Extremal Sobolev Inequalities and Applications

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1 Sobolev Spaces

Definition 1.1 *Let Ω be an open subset of $\mathbb{R}^n$, and fix a natural number k and a real $1 \leq p \leq \infty$. The Sobolev space $H_k^p(\Omega)$ is the space of all functions $u \in L^p(\Omega)$ such that all the weak derivatives of u of order $\leq k$ are in $L^p(\Omega)$.*

If $u \in H_k^p(\Omega)$ we set

$$\|u\|_{k,p} := \sum_{0 \leq |\alpha| \leq k} \left(\int_\Omega |\nabla^\alpha u|^p \right)^{1/p},$$

then $H_k^p(\Omega)$ becomes a Banach space. If $p = 2$ then it is also a Hilbert space, with the scalar product

$$(u, v)_{k,2}^2 := \sum_{0 \leq |\alpha| \leq k} \int_\Omega \nabla^\alpha u \nabla^\alpha v.$$

We will always write $\|u\|_p := \|u\|_{0,p}$. It is also true that the Sobolev space $H_k^p(\Omega)$ is the completion of $\{u \in C^\infty(\Omega) \mid \|u\|_{k,p} < \infty\}$ with respect to the norm $\|\cdot\|_{k,p}$ (this was first proved by Meyers and Serrin [14]).

Proposition 1.2 *The space $C_c^\infty(\mathbb{R}^n)$ of smooth functions with compact support is dense in $H_k^p(\mathbb{R}^n)$.*

Proof
Let $f : \mathbb{R} \to \mathbb{R}$ be any smooth function that is identically 1 for $t \leq 0$ and identically 0 for $t \geq 1$. Since $H_k^p(\mathbb{R}^n)$ is the completion of $C^\infty(\mathbb{R}^n)$, it is enough to show that every function $\varphi \in C^\infty(\mathbb{R}^n) \cap H_k^p(\mathbb{R}^n)$ can be approximated in $H_k^p(\mathbb{R}^n)$ by functions in $C_c^\infty(\mathbb{R}^n)$. Consider the sequence

$$\varphi_j(x) := \varphi(x) f(|x| - j).$$

We have that $\varphi_j \in C_c^\infty(\mathbb{R}^n)$: in fact $|x|$ is not differentiable at $x = 0$, but $f(t)$ is identically 1 for $t \leq 0$ so that φ_j is smooth for $j > 0$. As $j \to \infty$, $\varphi_j(x) \to \varphi(x)$ for every $x \in \mathbb{R}^n$, and $|\varphi_j(x)| \leq |\varphi(x)|$ which belongs to $L^p(\mathbb{R}^n)$, so by Lebesgue dominated convergence theorem we have $\|\varphi_j - \varphi\|_p \to 0$.

For every fixed k and every multiindex α of length k we have $\nabla^\alpha \varphi_j(x) \to \nabla^\alpha \varphi(x)$ as $j \to \infty$, and by induction

$$|\nabla^\alpha \varphi_j(x)| \leq |\nabla^\alpha \varphi(x)| + C \sum_{0 \leq |l| \leq k-1} |\nabla^l \varphi(x)| \cdot |\nabla^{k-|l|} f_j(x)|,$$

for some constant C, where $f_j(x) = f(|x| - j)$. We note that $f^{(s)}(t)$ and $|\nabla^s |x||$ are bounded for $s \geq 1$, so we get

$$|\nabla^\alpha \varphi_j(x)| \leq |\nabla^\alpha \varphi(x)| + C' \sum_{0 \leq |l| \leq k-1} |\nabla^l \varphi(x)|.$$

But the right hand side above belongs to L^p, so again by Lebesgue dominated convergence, $\|\nabla^\alpha(\varphi_j - \varphi)\|_p \to 0$. $\square$

This is false for general domains $\Omega \subset \mathbb{R}^n$, and we denote by $H_0^{k,p}(\Omega)$ the closure of $C_c^\infty(\Omega)$ with respect to $\|\cdot\|_{k,p}$.

Now let (M, g) be a smooth Riemannian manifold of dimension n, connected and without boundary, and let $u : M \to \mathbb{R}$ be a smooth function. Then for k a natural number, we let $\nabla^k u$ be the k-th total covariant derivative of u and $|\nabla^k u|$ be its norm with respect to g. In a local chart this is

$$|\nabla^k u|^2 = g^{i_1 j_1} \dots g^{i_k j_k} \nabla_{i_1} \dots \nabla_{i_k} u \nabla_{j_1} \dots \nabla_{j_k} u.$$

If $p \geq 1$ is a real number, we set

$$\|u\|_{k,p} := \sum_{j=0}^{k} \left(\int_M |\nabla^j u|^p dV \right)^{1/p}.$$

Definition 1.3 *The Sobolev space $H_k^p(M)$ is the completion of*

$$\{u \in C^\infty(M) \mid \|u\|_{k,p} < \infty\}$$

with respect to the norm $\|\cdot\|_{k,p}$.

We have that $H_k^p(M)$ is Banach space and if $p = 2$ then it is also a Hilbert space, with the scalar product

$$(u, v)_{k,2}^2 := \sum_{j=0}^{k} \int_M \langle \nabla^j u, \nabla^j v \rangle dV,$$

where $\langle \cdot, \cdot \rangle$ is the pairing induced by g. If M is compact and h is another Riemannian metric on M, then there is a constant $C > 0$ such that

$$\frac{1}{C} g \leq h \leq C g,$$

because this is true in every chart and we can cover M with finitely many charts. Also this is true for the covariant derivatives of g and h up to any finite order k. Then the Sobolev norms with respect to g and h are also equivalent, so they define the same Sobolev space. Hence we have proved the

Proposition 1.4 *If M is compact then the Sobolev spaces $H_k^p(M)$ do not depend on the Riemannian metric.*

By definition of Sobolev spaces we have that $C^\infty(M)$ is dense in $H_k^p(M)$, so we can ask when does this happen for $C_c^\infty(M)$. Of course if M is compact these two spaces coincide, but if M is just complete, in general $C_c^\infty(M)$ is NOT dense in $H_k^p(M)$. Nevertheless the following is true:

Proposition 1.5 *If (M,g) is a complete Riemannian manifold, then $C_c^\infty(M)$ is dense in $H_1^p(M)$.*

Proof
We notice that we cannot proceed like in Proposition 1.2 because in general the distance function $d(x,P)$ for a fixed $P \in M$ is only Lipschitz in x. So let's define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(t) = 1$ for $t \leq 0$, $f(t) = 1 - t$ for $0 \leq t \leq 1$ and $f(t) = 0$ for $t \geq 1$, so that f is Lipschitz and $|f'| \leq 1$. It is enough to show that we can approximate any $\varphi \in C^\infty(M) \cap H_1^p(M)$ by smooth functions with compact support. Fix $P \in M$ and define

$$\varphi_j(x) := \varphi(x) f(d(x,P) - j).$$

Then each of the φ_j is Lipschitz, so by Rademacher's theorem is differentiable a.e., has compact support and so is bounded. But $\nabla\varphi_j$ is also bounded, because

$$|\nabla\varphi_j(x)| \leq |\nabla\varphi(x)| + |\varphi(x)| \sup_{t\in[0,1]} |f'(t)| \leq |\nabla\varphi(x)| + |\varphi(x)|,$$

where we have used that $|\nabla d| = 1$ a.e. Hence all the φ_j belong to $H_1^p(M)$. Exactly like in the proof of Proposition 1.2 we can prove that $\varphi_j \to \varphi$ in $H_1^p(M)$. We now have to show that we can approximate each φ_j, but this is easy: by definition there are functions $\varphi_j^k \in C^\infty(M)$ that converge to φ_j in $H_1^p(M)$ as $k \to \infty$. Now pick $\alpha_j \in C_c^\infty(M)$ that is identically 1 on the support of φ_j; then we have that $\alpha_j\varphi_j^k \in C_c^\infty(M)$ converge to φ_j in $H_1^p(M)$, and we have finished. □

2 The Sobolev Inequalities

2.1 The Euclidean case

We have the following fundamental

Theorem 2.1 (Sobolev Embedding) *Assume $n \geq 2$, let k, l be two natural numbers, $k > l$, and p, q two real numbers $1 \leq q < p$ satisfying*

$$\frac{1}{p} = \frac{1}{q} - \frac{k-l}{n}.$$

Then

$$H^q_k(\mathbb{R}^n) \subset H^p_l(\mathbb{R}^n)$$

and the identity operator is continuous. If $n = 1$ *then for every natural numbers* $k > l$ *and* p, q *real numbers* $1 \leq q \leq p \leq \infty$ *we have a continuous embedding*

$$H^q_k(\mathbb{R}) \subset H^p_l(\mathbb{R}).$$

Proof
The proof consists of several steps. First assume $n \geq 2$.

