira. The proof also uses Hirzebruch-Riemann-Roch formula, Serre duality, and Kodaira vanishing, to apply the Miyaoka-Yau inequality.

Theorem 3.7 (Hirzebruch-Kodaira)**.** *Let X^n be a compact complex manifold, and $L \to X$ a holomorphic line bundle. Suppose the following conditions hold:*

1. *L is positive (so X is projective by Kodaira embedding),*
2. *$\int_X c_1(L)^n = 1$, and*
3. *$\dim_{\mathbb{C}} H^0(X, L) = n + 1$.*

Then X is biholomorphic to $\mathbb{P}^n$. Moreover, any basis $\{s_0, ..., s_n\}$ of $H^0(X, L)$ defines such a biholomorphic map $X \cong \mathbb{P}^n$ via $x \mapsto [s_0(x) : \cdots : s_n(x)]$.

Proof. Let L be a line bundle satisfying the conditions above. Fix a basis $\{s_1, ..., s_{n+1}\}$ of $H^0(X, L)$.

Define $D_j := \{s_j = 0\}$, which is a closed analytic hypersurface for each j. Indeed $D_j \neq \emptyset$, for otherwise s_j is nowhere vanishing global section, so that $L \cong \mathcal{O}_X$, and $c_1(L) = 0$, contradiction to the conditions above.

Define

$$V_{n-j} = D_1 \cap \cdots \cap D_j,$$

for each $j = 0, 1, ..., n$. In particular $V_n = X$. V_j are hence closed analytic subvarieties. Observe the following.

Claim 3.8. *For each $j = 0, 1, ..., n$,*

1. *V_{n-j} is irreducible, $\dim V_{n-j} = n - j$, and $[V_{n-j}] \in H_{2n-2j}(X, \mathbb{Z})$ is Poincaré dual to $c_1(L)^j \in H^{2j}(X, \mathbb{Z})$.*
2. *There is exact sequence*

$$0 \to \mathrm{Span}_{\mathbb{C}}\{s_1, ..., s_j\} \to H^0(X, L) \to H^0(V_{n-j}, L|_{V_{n-j}}),$$

where the last map $H^0(X, L) \to H^0(V_{n-j}, L|_{V_{n-j}})$ is restriction.

We first assume the claim above and finish the proof. Letting $j = n$, we see that $V_0 = D_1 \cap \cdots \cap D_n$ is a single point, since $\int_X c_1(L)^n = 1$, and hence $H^0(V_0, L|_{V_0}) \cong \mathbb{C}$. Thus s_{n+1} does NOT vanish at V_0 by the exact sequence. The zero locus of $H^0(X, L)$ is empty, so we can define a holomorphic map

$$f : X \to \mathbb{P}^n, \quad x \mapsto [s_1(x) : \cdots : s_{n+1}(x)].$$

Finally, we show that f is bijective, and hence f is biholomorphism. The idea is to view f as the map sending x to the hyperplane

$$\{s \in H^0(X, L) \mid s(x) = 0\} \subset H^0(X, L) \cong \mathbb{C}^{n+1}.$$

To see this, we define another holomorphic map

$$\hat{f} : X \to \mathbb{P}(H^0(X, L)^*), \quad x \mapsto \{s \in H^0(X, L) \mid s(x) = 0\},$$

where we identify each hyperplane in $H^0(X, L)$ with a line in $H^0(X, L)^*$ (canonically this is the line of linear functionals $H^0(X, L) \to \mathbb{C}$ vanishing on the hyperplane). Indeed $\{s \in H^0(X, L) \mid s(x) = 0\}$ is a hyperplane because the zero locus of $H^0(X,L)$ is empty. Now $\hat{f}$ is defined free of choice of basis for $H^0(X,L)$. Picking any basis $\{s_1,...,s_{n+1}\}$ of $H^0(X,L)$, we have commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\hat{f}} & \mathbb{P}(H^0(X,L)^*) \\ & \searrow^{f} & \downarrow \cong \\ & & \mathbb{P}^n \end{array}$$

Thus f is bijective $\iff$ $\hat{f}$ is bijective. To see $\hat{f}$ is bijective, let $H \leqslant H^0(X,L)$ be any hyperplane. Now we can choose a basis $\{s_1,...,s_{n+1}\}$ of $H^0(X,L)$ such that $H = \mathrm{Span}_{\mathbb{C}}\{s_1,...,s_n\}$. Since $\hat{f}(x) = H$ if and only if $x \in V_0$ under this basis, Claim 3.8 yields a single point (hence both existence and uniqueness) $V_0 = \{x\}$ such that $\hat{f}(x) = H$. This completes the proof.

□

Proof of Claim 3.8. We prove by induction on $j = 0,1,...,n$.

The base case $j = 0$ is trivial: $V_n = X$ is trivially irreducible, and $[X]$ is Poincaré dual to $1 = c_1(L)^0 \in H^0(X,\mathbb{Z})$.

Assume the claim is true for $j-1$, we prove for j. By induction hypothesis, V_{n-j+1} is irreducible, $\dim V_{n-j+1} = n-j+1$, and $[V_{n-j+1}]$ is Poincaré dual to $c_1(L)^{j-1} \in H^{2j-2}(X,\mathbb{Z})$. There is exact sequence

$$0 \to \mathrm{Span}_{\mathbb{C}}\{s_1,...,s_{j-1}\} \to H^0(X,L) \to H^0(V_{n-j+1}, L|_{V_{n-j+1}}).$$

It follows that s_j does not vanish identically on V_{n-j+1}. Thus

$$V_{n-j} = \{x \in V_{n-j+1} \mid s_j|_{V_{n-j+1}}(x) = 0\}$$

is an analytic hypersurface of V_{n-j+1}.