Step 1 (Gagliardo-Nirenberg Inequality [10],[16]) We prove that every $\varphi \in C^\infty_c(\mathbb{R}^n)$ we have

$$\|\varphi\|_{n/(n-1)} \leq \frac{1}{2} \prod_{i=1}^{n} \left\| \frac{\partial \varphi}{\partial x_i} \right\|_1^{1/n}. \tag{2.1}$$

Pick a point $P = (y_1, \dots, y_n)$ in $\mathbb{R}^n$, call $D_{x_{i_1},\dots,x_{i_j}}$ the j-plane through P parallel to the one generated by the coordinated axes $x_{i_1}, \dots, x_{i_j}$, so for example $D_{x_1,\dots,x_n} = \mathbb{R}^n$. Since φ has compact support, we can apply the fundamental theorem of calculus to get

$$\varphi(P) = \int_{-\infty}^{y_1} \frac{\partial \varphi}{\partial x_1}(x_1, y_2, \dots, y_n) dx_1 = -\int_{y_1}^{+\infty} \frac{\partial \varphi}{\partial x_1}(x_1, y_2, \dots, y_n) dx_1$$

$$|\varphi(P)| \leq \frac{1}{2} \int_{D_{x_1}} |\partial_{x_1} \varphi|(x_1, y_2, \dots, y_n) dx_1.$$

Doing the same for all the other coordinates, multiplying them all together and taking the $(n-1)$-th root we get

$$|\varphi(P)|^{\frac{n}{n-1}} \leq \frac{1}{2^{n/(n-1)}} \left(\int_{D_{x_1}} |\partial_{x_1} \varphi| dx_1 \cdots \int_{D_{x_n}} |\partial_{x_n} \varphi| dx_n \right)^{\frac{1}{n-1}}$$

Now we integrate this inequality for $y_1 \in \mathbb{R}$: the first integral does not depend on y_1 so it can be taken out. Then we apply Hölder's inequality $n-2$ times to the remaining terms this way:

$$\int_{\mathbb{R}} f_1^{\frac{1}{n-1}} \dots f_{n-1}^{\frac{1}{n-1}} \leq \left(\int_{\mathbb{R}} f_1 \right)^{\frac{1}{n-1}} \dots \left(\int_{\mathbb{R}} f_{n-1} \right)^{\frac{1}{n-1}}.$$

We get

$$\int_{D_{x_1}} |\varphi(y_1, y_2, \dots, y_n)|^{\frac{n}{n-1}} dy_1 \leq$$

$$\frac{1}{2^{n/(n-1)}} \left(\int_{D_{x_1}} |\partial_{x_1} \varphi|(x_1, y_2, \dots, y_n) dx_1 \int_{D_{x_1,x_2}} |\partial_{x_2} \varphi|(y_1, x_2, y_3, \dots, y_n) dy_1 dx_2 \right.$$

$$\left. \cdots \int_{D_{x_1,x_n}} |\partial_{x_n} \varphi|(y_1, y_2, \dots, x_n) dy_1 dx_n \right)^{\frac{1}{n-1}}.$$

Integration of $y_2, \dots, y_n$ over $\mathbb{R}$ and the use of Hölder's inequality again, leads to

$$\int_{\mathbb{R}^n} |\varphi|^{\frac{n}{n-1}} \le \frac{1}{2^{n/(n-1)}} \left(\int_{\mathbb{R}^n} |\partial_{x_1}\varphi| \dots \int_{\mathbb{R}^n} |\partial_{x_n}\varphi| \right)^{\frac{1}{n-1}},$$

which is exactly (2.1).

Step 2 (Sobolev Inequality) We prove that there exists a constant $K(n,q)$ such that for every $\varphi \in H_1^q(\mathbb{R}^n)$ we have

$$\|\varphi\|_p \le K(n,q)\|\nabla\varphi\|_q, \tag{2.2}$$

where $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$, and $1 \le q < n$.
By Proposition 1.2 it is enough to prove (2.2) for $\varphi \in C_c^\infty(\mathbb{R}^n)$. First of all for every i we have $|\partial\varphi/\partial x_i| \le |\nabla\varphi|$ so by (2.1)

$$\|\varphi\|_{n/(n-1)} \le \frac{1}{2}\|\nabla\varphi\|_1.$$

This gives us the Sobolev inequality for $q = 1$. Now let $1 < q < n$, $p = nq/(n-q)$, and set $u := |\varphi|^{p(n-1)/n}$. Then, using (2.1) and Hölder's inequality we get

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |\varphi|^p\right)^{(n-1)/n} &= \left(\int_{\mathbb{R}^n} |u|^{n/(n-1)}\right)^{(n-1)/n} \le \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u| \\ &= p\frac{n-1}{2n}\int_{\mathbb{R}^n} |\varphi|^{p'}|\nabla\varphi| \le p\frac{n-1}{2n}\left(\int_{\mathbb{R}^n} |\varphi|^{p'q'}\right)^{1/q'}\left(\int_{\mathbb{R}^n} |\nabla\varphi|^q\right)^{1/q},\end{aligned}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$, $p' = (p(n-1)/n) - 1$. So

$$\frac{1}{q'} = 1 - \frac{1}{q} = 1 - \frac{1}{p} - \frac{1}{n} = \frac{pn - n - p}{pn}$$

$$p' = \frac{pn - n - p}{n},$$

hence $p'q' = p$ and we get

$$\|\varphi\|_p^{p(n-1)/n} \le p\frac{n-1}{2n}\|\varphi\|_p^{p/q'}\|\nabla\varphi\|_q$$

so dividing by $\|\varphi\|_p^{p/q'}$ and computing

$$\frac{p}{q'} = \frac{pn - n - p}{n} = (p(n-1)/n) - 1$$

we get finally

$$\|\varphi\|_p \le p\frac{n-1}{2n}\|\nabla\varphi\|_q,$$

which is (2.2).

Now (2.2) tells us that we have a continuous embedding

$$H_1^q(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) = H_0^p(\mathbb{R}^n)$$

where $1 \leq q < n$, $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$. So we have proved the Sobolev embedding in the case $k = 1$.

Step 3 We prove that if the Sobolev embedding holds for any $1 \leq q < n$ and $k = 1$ then it holds for any k, so that if $1 \leq q < p_l$ and $1/p_l = 1/q - (k-l)/n$ then $H_k^q(\mathbb{R}^n)$ is continuously embedded in $H_l^{p_l}(\mathbb{R}^n)$.

By definition of Sobolev spaces it is enough to prove that there is a constant $C > 0$ such that for every $\varphi \in C^\infty(\mathbb{R}^n) \cap H_k^q(\mathbb{R}^n)$ we have

$$\|\varphi\|_{l,p_l} \leq C\|\varphi\|_{k,q}.$$

Notice that here we don't need φ to have compact support, so this step will work also for complete Riemannian manifolds. The first step is Kato's inequality: for every smooth function ψ and every multiindex r we have

$$|\nabla|\nabla^r\psi|| \leq |\nabla^{r+1}\psi|,$$

where $|\nabla^r\psi| \neq 0$. This is true in more generality: if $E \to M$ is a vector bundle over a Riemannian manifold M, with metric and compatible connection ∇, and if ξ is a section of E then

$$|d|\xi|| \leq |\nabla\xi|$$

where $\xi \neq 0$. The proof is very simple:

$$2\big|d|\xi|\big||\xi| = |d(|\xi|^2)| = 2|\langle\nabla\xi, \xi\rangle| \leq 2|\nabla\xi||\xi|.$$

Now that we have Kato's inequality, since $H_1^q(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ there is a constant A such that for all $\psi \in H_1^q(\mathbb{R}^n)$ we have

$$\|\psi\|_p \leq A(\|\nabla\psi\|_q + \|\psi\|_q).$$

Apply this to $\psi = |\nabla^r\varphi|$ with $r = k-1, k-2, \ldots, 0$ which all belong to $H_1^q(\mathbb{R}^n)$, and get

$$\|\nabla^r\varphi\|_p \leq A(\|\nabla|\nabla^r\varphi|\|_q + \|\nabla^r\varphi\|_q) \leq A(\|\nabla^{r+1}\varphi\|_q + \|\nabla^r\varphi\|_q),$$

where we have also used Kato's inequality. Now add all these k inequalities and get

$$\|\varphi\|_{k-1,p} \leq 2A\|\varphi\|_{k,q}.$$

By definition we have $p = p_{k-1}$. We have just shown that we have a continuous inclusion $H^q_k(\mathbb{R}^n) \subset H^{p_{k-1}}_{k-1}(\mathbb{R}^n)$. Now iterate the reasoning above to get a chain of continuous inclusions

$$H^q_k(\mathbb{R}^n) \subset H^{p_{k-1}}_{k-1}(\mathbb{R}^n) \subset H^{p_{k-2}}_{k-2}(\mathbb{R}^n) \subset \cdots \subset H^{p_{k-(k-l)}}_{k-(k-l)}(\mathbb{R}^n) = H^{p_l}_l(\mathbb{R}^n).$$

Step 4 Now assume $n = 1$. Exactly as in step 1, for every $\varphi \in C^\infty_c(\mathbb{R})$ and for every $x \in \mathbb{R}$ we have

$$|\varphi(x)| \leq \frac{1}{2} \int_{\mathbb{R}} \left| \frac{\partial \varphi}{\partial y} \right| dy.$$

This immediately implies that

$$H^1_1(\mathbb{R}) \subset L^\infty(\mathbb{R}).$$

Now assume that $\varphi \in C^\infty_c(\mathbb{R})$ and $p \geq 1$. By the Markov inequality

$$\mathrm{Vol}(\{x \mid \varphi(x) \geq 1\}) \leq \|\varphi\|_1 < \infty,$$

so

$$\int_{\mathbb{R}} |\varphi|^p = \int_{\{\varphi \geq 1\}} |\varphi|^p + \int_{\{\varphi < 1\}} |\varphi|^p \leq (\sup_{\mathbb{R}} |\varphi|)^p \|\varphi\|_1 + \int_{\mathbb{R}} |\varphi|,$$

$$\|\varphi\|_p \leq \frac{1}{2} \|\nabla \varphi\|_1 \|\varphi\|_1^{\frac{1}{p}} + \|\varphi\|_1^{\frac{1}{p}},$$

hence

$$H^1_1(\mathbb{R}) \subset L^p(\mathbb{R}).$$

Now let $q > 1$, $\varphi \in C^\infty_c(\mathbb{R})$ and set $u = |\varphi|^q$. Then

$$|\varphi|^q = u \leq \frac{1}{2} \int_{\mathbb{R}} |\nabla u| = \frac{q}{2} \int_{\mathbb{R}} |\varphi|^{q-1} |\nabla \varphi| \leq \frac{q}{2} \left(\int_{\mathbb{R}} |\varphi|^{(q-1)q'} \right)^{\frac{1}{q'}} \left(\int_{\mathbb{R}} |\nabla \varphi|^q \right)^{\frac{1}{q}},$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. Then $(q-1)q' = q$, so

$$|\varphi|^q \leq \frac{q}{2} \|\varphi\|_q^{q-1} \|\nabla \varphi\|_q,$$

hence

$$H^q_1(\mathbb{R}) \subset L^\infty(\mathbb{R}),$$

and if $p \geq q$ we proceed as above using Markov inequality to get

$$H^q_1(\mathbb{R}) \subset L^p(\mathbb{R}).$$

The last step when $k > l > 0$ follows exactly as in step 3. □

2.2 The compact manifold case

Theorem 2.2 (Sobolev Embedding) *Let M be a compact Riemannian manifold of dimension n. Let k, l be two natural numbers, $k > l$, and p, q two real numbers $1 \leq q < p$ satisfying*

$$\frac{1}{p} = \frac{1}{q} - \frac{k-l}{n}.$$

Then

$$H_k^q(M) \subset H_l^p(M)$$

and the identity operator is continuous.

Proof
Since the proof of the Step 3 of the Sobolev embedding on $\mathbb{R}^n$ carries on word by word to this context, it is enough to prove that we have a continuous embedding

$$H_1^q(M) \subset L^p(M) = H_0^p(M)$$

where $1 \leq q < n$, $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$, and so it is enough to prove an inequality of the form

$$\|\varphi\|_p \leq C(\|\nabla\varphi\|_q + \|\varphi\|_q) \tag{2.3}$$

for every $\varphi \in C^\infty(M)$. Let $(\Omega_i, \eta_i)_{1 \leq i \leq N}$ be a finite cover of M with coordinate charts such that for all $1 \leq m \leq N$

$$\frac{1}{2}\delta_{ij} \leq g_{ij}^m \leq 2\delta_{ij},$$

where g_{ij}^m are the components of g in the chart Ω_m. Let $\{\alpha_i\}$ be a partition of unity subordinate to this covering. If we prove that there is a constant C such that

$$\|\alpha_i\varphi\|_p \leq C(\|\nabla(\alpha_i\varphi)\|_q + \|\alpha_i\varphi\|_q) \tag{2.4}$$

then since $|\nabla(\alpha_i\varphi)| \leq |\nabla\varphi| + |\varphi| \cdot |\nabla\alpha_i|$, we'd get

$$\|\varphi\|_p = \left\|\sum_{i=1}^N \alpha_i\varphi\right\|_p \leq \sum_{i=1}^N \|\alpha_i\varphi\|_p \leq CN\left(\|\nabla\varphi\|_q + (1 + \max_i \sup_M |\nabla\alpha_i|)\|\varphi\|_q\right),$$

which is of the form (2.3). So we have to prove (2.4). On the compact set $K_i = \operatorname{supp} \alpha_i \subset \Omega_i$ the metric tensor and all its derivatives of all orders are bounded, in the coordinates η_i. So we get

$$\varphi \in H_1^q(M) \iff (\alpha_i\varphi \in H_1^q(M), \forall i) \iff (\alpha_i\varphi \circ \eta_i^{-1} \in H_1^q(\mathbb{R}^n), \forall i),$$

where we defined $\alpha_i\varphi \circ \eta_i^{-1}$ to be zero outside $\eta_i(K_i)$. Then we have

$$\left(\int_M |\alpha_i\varphi|^p dV\right)^{1/p} \leq 2^{n/2}\left(\int_{\mathbb{R}^n} |\alpha_i\varphi \circ \eta_i^{-1}(x)|^p dx\right)^{1/p}$$

$$\left(\int_M |\nabla(\alpha_i\varphi)|^q dV\right)^{1/q} \geq 2^{-(n+1)/2}\left(\int_{\mathbb{R}^n} |\nabla(\alpha_i\varphi\circ\eta_i^{-1})(x)|^q dx\right)^{1/q}$$

Now Theorem 2.1 tells us that there is a constant $C > 0$ such that

$$\left(\int_{\mathbb{R}^n} |\alpha_i\varphi\circ\eta_i^{-1}(x)|^p dx\right)^{1/p} \leq C\left(\int_{\mathbb{R}^n} |\nabla(\alpha_i\varphi\circ\eta_i^{-1})(x)|^q dx\right)^{1/q}$$

and putting together these 3 inequalities we get (2.4). This finishes the proof. □

2.3 The best constants

Theorem 2.3 (Aubin, Talenti [2],[20]) *The best constant in the Sobolev inequality* (2.2) *on* $\mathbb{R}^n$ *is*

$$K(n,q) = \frac{1}{n}\left(\frac{n(q-1)}{n-q}\right)^{1-\frac{1}{q}}\left(\frac{\Gamma(n+1)}{\Gamma(n/q)\Gamma(n+1-n/q)\omega_{n-1}}\right)^{\frac{1}{n}}$$

for $q > 1$*, and*

$$K(n,1) = \frac{1}{n}\left(\frac{n}{\omega_{n-1}}\right)^{\frac{1}{n}}$$

Recall that $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(x+1) = x\Gamma(x)$, $\Gamma(n) = (n-1)!$ and

$$\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

In particular we get

$$\omega_{2n} = \frac{(4\pi)^n(n-1)!}{(2n-1)!}$$

$$\omega_{2n+1} = \frac{2\pi^{n+1}}{n!}.$$

3 The Logarithmic Sobolev Inequalities

Theorem 3.1 ([6]) *If* $f \in H_1^2(\mathbb{R}^n)$ *with* $\|f\|_2 = 1$*,* $|f| > 0$ *a.e., then*

$$\int_{\mathbb{R}^n} |f|^2 \log|f| \leq \frac{n}{4}\log\left(\frac{2}{\pi e n}\int_{\mathbb{R}^n} |\nabla f|^2\right). \tag{3.1}$$

Proof
We set $p = \frac{2n}{n-2}$ and apply the Sobolev inequality to get

$$\left(\int_{\mathbb{R}^n} |f|^p\right)^{2/p} \leq K(n,2)^2\int_{\mathbb{R}^n} |\nabla f|^2.$$

Using Jensen's inequality we get

$$\log \int_{\mathbb{R}^n} |f|^p \geq (p-2) \int_{\mathbb{R}^n} |f|^2 \log |f|$$

and putting together these two inequalities we get

$$(p-2) \int_{\mathbb{R}^n} |f|^2 \log |f| \leq \frac{p}{2} \log \left(K(n,2)^2 \int_{\mathbb{R}^n} |\nabla f|^2 \right).$$

Since $\frac{p}{2(p-2)} = \frac{n}{4}$ we get

$$\int_{\mathbb{R}^n} |f|^2 \log |f| \leq \frac{n}{4} \log \left(K(n,2)^2 \int_{\mathbb{R}^n} |\nabla f|^2 \right). \tag{3.2}$$

This is almost what we want to prove, but we want a better constant. To achieve this we have to let n go to infinity. First we compute the asymptotic behaviour of $K(n,2)^2$ for n big. By Theorem 2.3 we have that

$$\begin{aligned} K(n,2)^2 &= \frac{1}{n^2} \left(\frac{n}{n-2} \right) \left(\frac{\Gamma(n+1)}{\Gamma(n/2)\Gamma(n/2+1)\omega_{n-1}} \right)^{\frac{2}{n}} \\ &= \frac{1}{n(n-2)} \left(\frac{2\Gamma(n)}{\Gamma(n/2)^2 \omega_{n-1}} \right)^{\frac{2}{n}} = \frac{1}{\pi n(n-2)} \left(\frac{\Gamma(n)}{\Gamma(n/2)} \right)^{\frac{2}{n}} \end{aligned}$$

and by Stirling's formula we have

$$\left(\frac{\Gamma(n)}{\Gamma(n/2)} \right)^{\frac{2}{n}} \sim 2ne^{-1}$$

so

$$K(n,2)^2 \sim \frac{2}{\pi e n}.$$

Now we use this asymptotic behaviour in the following way: set $m = nl$ with $l \geq 0$, and for $x \in \mathbb{R}^m$ set $F(x) = \prod_{k=1}^{l} f(x_k)$ where each x_k is in $\mathbb{R}^n$. Since $\|f\|_2 = 1$ we have $\|F\|_2 = 1$ so we can apply inequality (3.2) to F and get

$$l \int_{\mathbb{R}^n} |f|^2 \log |f| \leq \frac{nl}{4} \log \left(lK(nl,2)^2 \int_{\mathbb{R}^n} |\nabla f|^2 \right).$$

Now we let $l \to \infty$, and we have $lK(nl,2)^2 \to \frac{2}{\pi e n}$, so we have proved (3.1). □

Define the Gaussian measure on $\mathbb{R}^n$ by $d\mu = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} dx$. Then we have the following

Theorem 3.2 (Gross [11]) *If $g \in H_1^2(\mathbb{R}^n, d\mu)$, $\int_{\mathbb{R}^n} |g|^2 d\mu = 1$, $|g| > 0$ a.e. then*

$$\int_{\mathbb{R}^n} |g|^2 \log |g| d\mu \leq \int_{\mathbb{R}^n} |\nabla g|^2 d\mu \tag{3.3}$$

Proof
We will show that (3.3) is actually equivalent to (3.1). First of all set $f(x) = (2\pi)^{-\frac{n}{4}} e^{-\frac{|x|^2}{4}} g(x)$, so that $\|f\|_2 = \int_{\mathbb{R}^n} |g|^2 d\mu = 1$. Now compute

$$\nabla g = (2\pi)^{\frac{n}{4}} e^{\frac{|x|^2}{4}} \left(\nabla f + \frac{f \cdot x}{2} \right),$$

$$|\nabla g|^2 = (2\pi)^{\frac{n}{2}} e^{\frac{|x|^2}{2}} \left(|\nabla f|^2 + \frac{|f|^2 |x|^2}{4} + f \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} x^i \right),$$

and using integration by parts

$$\sum_{i=1}^{n} \int_{\mathbb{R}^n} f \frac{\partial f}{\partial x^i} x^i = - \sum_{i=1}^{n} \int_{\mathbb{R}^n} f \frac{\partial f}{\partial x^i} x^i - \sum_{i=1}^{n} \int_{\mathbb{R}^n} |f|^2$$

so $\sum_{i=1}^{n} \int_{\mathbb{R}^n} f \frac{\partial f}{\partial x^i} x^i = -\frac{n}{2}$. Substituting into (3.3) we get

$$\int_{\mathbb{R}^n} |f|^2 \left(\log |f| + \frac{n}{4} \log(2\pi) + \log \left(e^{\frac{|x|^2}{4}} \right) \right) \leq -\frac{n}{2} + \int_{\mathbb{R}^n} \left(|\nabla f|^2 + \frac{|f|^2 |x|^2}{4} \right)$$

which simplifies to

$$\int_{\mathbb{R}^n} |f|^2 \log |f| + \frac{n}{4} \log(2\pi e^2) \leq \int_{\mathbb{R}^n} |\nabla f|^2.$$