Since $V_{n-j} = V_{n-j+1} \cap D_j$, $[V_{n-j}]$ is Poincaré dual (PD) to

$$PD([V_{n-j+1}]) \wedge PD([D_j]) = c_1(L)^{j-1} \wedge c_1(L) = c_1(L)^j \in H^{2j}(X,\mathbb{Z}),$$

where $c_1(L) = c_1(\mathcal{O}(Z(s_j))) = c_1(\mathcal{O}([D_j])) = [D_j]$ under Poincaré duality (see Huybrechts §4).

V_{n-j} is irreducible. Suppose otherwise, $V_{n-j} = U_1 \cup U_2$, where U_1, U_2 are non-empty analytic subvarieties. Then

$$\begin{aligned} 1 &= \int_X c_1(L)^n \\ &= \int_X c_1(L)^j \wedge c_1(L)^{n-j} \\ &= \int_{V_{n-j}} c_1(L)^{n-j} \\ &= \int_{U_1} c_1(L)^{n-j} + \int_{U_2} c_1(L)^{n-j}. \end{aligned} \tag{3.1}$$

Recall that L is positive, so $c_1(L)$ can be represented by a Kähler metric ω. Then

$$\int_{U_i} c_1(L)^{n-j} = \int_{U_i} \omega^{n-j} = \mathrm{Vol}(U_i,\omega) > 0, \quad i = 1,2. \tag{3.2}$$

Meanwhile, $[U_i] \in H_{2n-2j}(X,\mathbb{Z})$, so

$$\int_{U_i} c_1(L)^{n-j} = \int_X PD([U_i]) \wedge c_1(L)^{n-j} \in \mathbb{Z}, \quad i = 1,2. \tag{3.3}$$

Thus (3.1), (3.2), and (3.3) yield a contradiction.

To show exactness of the sequence

$$0 \to \mathrm{Span}_{\mathbb{C}}\{s_1, ..., s_j\} \to H^0(X, L) \to H^0(V_{n-j}, L|_{V_{n-j}}), \tag{3.4}$$

note first that we have exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(V_{n-j+1}, \mathcal{O}) & \xrightarrow{f} & H^0(V_{n-j+1}, L) & \xrightarrow{g} & H^0(V_{n-j}, L) \\ & & \downarrow \cong & & \uparrow & \nearrow & \\ & & \mathbb{C} & & H^0(X, L) & & \end{array}$$

where f is multiplication by $s_j|_{V_{n-j+1}}$, and g is restriction. f is injective since s_j does not vanish identically on V_{n-j+1} as shown above. The exactness follows from the restriction short exact sequence

$$0 \to \mathcal{O}_{V_{n-j+1}} \xrightarrow{\otimes s_j} \mathcal{O}_{V_{n-j+1}} \otimes L \to \mathcal{O}_{V_{n-j}} \otimes L \to 0,$$

which adapts from the twist of the structure sheaf

$$0 \to \mathcal{I}_{V_{n-j}|V_{n-j+1}} \to \mathcal{O}_{V_{n-j+1}} \to \mathcal{O}_{V_{n-j}} \to 0.$$

Suppose $s \in H^0(X, L)$ restricts to the zero section in $H^0(V_{n-j}, L)$. By commutativity of restriction, $g(s|_{V_{n-j+1}}) = 0$. Thus by exactness above, $s|_{V_{n-j+1}} = \lambda \cdot s_j|_{V_{n-j+1}}$ for some $\lambda \in \mathbb{C} \cong H^0(V_{n-j+1}, \mathcal{O})$. Then $s - \lambda \cdot s_j \in H^0(X, L)$ restricts to the zero section in $H^0(V_{n-j+1}, L)$. By induction hypothesis, $s - \lambda \cdot s_j \in \mathrm{Span}_{\mathbb{C}}\{s_1, ..., s_{j-1}\}$, so that $s \in \mathrm{Span}_{\mathbb{C}}\{s_1, ..., s_j\}$. This proves the exactness of sequence (3.4).

This completes the induction and the proof. □

We have the following extension of Theorem 3.6 to general dimension.

Theorem 3.9. *Let X^n be a compact Kähler manifold. If X is homeomorphic to $\mathbb{P}^n$, then X is biholomorphic to $\mathbb{P}^n$.*

We can ask further the following question.

Question 3.10. *Let X^n be a compact complex manifold. If X is diffeomorphic to $\mathbb{P}^n$, then X is biholomorphic to $\mathbb{P}^n$?*

The answer is yes for $n = 1, 2$, yet unknown for $n \geq 3$. If it is true for $n = 3$, then **it follows that** S^6 is not complex manifold. This is also unknown.