Now fix $\delta > 0$ and change $f(x)$ with $\delta^{\frac{n}{2}} f(\delta x)$ in this last inequality, to get

$$\int_{\mathbb{R}^n} |f|^2 \log |f| + \frac{n}{4} \log(2\pi e^2) \leq \delta^2 \int_{\mathbb{R}^n} |\nabla f|^2 - \frac{n}{2} \log \delta. \tag{3.4}$$

We have just shown that (3.3) is equivalent to (3.4) for all $\delta > 0$. But the right hand side of (3.4) achieves its minimum for

$$\delta_{\min} = \sqrt{\frac{n}{4 \int_{\mathbb{R}^n} |\nabla f|^2}},$$

so having (3.4) for all $\delta > 0$ is equivalent to having (3.4) for $\delta_{\min}$, which is

$$\int_{\mathbb{R}^n} |f|^2 \log |f| + \frac{n}{4} \log(2\pi e^2) \leq \frac{n}{4} - \frac{n}{4} \log \left(\frac{n}{4 \int_{\mathbb{R}^n} |\nabla f|^2} \right)$$

and this is precisely (3.1). □

Notice that the constant of the Gross logarithmic Sobolev inequality does not depend on n.

4 The Moser-Trudinger Inequality

Let D be a bounded domain in $\mathbb{R}^n$. Then, using Hölder's inequality, for every $q \in [1, n)$ we have a continuous embedding

$$H_1^n(D) \subset H_1^q(D),$$

and now by Sobolev embedding, we have

$$H_1^q(D) \subset L^p(D)$$

where $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$. Since q is arbitrarily close to n we get continuous embeddings

$$H_1^n(D) \subset L^p(D)$$

for all $p \in [1, \infty)$. The point is that we don't get an embedding into $L^\infty(D)$ as the following example shows. Let $D = \{x \in \mathbb{R}^2 \mid 0 < |x| < 1/e\}$ and define $f : D \to \mathbb{R}$ by $f(x) = \log|\log|x||$. Then $|f|^2$ is integrable and

$$\|\nabla f\|_2^2 = 2\pi \int_0^{1/e} \frac{dr}{r|\log r|^2} = 2\pi,$$

so that $f \in H_1^2(D)$, but f is not bounded on D. On the other hand

$$\|e^f\|_1 = 2\pi \int_0^{1/e} r|\log r| dr < \infty.$$

This is a general phenomenon as we will soon see.

Theorem 4.1 (Trudinger [22]) *Let D be a bounded domain in $\mathbb{R}^n$. Then there exist constants $C, \alpha > 0$, with C depending only on n, such that every $\varphi \in H_0^{1,n}(D)$ with $\|\nabla\varphi\|_n \le 1$ satisfies*

$$\int_D e^{\alpha|\varphi|^{n/(n-1)}} \le C\mathrm{Vol}(D). \tag{4.1}$$

Proof

First assume that $\varphi \in C_c^\infty(D)$. Fix $x \in D$ and use polar coordinates (r, θ) centered at x. Let $y \in \mathbb{R}^n$, $r = |x - y|$, and write

$$\varphi(x) = -\int_0^\infty \frac{\partial\varphi(r,\theta)}{\partial r} dr = -\int_0^\infty |x-y|^{1-n} \frac{\partial\varphi}{\partial r} r^{n-1} dr,$$

$$|\varphi(x)| \le \int_0^\infty |x-y|^{1-n} |\nabla\varphi| r^{n-1} dr$$

and integrate over S^{n-1} to get

$$|\varphi(x)| \le \frac{1}{\omega_{n-1}} \int_D |x-y|^{1-n} |\nabla\varphi(y)| dy.$$

By density this holds for every $\varphi \in H_0^{1,n}(D)$ and a.e. $x \in D$. Now fix $p \geq n$ and set $1/k = 1/p - 1/n + 1$, so that $k \geq 1$, $f(x,y) := |x-y|^{1-n}$, $g(y) := |\nabla\varphi(y)|$ and write

$$fg = (f^k g^n)^{\frac{1}{p}} (f^k)^{\frac{1}{k}-\frac{1}{p}} (g^n)^{\frac{1}{n}-\frac{1}{p}}.$$

Since $1/p + (1/k - 1/p) + (1/n - 1/p) = 1$ we can apply Hölder's inequality to get

$$\int_D f(x,y)g(y)dy \leq \left(\int_D f^k(x,y)g^n(y)dy\right)^{\frac{1}{p}} \left(\int_D f^k(x,y)dy\right)^{\frac{1}{k}-\frac{1}{p}} \left(\int_D g^n(y)dy\right)^{\frac{1}{n}-\frac{1}{p}}.$$

From this we get

$$\|\varphi\|_p = \left(\int_D |\varphi(x)|^p dx\right)^{\frac{1}{p}} \leq \frac{1}{\omega_{n-1}} \left(\int_D \left(\int_D f(x,y)g(y)dy\right)^p dx\right)^{\frac{1}{p}}$$

$$\leq \frac{1}{\omega_{n-1}} \left(\int_D \left(\int_D f^k(x,y)g^n(y)dy\right) \left(\int_D f^k(x,y)dy\right)^{\frac{p}{k}-1} dx\right)^{\frac{1}{p}} \left(\int_D g^n(y)dy\right)^{\frac{1}{n}-\frac{1}{p}}$$

$$\leq \frac{1}{\omega_{n-1}} \sup_{x\in D} \left(\int_D f^k(x,y)dy\right)^{\frac{1}{k}-\frac{1}{p}} \left(\int_D \int_D f^k(x,y)g^n(y)dydx\right)^{\frac{1}{p}} \left(\int_D g^n(y)dy\right)^{\frac{1}{n}-\frac{1}{p}}$$

$$\leq \frac{1}{\omega_{n-1}} \sup_{x\in D} \left(\int_D f^k(x,y)dy\right)^{\frac{1}{k}} \left(\int_D g^n(y)dy\right)^{\frac{1}{p}} \left(\int_D g^n(y)dy\right)^{\frac{1}{n}-\frac{1}{p}}$$

$$= \frac{1}{\omega_{n-1}} \sup_{x\in D} \left(\int_D f^k(x,y)dy\right)^{\frac{1}{k}} \|\nabla\varphi\|_n = \frac{1}{\omega_{n-1}} \sup_{x\in D} \left(\int_D |x-y|^{k(1-n)}dy\right)^{\frac{1}{k}} \|\nabla\varphi\|_n.$$

Let B be the ball with center x and the same volume as D, say that its radius is R. Then by spherical symmetrization we have that

$$\left(\int_D |x-y|^{k(1-n)}dy\right)^{\frac{1}{k}} \leq \left(\int_B |x-y|^{k(1-n)}dy\right)^{\frac{1}{k}}$$

and the last term is independent of x, so that we have

$$\sup_{x\in D} \left(\int_D |x-y|^{k(1-n)}dy\right)^{\frac{1}{k}} \leq \omega_{n-1}^{1/k} \left(\int_0^R r^{(k-1)(1-n)}dr\right)^{\frac{1}{k}} = \omega_{n-1}^{1/k} \left(\frac{R^{k+n-kn}}{k+n-kn}\right)^{\frac{1}{k}}$$

$$= \omega_{n-1}^{1/k} R^{\frac{k+n-kn}{k}} \frac{1}{(k+n-kn)^{1/k}}.$$

Now

$$\frac{1}{(k+n-kn)^{1/k}} = \left(\frac{p+1-p/n}{n}\right)^{\frac{n-1}{n}+\frac{1}{p}} \leq Cp^{\frac{n-1}{n}}$$

where $C > 0$ only depends on n, so putting all together

$$\|\varphi\|_p \leq C\|\nabla\varphi\|_n p^{\frac{n-1}{n}} R^{\frac{k+n-kn}{k}}.$$

Notice that

$$\|\varphi\|_p^p \leq C^p \|\nabla\varphi\|_n^p p^{\frac{p(n-1)}{n}} R^n \leq C^p \|\nabla\varphi\|_n^p p^{\frac{p(n-1)}{n}} \mathrm{Vol}(D)$$

for $p \geq n$. By changing the constant we may assume that we have such an inequality also for $p = \frac{kn}{n-1}$, $1 \leq k \leq n-1$. Then

$$\int_D e^{\alpha|\varphi|^{n/(n-1)}} = \sum_{p=0}^{\infty} \frac{\alpha^p}{p!} \int_D |\varphi|^{\frac{pn}{n-1}} \leq \mathrm{Vol}(D) \sum_{p=0}^{\infty} \frac{\alpha^p}{p!} \left(C\|\nabla\varphi\|_n\right)^{\frac{pn}{n-1}} \left(\frac{pn}{n-1}\right)^p$$
$$= \mathrm{Vol}(D) \sum_{p=0}^{\infty} \frac{\left(\alpha (eC\|\nabla\varphi\|_n)^{\frac{n}{n-1}} \frac{n}{n-1}\right)^p \left(pe^{-\frac{n}{n-1}}\right)^p}{p!}.$$

Since $e^{\frac{n}{n-1}} > e$ we have, using Stirling's formula, that the sum

$$\sum_{p=0}^{\infty} \frac{\left(pe^{-\frac{n}{n-1}}\right)^p}{p!}$$

converges, so if we choose α small enough so that

$$\alpha (eC\|\nabla\varphi\|_n)^{\frac{n}{n-1}} \frac{n}{n-1} < 1$$

we have finished. This is possible since by hypothesis we have

$$\|\nabla\varphi\|_n \leq 1.$$

□

Corollary 4.2 *Let D be a bounded domain in $\mathbb{R}^n$. Then there exist constant $\mu, C > 0$ with C depending only on n, such that every $\varphi \in H_0^{1,n}(D)$ satisfies*

$$\int_D e^{\varphi} \leq C \mathrm{Vol}(D) \exp(\mu \|\nabla\varphi\|_n^n). \tag{4.2}$$

Proof
Start with Young's inequality: if u, v are two real numbers and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$uv \leq \frac{|u|^p}{p} + \frac{|v|^q}{q}.$$

Also for every $\varepsilon > 0$ we have

$$uv = (u\varepsilon)(v/\varepsilon) \leq \varepsilon^p \frac{|u|^p}{p} + \varepsilon^{-q} \frac{|v|^q}{q}.$$

Apply this with $u = \varphi/\|\nabla\varphi\|_n$, $v = \|\nabla\varphi\|_n$, $p = \frac{n}{n-1}$, $q = n$, $\varepsilon^p/p = \alpha$ and get

$$\varphi \leq \frac{\alpha|\varphi|^{\frac{n}{n-1}}}{\|\nabla\varphi\|_n^{\frac{n}{n-1}}} + \frac{\varepsilon^{-n}}{n}\|\nabla\varphi\|_n^n.$$

Take this inequality, exponentiate it and integrate it over D. Since $\|\nabla u\|_n = 1$ we can apply (4.1) to the first term and get

$$\int_D e^{\varphi} \leq C\text{Vol}(D)\exp(\mu\|\nabla\varphi\|_n^n).$$

□

The best constants in these inequalities were calculated by J.Moser

Theorem 4.3 (Moser [15]) *The best constant for the inequality* (4.1) *is*

$$\alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}.$$

This means that (4.1) *holds for* $\alpha = \alpha_n$ *and if* $\alpha > \alpha_n$ *the left hand side is finite but can be made arbitrarily large. The best constant for the inequality* (4.2) *is*

$$\mu_n = (n-1)^{n-1}n^{1-2n}\omega_{n-1}^{-1}.$$

Let's examine the case of compact Riemannian manifolds.

Theorem 4.4 (Aubin [4]) *Let* M *be a compact Riemannian manifold of dimension* n*. Then there exist constants*

$$C, \alpha, \mu, \nu > 0$$

such that for all $\varphi \in H_1^n(M)$ *we have*

$$\int_M e^{\varphi}dV \leq C\exp(\mu\|\nabla\varphi\|_n^n + \nu\|\varphi\|_n^n), \tag{4.3}$$

and for all $\varphi \in H_1^n(M)$ *with* $\|\nabla\varphi\|_n \leq 1$ *we have*

$$\int_M e^{\alpha|\varphi|^{n/(n-1)}}dV \leq C. \tag{4.4}$$

Theorem 4.5 (Cherrier [7]) *For a compact Riemannian manifold of dimension* n *the best constants in the inequalities* (4.4) *and* (4.3) *are the same* α_n *and* μ_n *as before.*

Theorem 4.6 (Moser [15]) *Consider* S^2 *with the canonical metric. Every* $\varphi \in H_1^2(S^2)$ *with* $\int_{S^2}\varphi dV = 0$ *satisfies*

$$\int_{S^2} e^{\varphi}dV \leq C\exp(\mu_2\|\nabla\varphi\|_2^2),$$

where $\mu_2 = \frac{1}{16\pi}$.

As a corollary we can easily see that every $\varphi \in H_1^2(S^2)$ satisfies

$$\log \int_{S^2} e^{\varphi} dV \leq \frac{1}{16\pi} \int_{S^2} |\nabla \varphi|^2 dV + \frac{1}{4\pi} \int_{S^2} \varphi dV + C. \tag{4.5}$$

We have the following generalization to higher derivatives: If u is a real function defined in $\mathbb{R}^n$ define

$$D^m u = \begin{cases} \triangle^{m/2} u & \text{if } m \text{ even} \\ \nabla \triangle^{(m-1)/2} u & \text{if } m \text{ odd} \end{cases} \tag{4.6}$$

Theorem 4.7 (Adams [1]) *If m is a positive integer, $m < n$ then there is a constant $C(m,n)$ such that for all $u \in C^m(\mathbb{R}^n)$ supported in D a bounded domain, with $\|D^m u\|_p \leq 1$, $p = n/m$, we have*

$$\int_D e^{\beta |u|^q} \leq C \mathrm{Vol}(D), \tag{4.7}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, for all $\beta \leq \beta_0(n,m)$

$$\beta_0(n,m) = \begin{cases} \dfrac{n}{\omega_{n-1}} \left(\dfrac{\pi^{n/2} 2^m \Gamma(\frac{m+1}{2})}{\Gamma(\frac{n-m+1}{2})} \right)^q & \text{if } m \text{ odd} \\ \dfrac{n}{\omega_{n-1}} \left(\dfrac{\pi^{n/2} 2^m \Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})} \right)^q & \text{if } m \text{ even} \end{cases} \tag{4.8}$$

Moreover if $\beta > \beta_0(n,m)$ then there is no such inequality.

Finally we have the

Theorem 4.8 (Fontana [9]) *Let M be a compact Riemannian manifold of dimension n, and let m be a positive integer, $m < n$. Then there is a constant $C(m,M)$ such that for all $u \in C^m(M)$ with $\|D^m u\|_p \leq 1$, $p = n/m$, and $\int_M u dV = 0$ we have*

$$\int_D e^{\beta |u|^q} \leq C, \tag{4.9}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, for all $\beta \leq \beta_0(n,m)$ give in the previous theorem. Moreover if $\beta > \beta_0(n,m)$ then there is no such inequality.

5 Applications

5.1 The Ricci Flow

The first application we will give of the previous material is due to G.Perelman. He used the logarithmic Sobolev inequality to prove a technical result about the Ricci flow.

Let (M, g) be a compact Riemannian manifold of dimension n, define

$$\mathcal{W}(g, f, \tau) = \int_M [\tau(|\nabla f|^2 + R) + f - n](4\pi\tau)^{-\frac{n}{2}} e^{-f} dV,$$

where $f \in C^\infty(M)$, $\tau \in \mathbb{R}$, $\tau > 0$, that satisfy

$$\int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV = 1. \tag{5.1}$$

We immediately see that for every $\alpha > 0$ we have

$$\mathcal{W}(g, f, \tau) = \mathcal{W}(\alpha g, f, \alpha\tau).$$

Suppose now that g, f, τ depend also smoothly on time $t \in [0, T)$ and satisfy

$$\begin{cases} \dfrac{\partial}{\partial t} g_{ij} = -2R_{ij} \\ \dfrac{\partial}{\partial t} f = -\triangle f + |\nabla f|^2 - R + \dfrac{n}{2\tau} \\ \dfrac{\partial}{\partial t} \tau = -1 \end{cases} \tag{5.2}$$

We say that g moves along the *Ricci flow*. Then we can compute (see [13])

$$\frac{\partial}{\partial t}\mathcal{W} = \int_M 2\tau \left| R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij} \right|^2 (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV \geq 0. \tag{5.3}$$

We now let

$$\mu(g, \tau) = \inf \mathcal{W}(g, f, \tau)$$

where the inf is taken over all f satisfying (5.1), and

$$\nu(g) = \inf \mu(g, \tau)$$

where the inf is taken over all $\tau > 0$. We want to show that there always exists a smooth minimizer $\bar{f}$ of $\mu(g, \tau)$. Set

$$\Phi = e^{-\frac{f}{2}} (4\pi\tau)^{-\frac{n}{4}}$$

so that we can write

$$\mathcal{W}(g, f, \tau) = \int_M \left[4\tau |\nabla \Phi|^2 - \Phi^2 \log \Phi^2 + \Phi^2 \left(\tau R - n - \frac{n}{4} \log 4\pi\tau \right) \right] dV$$

$$\int_M \Phi^2 = 1.$$

Then a theorem of O.S.Rothaus [18] assures us that there is a smooth minimizer $\bar{f}$ for $\mathcal{W}(g, f, \tau)$, and that the corresponding Φ satisfies

$$-4\tau\triangle\Phi - \Phi \log \Phi^2 = \Phi \left(\mu(g, \tau) - \tau R + n + \frac{n}{4} \log 4\pi\tau \right). \tag{5.4}$$

This implies that $\nu(g)$ is nondecreasing along the Ricci flow: consider a time interval $[t_0, t_1]$ and the minimizer $\bar{f}(t_1)$, so that

$$\mu(g(t_1), \tau(t_1)) = \mathcal{W}(g(t_1), \bar{f}(t_1), \tau(t_1)).$$

Solve the backward heat equation for f on $[t_0, t_1]$ to obtain a solution $f(t)$ satisfying $f(t_1) = \bar{f}(t_1)$. Then since $\mathcal{W}$ is nondecreasing we get

$$\mathcal{W}(g(t_0), f(t_0), \tau(t_0)) \leq \mathcal{W}(g(t_1), f(t_1), \tau(t_1)).$$

But if $\bar{f}(t_0)$ is the minimizer of μ at time t_0 we have

$$\mu(g(t_0), \tau(t_0)) = \mathcal{W}(g(t_0), \bar{f}(t_0), \tau(t_0)) \leq \mathcal{W}(g(t_0), f(t_0), \tau(t_0)),$$

so that $\mu(g(t_0), \tau(t_0)) \leq \mu(g(t_1), \tau(t_1))$. It follows that also $\nu(g)$ is nondecreasing along the flow.

Let's compute $\mathcal{W}$ in one explicit example. On $\mathbb{R}^n$ with the canonical metric, constant in time, fix $t_0 > 0$, set $\tau = t_0 - t$ and

$$f(t, x) = \frac{|x|^2}{4\tau},$$

so that $(4\pi\tau)^{-\frac{n}{2}} e^{-f}$ is the fundamental solution of the backward heat equation, that starts at $t = t_0$ as a δ-function at 0. Then it is readily verified that (g_{can}, f, τ) satisfy (5.2). We can compute that

$$\tau(|\nabla f|^2 + R) + f - n = \tau \frac{|x|^2}{4\tau^2} + \frac{|x|^2}{4\tau} - n = \frac{|x|^2}{2\tau} - n.$$

Now we have the well-known Gaussian integral

$$\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4\tau}} dx = (4\pi\tau)^{\frac{n}{2}},$$

and differentiating this with respect to τ we get

$$\int_{\mathbb{R}^n} \frac{|x|^2}{4\tau^2} e^{-\frac{|x|^2}{4\tau}} dx = (4\pi\tau)^{\frac{n}{2}} \frac{n}{2\tau}.$$

Hence

$$\mathcal{W}(g_{can}, f, \tau) = \int_{\mathbb{R}^n} (4\pi\tau)^{-\frac{n}{2}} \left(\frac{|x|^2}{2\tau} - n \right) e^{-\frac{|x|^2}{4\tau}} dx = n - n = 0,$$

for all $t \in [0, t_0)$.