3.3 Degenerations of Ricci-Flat Kähler Metrics

Let X^n be a compact Calabi-Yau manifold, i.e. X is Kähler and $c_1(X) = 0 \in H^2(X, \mathbb{R})$. By Calabi-Yau Theorem, the Ricci-flat Kähler metrics on X are parametrized bijectively by the set

$$\mathcal{C}_X := \{[\omega] \in H^{1,1}(X, \mathbb{R}) \mid \omega \text{ Kähler metric on } X\}.$$

Recall that by Hodge Theory (and $\partial\overline{\partial}$ lemma as a consequence),

$$\begin{aligned} H^{1,1}(X, \mathbb{R}) &= H^{1,1}(X) \cap H^2(X, \mathbb{R}) \subset H^2(X, \mathbb{C}) \\ &= \{[\alpha] \in H^2(X, \mathbb{R}) \mid [\alpha] \text{ contains a closed real (1, 1)-form}\} \\ &= \frac{\{d\text{-closed real (1, 1)-forms}\}}{i\partial\overline{\partial}C^\infty(X, \mathbb{R})}. \end{aligned}$$

(The last line is also the definition of Bott-Chern cohomology for non-Kähler manifolds). We first derive some basic properties of $\mathcal{C}_X$.

Proposition 3.11. *$\mathcal{C}_X \subset H^{1,1}(X, \mathbb{R})$ is an open convex cone, called the* ***Kähler cone*** *of X.*

$\mathcal{C}_X$ is clearly a convex cone. To see openness, consider an $\mathbb{R}$-basis of $H^{1,1}(X, \mathbb{R})$ and use $\partial\overline{\partial}$-lemma.

Example 3.12. Consider $n = 1$. Riemann surfaces are always Kähler by existence of Hermitian metrics. Then

$$\mathbb{C} \cong H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X),$$

so that $H^{1,1}(X) \cong \mathbb{C}$, and $H^{1,1}(X, \mathbb{R})$ is a real line in $\mathbb{C}$. Then $\mathcal{C}_X$ is a half-line. Also, $[\alpha] \in H^{1,1}(X, \mathbb{R})$ belongs to $\mathcal{C}_X$ if and only if $\int_X \alpha > 0$.

Example 3.13. Let $X^n := \mathbb{C}^n/\Lambda$ be a torus. X is a Lie group, and X acts on itself by translations. By averaging forms, we see that every class $[\alpha] \in H^{1,1}(X, \mathbb{R})$ has a **unique** representative α which is a constant real (1, 1)-form. Writing $\alpha = i\alpha_{i\bar{j}} dz_i \wedge d\bar{z}_j$, we can associate the form with a Hermitian matrix $(\alpha_{i\bar{j}})$. Thus $[\alpha] \in \mathcal{C}_X$ if and only if this unique invariant representative is positive, i.e. the associated matrix is positive definite. Therefore,

$$\mathcal{C}_X = \mathrm{Herm}^+(n) \subset \mathrm{Herm}(n) = H^{1,1}(X, \mathbb{R}),$$

which is indeed an open convex cone.

A natural question to ask is: which classes in $H^{1,1}(X, \mathbb{R})$ lie on $\partial\mathcal{C}_X := \overline{\mathcal{C}_X} \setminus \mathcal{C}_X$?

Definition 3.14. We call $\partial\mathcal{C}_X$ the **numerically effective** (NEF) cone, and a class in $H^{1,1}(X, \mathbb{R})$ is called NEF if it belongs to $\partial\mathcal{C}_X$.

Example 3.15. Continuing on the torus example above, the NEF cone is the subset of positive-semidefinite Hermitian matrices under the identification $\mathrm{Herm}(n) = H^{1,1}(X, \mathbb{R})$.

Recall first the definition of (semi-)positive real (1, 1)-forms on complex manifold X (see Huybrechts Def.4.3.14). If $[\alpha] \in H^{1,1}(X, \mathbb{R})$ contains a closed real (1, 1)-type representative $\alpha \geq 0$, then $[\alpha] \in \overline{\mathcal{C}_X}$, because $\alpha + \varepsilon\omega > 0$ for any $\varepsilon > 0$ and any Kähler metric ω, so that $[\alpha] + \varepsilon[\omega] \in \mathcal{C}_X$. The converse is not true (too strong), however. Fujita made the conjecture, and Demailly-Peternell-Schneider gave counterexamples. The correct statement is:

Proposition 3.16 (Characterization of NEF cone)**.** *Let (X^n, ω) be a compact Kähler manifold. Let α be a closed real $(1, 1)$-form on X. Then*

$$[\alpha] \in \overline{\mathcal{C}_X} \iff \textit{for any } \varepsilon > 0, \textit{ there exists } \varphi_\varepsilon \in C^\infty(X, \mathbb{R}) \textit{ such that } \alpha + i\partial\overline{\partial}\varphi_\varepsilon > -\varepsilon\omega \textit{ on } X.$$

Proof. $\Leftarrow$: The condition means that $[\alpha] + \varepsilon[\omega] \in \mathcal{C}_X$ for any $\varepsilon > 0$. Letting $\varepsilon \to 0$, we see $[\alpha] \in \overline{\mathcal{C}_X}$.

$\Rightarrow$: Since $[\alpha] \in \overline{\mathcal{C}_X}$, we can find a sequence of Kähler metrics $\{\omega_i\}$ such that $[\omega_i] \to [\alpha]$ in $H^{1,1}(X, \mathbb{R})$. Define $[\beta_i] := -[\alpha] + [\omega_i]$, so $[\beta_i] \to 0 \in H^{1,1}(X, \mathbb{R})$. Thus we can find representative $\beta_i \in [\beta_i]$ that is closed real (1, 1)-form and $\alpha + \beta_i = \omega_i > 0$.