Theorem 5.1 *Start with an arbitrary metric g_{ij}. Then the function $\mu(g,\tau)$ is negative for small $\tau > 0$ and tends to zero as τ tends to zero.*

Proof
Assume $\bar{\tau} > 0$ is small so that the Ricci flow starting from g_{ij} exists on $[0,\bar{\tau}]$. Set $u = (4\pi\tau)^{-\frac{n}{2}} e^{-f}$ and compute its evolution

$$\frac{\partial}{\partial t} u = -\triangle u + Ru.$$

This is the conjugate heat equation in the following sense: if $\Box = \frac{\partial}{\partial t} - \triangle$ is the heat operator, with respect to the metric moving along the Ricci flow, and $\Box^* = -\frac{\partial}{\partial t} - \triangle + R$ then for any two functions $u, v \in C^\infty(M \times [0,T))$ we have

$$\frac{\partial}{\partial t}\int_M uv dV = \int_M (v\Box u - u\Box^* v) dV.$$

This can be easily proved remembering that $\frac{\partial}{\partial t} dV = -R dV$ and $\int_M (u\triangle v - v\triangle u) dV = 0$. Now solve the conjugate heat equation for u starting at $t = \bar{\tau}$ with a δ-function concentrated around some point, with total integral 1. Since the conjugate heat equation for u is now linear and R exists on $[0,\bar{\tau}]$, the solution we get is defined on all $[0,\bar{\tau}]$. Set $\tau(t) = \bar{\tau} - t$ and get an $f(t)$ from the $u(t)$ (this way we've got a global solution for f, which satisfies a nonlinear evolution equation). Then as $t \to \bar{\tau}$ the situation approaches the Euclidean one, for which we computed above that $\mathcal{W} = 0$. So $\mathcal{W}(g(t), f(t), \tau(t))$ tends to zero as $t \to \bar{\tau}$, and we have by monotonicity

$$\mu(g,\tau) \leq \mathcal{W}(g(0), f(0), \tau(0)) \leq \lim_{t\to\bar{\tau}} \mathcal{W}(g(t), f(t), \tau(t)) = 0.$$

To show that $\lim_{\tau\to 0}\mu(g,\tau) = 0$ we won't use the Ricci flow anymore, but we'll employ the Gross logarithmic Sobolev inequality. Assume that there is a sequence $\tau_k \to 0$ such that $\mu(g,\tau_k) \leq c < 0$ for all k and cover M with finitely many charts $U_1, \dots, U_N$ such that each U_j is a geodesic ball $B(p_j, \delta)$, for some $\delta > 0$. Let $g^\tau_{ij} = (2\tau)^{-1} g_{ij}$ and $g_k = g^{\tau_k}$. Then each (U_j, g_k, p_j) converges as $k \to \infty$ to $(\mathbb{R}^n, g_{can}, 0)$ in the C^∞ topology. Then we can easily compute that

$$\mathcal{W}(g,f,\tau) = \int_M \left[2|\nabla\Phi|^2_\tau - \Phi^2\log\Phi^2 + \Phi^2\left(\frac{R_\tau}{2} - n - \frac{n}{2}\log 2\pi\right)\right] dV_\tau$$

$$\Phi = e^{-\frac{f}{2}}(2\pi)^{-\frac{n}{4}}$$

$$\int_M \Phi^2 dV_\tau = 1,$$

where $dV_\tau = (2\tau)^{-\frac{n}{2}}dV$, $|\nabla\Phi|^2_\tau = 2\tau|\nabla\Phi|^2$, $R_\tau = 2\tau R$. Let φ_k be the minimizer realizing $\mu(g,\tau_k)$, which satisfies

$$\begin{cases} -2\triangle_k\varphi_k - 2\varphi_k\log\varphi_k = \left(\mu(g,\tau_k) - \dfrac{R_k}{2} + n + \dfrac{n}{2}\log 2\pi\right)\varphi_k \\ \displaystyle\int_M \varphi_k^2 dV_k = 1 \end{cases} \tag{5.5}$$

Write

$$F_k(\Phi) = 2|\nabla\Phi|^2_{\tau_k} - \Phi^2\log\Phi^2 + \Phi^2\left(\frac{R_{\tau_k}}{2} - n - \frac{n}{2}\log 2\pi\right)$$

so that

$$\frac{\int F_k(\lambda\Phi)dV_k}{\int(\lambda\Phi)^2 dV_k} = \frac{\int F_k(\Phi)dV_k}{\int\Phi^2 dV_k} - \log\lambda^2.$$

Since by hypothesis $\mu(g,\tau_k) \leq c < 0$, we know that

$$\int_M F_k(\varphi_k)dV_k \leq c < 0,$$

so that up to a subsequence

$$\int_{U_1} F_k(\varphi_k)dV_k \leq \frac{c}{N} < 0.$$

Clearly we also have $\int_{U_1}\varphi_k^2 dV_k \leq 1$. Let's fix the attention on U_1. Since g_k converges to g_{can} uniformly on compact sets of $\mathbb{R}^n$, elliptic PDE theory tells us that there is a subsequence of φ_k, still denoted φ_k that converges uniformly on compact sets of $\mathbb{R}^n$ to a limit φ_∞. The functions F_k on the other hand converge to the function

$$F(\Phi) = 2|\nabla\Phi|^2 - \Phi^2\log\Phi^2 - \Phi^2\left(n + \frac{n}{2}\log 2\pi\right),$$

and φ_∞ can't be identically zero because

$$\int_{\mathbb{R}^n} F(\varphi_\infty)dx = \lim_{k\to\infty}\int_{U_1} F_k(\varphi_k)dV_k \leq \frac{c}{N} < 0.$$

Set

$$\varepsilon^2 = \int_{\mathbb{R}^n}\varphi_\infty^2 dx, \tag{5.6}$$

so that

$$\int_{\mathbb{R}^n} F\left(\frac{\varphi_\infty}{\varepsilon}\right)dx \leq \frac{c}{N} + 2\log\varepsilon < \frac{c}{N}. \tag{5.7}$$

Let

$$\left(\frac{\varphi_\infty}{\varepsilon}\right)^2 = (2\pi)^{-\frac{n}{2}}e^{-f_\infty}.$$

Then by (5.6) we get

$$\int_{\mathbb{R}^n} e^{-f_\infty}(2\pi)^{-\frac{n}{2}}dx = 1$$

and by (5.7)

$$\int_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla f_\infty|^2 + f_\infty - n\right)(2\pi)^{-\frac{n}{2}}e^{-f_\infty}dx \le \frac{c}{N} < 0. \tag{5.8}$$

This last inequality is precisely the opposite of the Gross logarithmic Sobolev inequality (3.3). We verify this by setting

$$f_\infty = \frac{|x|^2}{2} - 2\log\phi.$$

Then

$$\nabla f_\infty = x - 2\frac{\nabla\phi}{\phi},$$

$$\frac{|\nabla f_\infty|^2}{2} = \frac{|x|^2}{2} + 2\frac{|\nabla\phi|^2}{\phi^2} - 2\frac{\langle\nabla\phi, x\rangle}{\phi},$$

$$\int_{\mathbb{R}^n} \phi^2(2\pi)^{-\frac{n}{2}}e^{-\frac{|x|^2}{2}}dx = 1.$$

The left hand side of inequality (5.8) becomes

$$\int_{\mathbb{R}^n}\left(\frac{|x|^2}{2} + 2\frac{|\nabla\phi|^2}{\phi^2} - 2\frac{\langle\nabla\phi, x\rangle}{\phi} + \frac{|x|^2}{2} - 2\log\phi - n\right)\phi^2(2\pi)^{-\frac{n}{2}}e^{-\frac{|x|^2}{2}}dx.$$

We can integrate by parts the third term to get

$$\begin{aligned}(2\pi)^{-\frac{n}{2}}\sum_{i=1}^n\int_{\mathbb{R}^n}\phi\frac{\partial\phi}{\partial x^i}x^ie^{-\frac{|x|^2}{2}} &= -(2\pi)^{-\frac{n}{2}}\sum_{i=1}^n\int_{\mathbb{R}^n}\phi\frac{\partial\phi}{\partial x^i}x^ie^{-\frac{|x|^2}{2}}\\ &- (2\pi)^{-\frac{n}{2}}\sum_{i=1}^n\int_{\mathbb{R}^n}\phi^2e^{-\frac{|x|^2}{2}} + (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\phi^2|x|^2e^{-\frac{|x|^2}{2}} =\\ &- (2\pi)^{-\frac{n}{2}}\sum_{i=1}^n\int_{\mathbb{R}^n}\phi\frac{\partial\phi}{\partial x^i}x^ie^{-\frac{|x|^2}{2}} - n + (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\phi^2|x|^2e^{-\frac{|x|^2}{2}},\end{aligned}$$

so

$$(2\pi)^{-\frac{n}{2}}\sum_{i=1}^n\int_{\mathbb{R}^n}\phi\frac{\partial\phi}{\partial x^i}x^ie^{-\frac{|x|^2}{2}} = -\frac{n}{2} + (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\phi^2\frac{|x|^2}{2}e^{-\frac{|x|^2}{2}}.$$