Let $\{[\alpha_1], ..., [\alpha_N]\}$ be an $\mathbb{R}$-basis for $H^{1,1}(X, \mathbb{R})$. We can thus write

$$[\beta_i] = \sum_{j=1}^{N} \lambda_{ij} [\alpha_j], \quad i = 1, 2, ...$$

$[\beta_i] \to 0$ means that $\sum_j |\lambda_{ij}| \to 0$ as $i \to \infty$. Thus we can choose for each $\varepsilon > 0$ some i_0 sufficiently large such that

$$\sum_{j=1}^{N} \lambda_{i_0 j} \alpha_j \leq \varepsilon\omega,$$

by compactness of X. Combined, we get

$$0 < \omega_{i_0} = \alpha + \beta_{i_0} = \alpha + \sum_{j=1}^{N} \lambda_{i_0 j} \alpha_j + i\partial\overline{\partial}\varphi_{i_0} \leq \alpha + i\partial\overline{\partial}\varphi_{i_0} + \varepsilon\omega \Rightarrow \alpha + i\partial\overline{\partial}\varphi_{i_0} > -\varepsilon\omega.$$

□

Corollary 3.17. *$\overline{\mathcal{C}_X} + \mathcal{C}_X = \mathcal{C}_X$ in $H^{1,1}(X, \mathbb{R})$.*

Proof. Let $[\alpha] \in \overline{\mathcal{C}_X}$, $[\beta] \in \mathcal{C}_X$. By definition, we can pick a representative $\beta = \omega$ for some Kähler metric ω. Let α be any closed real (1, 1)-form representing $[\alpha]$. By Proposition 3.16, there is some $\varphi \in C^\infty(X, \mathbb{R})$ such that $\alpha + i\partial\overline{\partial}\varphi > -\omega$. Then $[\alpha] + [\beta]$ has representative $\alpha + i\partial\overline{\partial}\varphi + \omega > 0$, which is hence a Kähler metric on X. Therefore, $[\alpha] + [\beta] \in \mathcal{C}_X$. □

Proposition 3.18. *$(-\mathcal{C}_X) \cap \mathcal{C}_X = \emptyset$. $(-\overline{\mathcal{C}_X}) \cap \overline{\mathcal{C}_X} = \{0\}$, i.e. the NEF cone is salient.*

Proof. If $(-\mathcal{C}_X) \cap \mathcal{C}_X \neq \emptyset$, then $0 \in \mathcal{C}_X$ by convexity. Thus there is a Kähler metric ω with $[\omega] = 0 \in H^{1,1}(X, \mathbb{R})$. Then

$$\mathrm{Vol}(X, \omega) = \int_X \omega^n = \int_X [\omega]^n = 0,$$

a contradiction. Thus $(-\mathcal{C}_X) \cap \mathcal{C}_X = \emptyset$.

Clearly $0 \in \overline{\mathcal{C}_X}$. To see $(-\overline{\mathcal{C}_X}) \cap \overline{\mathcal{C}_X} \subset \{0\}$, suppose $0 \neq [\alpha] \in (-\overline{\mathcal{C}_X}) \cap \overline{\mathcal{C}_X}$. Fix a Kähler metric ω. Then by Corollary 3.17, $[\omega] + t[\alpha] \in \mathcal{C}_X$ for all $t \in \mathbb{R}$. Pick a representative $\alpha \in [\alpha]$ which is a closed real (1, 1)-form. We can pick a Kähler form ω_t representing the class $[\omega] + t[\alpha]$, and by compactness $\omega_t > \varepsilon_t\omega$ for some $\varepsilon_t > 0$. This is essentially Proposition 3.16. Then

$$0 < \int_X (\omega + t\alpha) \wedge \omega^{n-1} = \int_X \omega^n + t \int_X \alpha \wedge \omega^{n-1}, \quad \forall t \in \mathbb{R}.$$

Hence

$$\int_X \alpha \wedge \omega^{n-1} = 0.$$

Similarly,

$$0 < \int_X (\omega + t_1\alpha) \wedge (\omega + t_2\alpha) \wedge \omega^{n-2} = \int_X \omega^n + t_1 t_2 \int_X \alpha^2 \wedge \omega^{n-2}, \quad \forall t_1, t_2 \in \mathbb{R},$$

so

$$\int_X \alpha^2 \wedge \omega^{n-2} = 0.$$

By Lefschetz Decomposition (Cor.3.1.2 of Huybrechts), we can write $\alpha = \beta + c\omega$, where β is primitive closed real (1, 1)-form. Then $\beta \wedge \omega^{n-1} = 0$, so

$$0 = \int_X \alpha \wedge \omega^{n-1} = c \int_X \omega^n \Rightarrow c = 0.$$

Thus

$$\int_X \beta^2 \wedge \omega^{n-2} = 0.$$

Since $[\alpha] \neq 0$, we have by Hodge-Riemann bilinear relation (Prop.3.3.15 of Huybrechts)

$$\int_X \beta^2 \wedge \omega^{n-2} < 0,$$

a contradiction. This completes the proof that $(-\overline{\mathcal{C}_X}) \cap \overline{\mathcal{C}_X} = \{0\}$.

□

Now consider $V^k \subset X^n$ a compact complex submanifold, or more generally a closed irreducible analytic subvariety of dimension $1 \le k \le n$. If $[\alpha] \in \mathcal{C}_X$, and we pick α to be a Kähler metric, then

$$\int_V \alpha^k = \int_V [\alpha]^k = \langle [V], [\alpha]^k \rangle = \mathrm{Vol}(V, \alpha) > 0.$$

Thus if $[\alpha] \in \overline{\mathcal{C}_X}$, then there exists a sequence $[\alpha_i] \in \mathcal{C}_X$ with $[\alpha_i] \to [\alpha]$, so that

$$\int_V \alpha^k = \int_V [\alpha]^k = \langle [V], [\alpha]^k \rangle = \lim_i \langle [V], [\alpha_i]^k \rangle \ge 0$$

by Poincaré duality.

In fact,

Theorem 3.19 (Demailly-Paun '01)**.** *Let X^n be a compact Kähler manifold. If $[\alpha] \in \overline{\mathcal{C}_X}$, then*

$$[\alpha] \in \mathcal{C}_X \iff \int_V [\alpha]^{\dim V} > 0 \text{ for all positive-dimensional closed irreducible analytic subvariety } V.$$

Therefore, if $[\alpha] \in \partial\mathcal{C}_X$, then there exists some irreducible analytic subvariety V of positive dimension, such that $\int_V [\alpha]^{\dim V} = 0$. Theorem 3.19 generalizes Nakai-Moishezon Theorem in algebraic geometry, which characterizes ample line bundles on a proper scheme.

Another concept we need for studying degenerations of Ricci-flat Kähler metrics is null locus, motivated from Theorem of Demailly-Paun above.

Definition 3.20. The **null locus** of a class $[\alpha] \in \overline{\mathcal{C}_X}$ is

$$\mathrm{Null}([\alpha]) := \bigcup_{\substack{V \subset X \\ \int_V [\alpha]^{\dim V} = 0}} V \subset X$$

where V ranges over all positive-dimensional closed irreducible analytic subvarieties.

Therefore, $\mathrm{Null}([\alpha]) = \emptyset \iff [\alpha] \in \mathcal{C}_X$ by Theorem of Demailly-Paun.

Example 3.21. Let $X^n = \mathbb{C}^n/\Lambda$ be a torus. We claim that for every $[\alpha] \in \partial\mathcal{C}_X$, one has $\mathrm{Null}([\alpha]) = X$. Recall from examples above on complex tori, that $\partial\mathcal{C}_X$ is the subset of positive-semidefinite Hermitian matrices with at least one zero eigenvalue, under the identification of each class in $H^{1,1}(X, \mathbb{R})$ with its unique constant representative. Thus the determinant vanish, and we integrate using this constant representative that

$$\int_X [\alpha]^n = \int_X 0 = 0.$$

Hence $\mathrm{Null}([\alpha]) = X$.

Theorem 3.22 (Collins-Tosatti '13)**.** *Let X^n be a compact Kähler manifold, $[\alpha] \in \partial\mathcal{C}_X$. Then* $\mathrm{Null}([\alpha])$ *is a closed analytic subvariety of X (not necessarily irreducible), and*

$$\mathrm{Null}([\alpha]) = X \iff \int_X [\alpha]^n = 0.$$

Remark 3.23. Theorem 3.22 and work of Chiose combine to give a new proof of Theorem 3.19.

We are now ready to start. Let X^n be a compact Kähler Calabi-Yau manifold. Let ω be a Ricci-flat Kähler metric on X. Let $[\alpha_t]$ be a C^0 path in $H^{1,1}(X,\mathbb{R})$, $t \in [0,1]$, such that $[\alpha_t] \in \mathcal{C}_X$ for $t \in (0,1]$ and $[\alpha_0] \in \partial\mathcal{C}_X$. By Calabi-Yau Theorem, for each $t \in (0,1]$, there exists a unique Ricci-flat Kähler metric ω_t in the class $[\alpha_t]$. We hope to understand the "degeneration" of (X, ω_t) as $t \to 0$.

Example 3.24. Consider $X^n = \mathbb{C}^n/\Lambda$ a torus. Under the canonical representation of $H^{1,1}(X,\mathbb{R})$ above, ω_t is a family of constant closed real (1, 1)-forms such that $\omega_t \to \omega_0$ as $t \to 0$. ω_0 is positive-semidefinite but not positive-definite. We call ω_0 the **degenerate tensor**.

In the general setting, we fix α_t a C^0 family in t of smooth closed real (1, 1)-forms on X, such that $\alpha_t \in [\alpha_t]$. This can be achieved by fixing a basis for $H^{1,1}(X,\mathbb{R})$.

On $t \in (0,1]$, since ω_t and ω are Ricci-flat, we have

$$\Delta_\omega \left(\log \left(\frac{\omega_t^n}{\omega^n} \right) \right) = 0 \quad \text{on } X,$$

so $\omega_t^n = c\omega^n$ on X for some constant c by maximum principle. We then integrate boths sides over X to find $c = \frac{\int_X \alpha_t^n}{\int_X \omega^n}$. In summary, we have a PDE problem

$$\begin{cases} \omega_t = \alpha_t + i\partial\overline{\partial}\varphi_t \\ \int_X \varphi_t \omega^n = 0 \\ (\alpha_t + i\partial\overline{\partial}\varphi_t)^n = \omega_t^n = \frac{\int_X \alpha_t^n}{\int_X \omega^n}\omega^n \end{cases} \tag{$*_t$}$$

and

$$0 < \int_X \alpha_t^n \to \int_X \alpha_0^n \quad \text{as } t \to 0.$$

For each $t \in (0,1]$, since ω_t is fixed, we have a unique solution φ_t for $*_t$. And

$$\mathrm{Vol}(X, \omega_t) = \int_X \omega_t^n = \int_X \alpha_t^n \to \int_X \alpha_0^n \geq 0.$$

Hence there are two cases:

I) $\int_X \alpha_0^n > 0 \iff \mathrm{Vol}(X, \omega_t) \geq c^{-1} > 0$ for some $c > 0$. We call this case **volume non-collapsed**.