Substituting this into the left hand side of (5.8) we get

$$\int_{\mathbb{R}^n}\left(2|\nabla\phi|^2 - 2\phi^2\log\phi\right)(2\pi)^{-\frac{n}{2}}e^{-\frac{|x|^2}{2}}dx \le \frac{c}{N} < 0,$$

which contradicts (3.3). So we must have that $\lim_{\tau\to 0}\mu(g,\tau)=0$. □

We have the following application of the previous theorem. If $g(t)$, $t\in[0,T)$, is a metric evolving along the Ricci flow, we say that is a *shrinking breather* if there exist an $0<\alpha<1$, two times $t_1<t_2$ and a diffeomorphism $h:M\to M$ such that

$$\alpha g(t_1)=h^*g(t_2).$$

If this holds for every t_1,t_2 we say that $g(t)$ is a *shrinking Ricci soliton.* This is equivalent to the existence of a one-form b and a number $\lambda<0$ such that

$$2R_{ij}(0)+2\lambda g_{ij}(0)+\nabla_i b_j+\nabla_j b_i=0.$$

If $b=\nabla f$ for some smooth function f we say that $g(t)$ is a gradient shrinking Ricci soliton. This means

$$R_{ij}(0)+\lambda g_{ij}(0)+\nabla_i\nabla_j f=0.$$

We want to prove the

Theorem 5.2 (Perelman [17]) *Every shrinking breather is a gradient shrinking Ricci soliton.*

Sketch of proof
Assume that $g(t)$ is a Ricci breather defined on $[0,T]$, so that there are $0<\alpha<1$, $t_1<t_2$ and h as above. Since

$$\mathcal{W}(g(t_2),f,\tau)=\mathcal{W}(\alpha g(t_1),f,\tau)=\mathcal{W}\left(g(t_1),f,\frac{\tau}{\alpha}\right)$$

we get $\nu(g(t_2))=\nu(g(t_1))$. Define $\lambda(g_{ij})$ to be the lowest nonzero eigenvalue of the operator $-4\triangle+R$, and

$$\bar\lambda(g_{ij})=\mathrm{Vol}(g_{ij})^{\frac{2}{n}}\lambda(g_{ij}).$$

Since we are on a shrinking breather we have that $\bar\lambda(g(t_1))=\bar\lambda(g(t_2))$. In [17] it is shown that if $g(t)$ moves along the Ricci flow, then $\bar\lambda(g(t))$ is nondecreasing whenever it is nonpositive, and that monotonicity is strict unless $g(t)$ is a Ricci soliton. Hence we are left with the case when $\bar\lambda(g(t))>0$ for all $t\in[t_1,t_2]$. It is not hard to see using (5.4) that $\bar\lambda(g_{ij})>0$ implies that

$$\lim_{\tau\to\infty}\mu(g,\tau)=+\infty,$$

because when τ is big, $\mu(g,\tau)$ is approximately $\tau\lambda(g_{ij})$. In particular this is true for $g_{ij}=g(t_2)$. Now apply theorem 5.1 to get that $\mu(g(t_2),\tau)<0$ for τ sufficiently small, and

$$\lim_{\tau\to 0}\mu(g(t_2),\tau)=0.$$

These things together imply that there is a $\tilde{\tau} > 0$ that realizes the infimum

$$\nu(g(t_2)) = \mu(g(t_2), \tilde{\tau}) < 0.$$

Now by the theorem of Rothaus, there is a function $\tilde{f}$ that realizes the infimum

$$\nu(g(t_2)) = \mu(g(t_2), \tilde{\tau}) = \mathcal{W}(g(t_2), \tilde{f}, \tilde{\tau}) < 0.$$

Now we flow $\tilde{f}$ by the backward heat flow to get a family $f(t)$, $t \in [t_1, t_2]$ and set $\tau(t) = \tilde{\tau} + t_2 - t$, so that (5.2) are satisfied. By monotonicity we get

$$\nu(g(t_2)) = \mathcal{W}(g(t_2), \tilde{f}, \tilde{\tau}) \geq \mathcal{W}(g(t_1), f(t_1), \tilde{\tau} + t_2 - t_1) \geq \nu(g(t_1)).$$

Since $\nu(g(t_2)) = \nu(g(t_1))$ these inequalities must be equalities, so that $\mathcal{W}$ is constant on $[t_1, t_2]$. But then formula (5.3) tells us that $g(t)$ is a gradient shrinking Ricci soliton on this interval. □

5.2 Kähler Geometry

Now we turn to the Moser-Trudinger inequality. Let us try to generalize (4.5) to higher dimensional varieties. Consider S^2 as the complex manifold $\mathbb{CP}^1$ with its canonical Kähler metric ω. Then ω is Kähler-Einstein, because

$$R_{i\bar{j}} = 2g_{i\bar{j}}.$$

We can generalize the Moser-Trudinger inequality in the following way. If (M, ω) is a compact Kähler manifold of complex dimension n, and

$$P(M, \omega) = \{\phi \in C^\infty(M, \mathbb{R}) \mid \omega_\phi = \omega + \sqrt{-1}\partial\overline{\partial}\phi > 0\}$$

is the space of Kähler potentials, we can define

$$J_\omega(\phi) = \frac{\sqrt{-1}}{V} \sum_{i=0}^{n-1} \frac{i+1}{n+1} \int_M \partial\phi \wedge \overline{\partial}\phi \wedge \omega^i \wedge \omega_\phi^{n-i-1},$$

where $V = \int_M \omega^n$. If $n = 1$ we get

$$J_\omega(\phi) = \frac{\sqrt{-1}}{2V} \int_M \partial\phi \wedge \overline{\partial}\phi = \frac{1}{2V} \int_M |\partial\phi|^2 \omega = \frac{1}{4V} \int_M |\nabla\phi|^2 \omega.$$

Now assume that $c_1(M) > 0$ and pick ω representing the first Chern class. By $\partial\overline{\partial}$-lemma there is a unique smooth real-valued function h_ω such that

$$\begin{cases} \operatorname{Ric}(\omega) = \omega + \sqrt{-1}\partial\overline{\partial}h_\omega \\ \displaystyle\int_M (e^{h_\omega} - 1)\omega^n = 0 \end{cases} \tag{5.9}$$

Define

$$F_\omega(\phi) = J_\omega(\phi) - \frac{1}{V}\int_M \phi\omega^n - \log\left(\frac{1}{V}\int_M e^{h_\omega-\phi}\omega^n\right).$$

It satisfies the following cocycle relation (see [21])

$$F_\omega(\phi) = F_\omega(\psi) + F_{\omega+\sqrt{-1}\partial\overline{\partial}\psi}(\phi-\psi). \tag{5.10}$$

We say that F_ω is bounded below on $P(M,\omega)$ if there is $C>0$ such that $F_\omega(\phi) \geq -C$ for all $\phi \in P(M,\omega)$. Then if M is Kähler-Einstein (i.e. $h_\omega = 0$), the statement that F_ω is bounded below means

$$\log\left(\frac{1}{V}\int_M e^{-\phi}\omega^n\right) \leq J_\omega(\phi) - \frac{1}{V}\int_M \phi\omega^n + C.$$

For S^2 this means that for every $\phi \in P(S^2,\omega)$

$$\log\left(\int_M e^{-\phi}\omega\right) \leq \frac{1}{16\pi}\int_M |\nabla\phi|^2\omega + \frac{1}{4\pi}\int_M(-\phi)\omega + C,$$

which is precisely (4.5) with $\phi = -\varphi$. Notice that this is still weaker than the result of Moser, because we are requiring that $\phi \in P(S^2,\omega)$.

Let (M,ω) be a Kähler-Einstein manifold with $c_1(M)>0$, and let Λ_1 be the space of eigenfunctions of $\triangle$ with eigenvalue 1. Then it is easy to see that there is a bijection between elements of Λ_1 (up to constants) and holomorphic vector fields: if $\triangle_1 u + u = 0$ then $X = g^{i\bar{j}}\frac{\partial u}{\partial \bar{z}^j}\partial_i$ is holomorphic, and if X is holomorphic then $i_X\omega = \overline{\partial}u$ with $\triangle_1 u + u = 0$. (see [19] for the details).

Theorem 5.3 (Bando-Mabuchi [5], Ding-Tian [8]) *If (M,ω) is a Kähler-Einstein manifold with $c_1(M)>0$, so that $\mathrm{Ric}(\omega)=\omega$, then F_ω is bounded below on $P(M,\omega)\cap\Lambda_1^\perp$ where the orthogonal complement is with respect to the L^2 scalar product. In particular if M has no nonzero holomorphic vector fields then F_ω is bounded below on the whole $P(M,\omega)$.*

Proof
Fix any $\phi \in P(M,\omega)$, and set $\omega' = \omega_\phi$. It is easy to prove that the solvability of the following complex Monge-Ampère equation

$$(\omega' + \sqrt{-1}\partial\overline{\partial}\psi)^n = e^{h_{\omega'}-\psi}\omega'^n$$

is equivalent to $\omega' + \sqrt{-1}\partial\overline{\partial}\psi$ being Kähler-Einstein. Let's introduce a time parameter t in the above equation:

$$(\omega' + \sqrt{-1}\partial\overline{\partial}\psi)^n = e^{h_{\omega'}-t\psi}\omega'^n. \tag{$*_t$}$$

Since ω is Kähler-Einstein there is a solution of $(*_1)$, namely $\psi = -\phi$. Suppose that we could get a whole family $\{\psi_t\}$ of solutions of $(*_t)$ for $t \in [0,1]$, that varies smoothly in t. Let's introduce a new functional

$$I_\omega(\phi) = \frac{1}{V}\int_M \phi(\omega^n - \omega_\phi^n) = \frac{\sqrt{-1}}{V}\sum_{i=0}^{n-1}\int_M \partial\phi \wedge \overline{\partial}\phi \wedge \omega^i \wedge \omega_\phi^{n-i-1}.$$