II) $\int_X \alpha_0^n = 0 \iff \mathrm{Vol}(X, \omega_t) \to 0$ as $t \to 0$. We call this case **volume collapsed**.

The method in Yau's proof no longer applies here, as the reference Kähler metric ω is now replaced by α_t, and we don't know about the geometry of (X, α_t) as $t \to 0$. In fact, Yau's estimates for φ_t blow up as $t \to 0$.

Conjecture 3.25. *There exists some constant $C > 0$ such that* $\sup_X |\varphi_t| \leq C$ *for all* $t \in (0,1]$.

The statement is true when $[\alpha_0] \in \partial\mathcal{C}_X$ contains a smooth representative $\alpha_0 \geq 0$. The torus case discussed above is one such example. It is necessary that X is Calabi-Yau, for there are counterexamples when X is not.

Let us assume there is indeed $\alpha_0 \geq 0$ such that $[\alpha_0] \in \partial\mathcal{C}_X$. Assume $\int_X \alpha_0^n > 0$, i.e. volume non-collapsed. To further simplify, assume $\alpha_t = \alpha_0 + t\omega$ for $t \in [0,1]$, so that $[\alpha_t] \in \mathcal{C}_X$ for $t \in (0,1]$. α_t is Kähler metric, but not necessarily Ricci-flat, so as above

$$\begin{cases} \omega_t = \alpha_t + i\partial\overline{\partial}\varphi_t \\ \int_X \varphi_t \omega^n = 0 \\ (\alpha_t + i\partial\overline{\partial}\varphi_t)^n = \omega_t^n = \frac{\int_X \alpha_t^n}{\int_X \omega^n}\omega^n \end{cases} \tag{3.5}$$

Using Conjecutre 3.25 which holds in this case, one gets

Theorem 3.26 (Tosatti '07, Collins-Tosatti '13)**.** *Under assumptions above, ω_t and φ_t have uniform $C^k(K)$ bounds independent of t, for all $k \geq 0$ and for all $K \Subset X \setminus \mathrm{Null}([\alpha_0])$.*

Proof Sketch. Since $\int_X \alpha_0^n > 0$, we have $\mathrm{Null}([\alpha_0]) \neq X$ by Theorem 3.22. One key claim in the proof, whose proof we omit, is the following.

Claim 3.27. *There exists a smooth function*

$$\psi : X \setminus \mathrm{Null}([\alpha_0]) \to \mathbb{R}$$

such that $\psi(x) \to -\infty$ as x goes to $\mathrm{Null}([\alpha_0])$, and $\alpha_0 + i\partial\overline{\partial}\psi \geq \varepsilon\omega$ on $X \setminus \mathrm{Null}([\alpha_0])$ for some $\varepsilon > 0$.

Next apply Tsuji's trick. Consider the quantity

$$Q := \log \operatorname{tr}_\omega \omega_t - A(\varphi_t - \psi),$$

for $t \in (0,1]$, $A > 0$ a constant to be determined. By claim above, Q is a smooth function on $X \setminus \mathrm{Null}([\alpha_0])$, and $Q \to -\infty$ near $\mathrm{Null}([\alpha_0])$. Thus Q attains its maximum, say at $x \in X \setminus \mathrm{Null}([\alpha_0])$. Compute at x:

$$\Delta_{\omega_t} Q \geq -C \operatorname{tr}_{\omega_t} \omega - C - A\Delta_{\omega_t}\varphi_t + A\Delta_{\omega_t}\psi, \tag{3.6}$$

using the calculations in the proof of Aubin-Yau Theorem (see (1.12)). Now

$$\Delta_{\omega_t}\varphi_t = \operatorname{tr}_{\omega_t}(i\partial\overline{\partial}\varphi_t) = \operatorname{tr}_{\omega_t}(\omega_t - \alpha_t) = n - \operatorname{tr}_{\omega_t}\alpha_t,$$

so

$$\begin{aligned} -A\Delta_{\omega_t}\varphi_t + A\Delta_{\omega_t}\psi &= -An + A\operatorname{tr}_{\omega_t}\alpha_t + A\operatorname{tr}_{\omega_t}(i\partial\overline{\partial}\psi) \\ &= -An + A\operatorname{tr}_{\omega_t}\left(\alpha_0 + t\omega + i\partial\overline{\partial}\psi\right) \\ &\geq -An + A\varepsilon\operatorname{tr}_{\omega_t}\omega. \end{aligned}$$

Combined, we get

$$\Delta_{\omega_t} Q \geq -C\operatorname{tr}_{\omega_t}\omega - C - An + A\varepsilon\operatorname{tr}_{\omega_t}\omega. \tag{3.7}$$

Choose $A \gg 1$ such that $A\varepsilon = C + 1$, and replace C if needed, to get

$$0 \geq \Delta_{\omega_t} Q(x) \geq \operatorname{tr}_{\omega_t}\omega(x) - C \Rightarrow \operatorname{tr}_{\omega_t}\omega(x) \leq C. \tag{3.8}$$

Using the simultaneous diagonalization trick and inequality (3.8),

$$\begin{aligned} \operatorname{tr}_\omega \omega_t(x) &\leq \frac{1}{(n-1)!}\left(\operatorname{tr}_{\omega_t}\omega(x)\right)^{n-1}\frac{\omega_t^n}{\omega^n}(x) \\ &\leq C\frac{\omega_t^n}{\omega^n}(x) \\ &\leq C\frac{\int_X \alpha_t^n}{\int_X \omega^n} \\ &\leq C, \end{aligned}$$

so $\log \operatorname{tr}_\omega \omega_t(x) \le C$. $\psi \to -\infty$ near $\mathrm{Null}([\alpha_0])$ and is smooth on $X \setminus \mathrm{Null}([\alpha_0])$, so $\psi \le C$ on $X \setminus \mathrm{Null}([\alpha_0])$. By Conjecture 3.25, $\sup_X |\varphi_t| \le C$ for all t. Combined, we get

$$Q(x) \le C - A\varphi_t(x) + A\psi(x) \le C,$$

so

$$Q \le C \quad \text{ on } X \setminus \mathrm{Null}([\alpha_0]).$$

Then

$$\log \operatorname{tr}_\omega \omega_t \le C + A(\varphi_t - \psi) \le C - A\psi \quad \text{ on } X \setminus \mathrm{Null}([\alpha_0]),$$

so

$$\operatorname{tr}_\omega \omega_t \le Ce^{-A\psi} \quad \text{ on } X \setminus \mathrm{Null}([\alpha_0]), \text{ for all } t \in (0,1].$$

On each $K \Subset X \setminus \mathrm{Null}([\alpha_0])$, we get bound on $|\psi|$, so that

$$\omega_t \le C_K \omega \quad \text{ on each } K \Subset X \setminus \mathrm{Null}([\alpha_0]), \text{ for all } t \in (0,1].$$

Applying the same trick again, on each $K \Subset X \setminus \mathrm{Null}([\alpha_0])$,

$$\operatorname{tr}_{\omega_t} \omega \le \frac{1}{(n-1)!} \left(\operatorname{tr}_\omega \omega_t(x)\right)^{n-1} \frac{\omega^n}{\omega_t^n}(x) \le C_K \frac{\int_X \omega^n}{\int_X \alpha_t^n} \le C_K,$$

for we assume that $\int_X \alpha_t^n \to \int_X \alpha_0^n > 0$. Combined, we get

$$C_K^{-1}\omega \le \omega_t \le C_K \omega \text{ on each } K \Subset X \setminus \mathrm{Null}([\alpha_0]), \text{ for all } t \in (0,1].$$

Finally, we can apply local higher order estimates (Theorem 1.27) to ω_t with some suitable fixed open cover of K to bound $\|\omega_t\|_{C^k(K,\omega)}$ with constants independent of t (**it depends on the choice of open cover, k, n, and ω**). Indeed ω is comparable with the Euclidean metric in each local coordinate.

Since α_t depends continuously on $t \in [0,1]$, we also have uniform $C^k(K,\omega)$ bound on α_t independent of t. Thus $i\partial\bar{\partial}\varphi_t = \omega_t - \alpha_t$ is bounded uniformly, as well as their trace

$$\Delta_\omega \varphi_t = \operatorname{tr}_\omega(\omega_t - \alpha_t).$$

Finally apply Schauder estimates to give uniform bound on $\|\varphi_t\|_{C^k(K,\omega)}$. This completes the proof. $\square$

With this uniform bound on ω_t, compactness results show that ω_t converges in the $C^\infty_{\mathrm{loc}}(X \setminus \mathrm{Null}([\alpha_0]))$ topology to some Ricci-flat Kähler metric ω_0 on $X \setminus \mathrm{Null}([\alpha_0])$, as $t \to 0$.

We introduce K3 surface as an example of the case of degenerations of Ricci-flat Kähler metrics discussed above.

Definition 3.28. A K3 surface X^2 is a 2-dimensional compact Kähler manifold that is Calabi-Yau (i.e. $c_1(X) = 0 \in H^2(X,\mathbb{R})$) and simply connected (i.e. $\pi_1(X) = \{1\}$).

Lemma 3.29. *For every K3 surface X, the canonical bundle $\mathcal{K}_X$ is isomorphic to the trivial line bundle $\mathcal{O}_X$.*

Proof. We know that

$$H_1(X,\mathbb{Z}) = \pi_1(X)_{abelian} = 0.$$

The Universal Coefficient Theorem in topology implies that i) the torsion of $H^2(X,\mathbb{Z})$ is isomorphic to the torsion of $H_1(X,\mathbb{Z})$, so $H^2(X,\mathbb{Z})$ is torsion-free; ii) $H^1(X,\mathbb{Z})$ is isomorphic to the free part of $H_1(X,\mathbb{Z})$, so $H^1(X,\mathbb{Z}) = 0$.

Since X is Kähler, Hodge thoery implies

$$0 = H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X) \Rightarrow H^{0,1}(X) = 0.$$

By the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0,$$

we have exact sequence

$$H^{0,1}(X) \to \operatorname{Pic}(X) \to H^2(X, \mathbb{Z}) \to H^{0,2}(X).$$

As $H^{0,1}(X) = 0$, the map $c_1 : \operatorname{Pic}(X) \to H^2(X, \mathbb{Z})$ is injective. Note that

$$c_1(\mathcal{K}_X) = -c_1(X) = 0 = c_1(\mathcal{O}_X) \in H^2(X, \mathbb{R}),$$

and $H^2(X, \mathbb{Z})$ is torsion-free, so $c_1(\mathcal{K}_X) = c_1(\mathcal{O}_X) = 0 \in H^2(X, \mathbb{Z})$, $\mathcal{K}_X \cong \mathcal{O}_X$. □

The statement is equivalent to the existence of a global holomorphic section $s \in H^0(X, \mathcal{K}_X)$ that is nowhere vanishing, i.e. a nowhere vanishing holomorphic 2-form ($\overline{\partial}$-closed (2, 0)-form) on X. By maximum principle, such s is unique up to scaling by $\mathbb{C}^*$.

Example 3.30. Smooth hypersurfaces $X = \{P = 0\} \subset \mathbb{P}^3$ where P is homogeneous polynomial in $\mathbb{C}[z_0, ..., z_3]$ of degree 4, are K3 surfaces. Proof. We know that $\mathcal{K}_X \cong \mathcal{O}(4-3-1)|_X \cong \mathcal{O}_X$, so $c_1(X) = -c_1(\mathcal{K}_X) = 0 \in H^2(X, \mathbb{R})$. By Lefschetz Hyperplane Theorem, $\pi_1(X) \cong \pi_1(\mathbb{P}^3) = \{1\}$.

We can use Hirzebruch-Riemann-Roch on K3 surfaces for more topological properties, like the Betti numbers. Let X be a K3 surface, $L := \mathcal{O}_X$. Then

$$\begin{aligned}
\chi(X, \mathcal{O}_X) &= \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X) - \dim_{\mathbb{C}} H^1(X, \mathcal{O}_X) + \dim_{\mathbb{C}} H^2(X, \mathcal{O}_X) && \dim_{\mathbb{C}} X = 2 \\
&= \frac{1}{2}\left(\int_X c_1(L)^2 + \int_X c_1(X) \wedge c_1(L)\right) + \frac{1}{12}\left(\int_X c_1(X)^2 + \chi(X)\right) \\
&= \frac{\chi(X)}{12} && c_1(X) = 0
\end{aligned}$$

Also, $H^0(X, \mathcal{O}_X) = \mathbb{C}$. By Hodge theory, (we know already that $H^1(X, \mathbb{Z}) = H_1(X, \mathbb{Z}) = 0$ by Universal Coefficient Theorem)

$$0 = H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X) \Rightarrow H^1(X, \mathcal{O}_X) = 0.$$

By Serre duality,

$$H^2(X, \mathcal{O}_X) \cong H^{0,2}(X) \cong H^{2,0}(X)^* \cong H^0(X, \mathcal{K}_X)^* \cong H^0(X, \mathcal{O}_X)^* \Rightarrow \dim_{\mathbb{C}} H^2(X, \mathcal{O}_X) = 1.$$

Therefore, $\chi(X) = 24$. On the other hand,

$$\begin{aligned}
24 &= \chi(X) \\
&= \dim H^0(X, \mathbb{R}) - \dim H^1(X, \mathbb{R}) + \dim H^2(X, \mathbb{R}) - \dim H^3(X, \mathbb{R}) + \dim H^4(X, \mathbb{R}) \\
&= 2 + \dim H^2(X, \mathbb{R})
\end{aligned}$$

by Poincaré duality. Thus the second Betti number $b_2 = \dim H^2(X, \mathbb{R}) = 22$. In summary, the Betti numbers of X are

$$b_0 = b_4 = 1, \quad b_1 = b_3 = 0, \quad b_2 = 22.$$

Moreover, by Hodge theory,

$$\mathbb{C}^{22} \cong H^2(X, \mathbb{C}) \cong H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X),$$

and we know from above that $\dim_{\mathbb{C}} H^{0,2}(X) = 1$, so

$$\dim_{\mathbb{C}} H^{1,1}(X) = 20.$$

Thus $H^{1,1}(X, \mathbb{R}) = H^{1,1}(X) \cap H^2(X, \mathbb{R}) \subset H^2(X, \mathbb{C})$ is isomorphic to $\mathbb{R}^{20}$ (consider $H^{1,1}(X, \mathbb{R})$ as the space of real harmonic (1, 1)-forms), and $\mathcal{C}_X$ is a cone in $\mathbb{R}^{20}$.

Example 3.31. Let X be a Kummer K3 surface. X is constructed as follows. Take a torus $Y = \mathbb{C}^2/\Lambda$. The map $\iota : Y \to Y$ via $(z_1, z_2) \mapsto (-z_1, -z_2)$ has 16 singular points. Resolve these singularities by a blow-up to get $\pi : X \to Y/\iota$. Take $[\alpha_0] = \pi^*[\omega_{\mathbb{C}^2}]$, then $\alpha_0 \geq 0$ and $\int_X \alpha_0^2 > 0$. Moreover, $\mathrm{Null}([\alpha_0])$ is the preimage under π of the 16 singular points on Y.