We now calculate the first variation of I_ω and J_ω along a smooth family $\{\phi_t\} \subset P(M,\omega)$. Set $\omega_t = \omega_{\phi_t}$, $\dot{\phi} = \frac{d}{dt}\phi_t$, and compute (see [21])

$$\frac{d}{dt}J_\omega(\phi_t) = \frac{1}{V}\int_M \dot{\phi}(\omega^n - \omega_t^n),$$

$$\frac{d}{dt}I_\omega(\phi_t) = \frac{1}{V}\int_M \dot{\phi}(\omega^n - \omega_t^n) - \frac{1}{V}\int_M \phi_t \triangle_t \dot{\phi}\omega_t^n,$$

where $\triangle_t$ is the laplacian of the metric ω_t. Now pick ψ_t as path, and differentiating $(*_t)$ with respect to t we get

$$n\sqrt{-1}\partial\overline{\partial}\dot{\psi} \wedge (\omega' + \sqrt{-1}\partial\overline{\partial}\psi_t)^{n-1} = (-\psi_t - t\dot{\psi})e^{h_{\omega'} - t\psi_t}\omega'^n = (-\psi_t - t\dot{\psi})\omega_t'^n$$

which means

$$\triangle_t\dot{\psi}\omega_t'^n = (-\psi_t - t\dot{\psi})\omega_t'^n. \tag{5.11}$$

Substituting this we get

$$\begin{aligned}\frac{d}{dt}\left(I_{\omega'}(\psi_t) - J_{\omega'}(\psi_t)\right) &= \frac{1}{V}\int_M \psi_t(\psi_t + t\dot{\psi})\omega_t'^n \\ &= -\frac{d}{dt}\left(\int_M \psi_t e^{h_{\omega'} - t\psi_t}\omega'^n\right) + \frac{1}{V}\int_M \dot{\psi}e^{h_{\omega'} - t\psi_t}\omega'^n.\end{aligned}$$

Since for every t we have

$$\int_M e^{h_{\omega'} - t\psi_t}\omega'^n = V,$$

differentiating this we get

$$\int_M (\psi_t + t\dot{\psi})e^{h_{\omega'} - t\psi_t}\omega'^n = 0,$$

which simplifies the above to

$$\frac{d}{dt}\left(I_{\omega'}(\psi_t) - J_{\omega'}(\psi_t)\right) = -\frac{d}{dt}\left(\int_M \psi_t\omega_t'^n\right) - \frac{1}{tV}\int_M \psi_t e^{h_{\omega'} - t\psi_t}\omega'^n.$$

Multiplying this by t we get

$$\frac{d}{de}\left(t(I_{\omega'}(\psi_t) - J_{\omega'}(\psi_t))\right) - \left(I_{\omega'}(\psi_t) - J_{\omega'}(\psi_t)\right) = -\frac{d}{dt}\left(\frac{t}{V}\int_M \psi_t\omega_t'^n\right).$$

Integrating this from 0 to t we get

$$t(I_{\omega'}(\psi_t) - J_{\omega'}(\psi_t)) - \int_0^t (I_{\omega'}(\psi_s) - J_{\omega'}(\psi_s))ds = -\frac{t}{V}\int_M \psi_t \omega_t'^n,$$

which is equivalent to

$$\int_0^t (I_{\omega'}(\psi_s) - J_{\omega'}(\psi_s))ds = t\left(-J_{\omega'}(\psi_t) + \frac{1}{V}\int_M \psi_t \omega'^n\right). \tag{5.12}$$

Now from the cocycle relation (5.10) we get

$$\begin{aligned} F_\omega(\phi) &= -F_{\omega'}(-\phi) = -F_{\omega'}(\psi_1) \\ &= -J_{\omega'}(\psi_1) + \frac{1}{V}\int_M \psi_1 \omega'^n + \log\left(\frac{1}{V}\int_M e^{h_{\omega'} - \psi_1}\omega'^n\right). \end{aligned} \tag{5.13}$$

Integrating $(*_1)$ over M we see that the last term is zero. Using (5.12) we get

$$F_\omega(\phi) = -J_{\omega'}(\psi_1) + \frac{1}{V}\int_M \psi_1 \omega'^n = \int_0^1 (I_{\omega'}(\psi_s) - J_{\omega'}(\psi_s))ds.$$

But the integrand is

$$I_{\omega'}(\psi_s) - J_{\omega'}(\psi_s) = \frac{\sqrt{-1}}{V}\sum_{i=0}^{n-1}\frac{n-i}{n+1}\int_M \partial\psi_s \wedge \overline{\partial}\psi_s \wedge \omega'^i \wedge \omega_s'^{n-i-1}$$

and each of the terms of the sum is nonnegative. Hence we have proved that

$$F_\omega(\phi) \geq 0.$$

Getting the family of solutions ψ_t is rather technical. We will assume that M has no nonzero holomorphic vector fields (so that $\Lambda_1 = 0$) and just give an idea of the general case. The family ψ_t is constructed using the continuity method. Define $E = \{t \in [0,1] \mid (*_s) \text{ is solvable for all } s \in [t,1]\}$. Then E is nonempty because $1 \in E$. If we can prove that E is open and closed in $[0,1]$, we'd have finished. To prove that E is open we have to prove that if $s \in E$ then we can solve $(*_t)$ for t close to s. Let ψ_s be a solution of $(*_s)$, so that

$$\omega_s'^n = e^{h_{\omega'} - s\psi_s}\omega'^n.$$

Then setting $\rho = \psi_t - \psi_s$ we can rewrite $(*_t)$ as

$$(\omega_s' + \sqrt{-1}\partial\overline{\partial}(\psi_t - \psi_s))^n = e^{h_{\omega'} - t\psi_t}\omega'^n = e^{h_{\omega'} - s\psi_s}e^{-s(\psi_t - \psi_s)}e^{-(t-s)\psi_t}\omega'^n,$$

$$(\omega_s' + \sqrt{-1}\partial\overline{\partial}\rho)^n = e^{-s\rho}e^{-(t-s)(\rho+\psi_s)}\omega_s'^n,$$

$$\log\frac{(\omega_s' + \sqrt{-1}\partial\overline{\partial}\rho)^n}{\omega_s'^n} + s\rho = -(t-s)(\rho + \psi_s).$$

So define operators

$$\Phi_s : C^{2,\frac{1}{2}}(M) \to C^{0,\frac{1}{2}}(M)$$

by

$$\Phi_s(\rho) = \log \frac{(\omega'_s + \sqrt{-1}\partial\overline{\partial}\rho)^n}{\omega'^n_s} + s\rho.$$

We want to solve the equation

$$\Phi_s(\rho) = -(t-s)(\rho + \psi_s)$$

for $|t-s|$ small. Notice that $\Phi_s(0) = 0$, so that by the implicit function theorem it is enough to prove that the differential of Φ_s at 0 is invertible (this gives us also that the family ψ_t is smooth in t). But this differential is

$$D\Phi_s(v) = \frac{\partial}{\partial t}\bigg|_{t=0} \Phi_s(tv) = \triangle_s v + sv,$$

so that we need to show that $\lambda_1(s)$, the first nonzero eigenvalue of $\triangle_s$, satisfies $\lambda_1(s) > s$. Compute

$$R'_{i\overline{j}}(s) = -\partial_i\partial_{\overline{j}} \log \omega'^n_s = -\partial_i\partial_{\overline{j}} \log \frac{\omega'^n_s}{\omega'^n} + R'_{i\overline{j}} = -\partial_i\partial_{\overline{j}}(h_{\omega'} - s\psi_s) + g'_{i\overline{j}} + \partial_i\partial_{\overline{j}} h_{\omega'}$$
$$= g'_{i\overline{j}} + s\partial_i\partial_{\overline{j}}\psi_s = g'_{i\overline{j}} + s(g'_{i\overline{j}}(s) - g'_{i\overline{j}}) = (1-s)g'_{i\overline{j}} + sg'_{i\overline{j}}(s) \geq sg'_{i\overline{j}}(s),$$

so by standard Bochner technique ([21]) we get $\lambda_1(s) \geq s$, and that the inequality is strict if $s < 1$. If $s = 1$ then recall that $\omega'_1 = \omega$ is Kähler-Einstein, so that $\mathrm{Ric}(\omega) = \omega$. Since we assume that there are no nonzero holomorphic vector fields, we have that $\lambda_1(1) > 1$, so that Φ_s is locally invertible around 0. Now standard elliptic regularity theory (Schauder estimates) tells us that the solution ρ we have found is in fact smooth, so E is open. To show that E is closed it is enough to establish an *a priori* bound $\|\psi\|_{C^3} \leq C$ for a solution of $(*_t)$. In fact if we have such a bound we can show that E is compact (hence closed): if $t_i \to \tau \in [0,1]$ and ψ_i is a sequence of solutions of $(*_{t_i})$ then $\|\psi_i\|_{C^3} \leq C$ implies that $\|\psi_i\|_{C^{2,\frac{3}{4}}} \leq C$ and by Ascoli-Arzelà's theorem we have a compact embedding $C^{2,\frac{3}{4}}(M) \subset C^{2,\frac{1}{2}}(M)$. So a subsequence of the ψ_i converges in $C^{2,\frac{1}{2}}(M)$ to a solution of $(*_\tau)$, which is smooth by Schauder estimates. Thanks to Yau's estimates [23], we can get a uniform bound $\|\psi\|_{C^3} \leq C$ if we have a uniform bound $\|\psi\|_\infty \leq C$.

Assume that ψ_t solves $(*_t)$, and let $G(x,y)$ be the Green function of (M, ω'), which has the following properties:

$$\begin{cases} \psi(x) = \dfrac{1}{V}\displaystyle\int_M \psi(y)\omega'^n(y) - \int_M \triangle\psi(y)G(x,y)\omega'^n(y) \\ \displaystyle\int_M G(x,y)\omega'^n(x) = 0 \ \forall y \in M \\ G(x,y) \geq -\gamma\dfrac{D^2}{V} = -A \end{cases} \tag{5.14}$$

if Ric $\geq K > 0$, $D = \mathrm{diam}_{\omega'}(M)$, and $\gamma = \gamma(n, KD^2) > 0$ is a constant. For a proof of the existence of G see [3],[19]. Since $\psi_t \in P(M, \omega')$ we get $n + \triangle\psi_t > 0$ so that

$$\begin{cases} \psi_t(x) = \frac{1}{V}\int_M \psi_t \omega'^n + \int_M (-\triangle\psi_t)(G+A)\omega'^n \leq \frac{1}{V}\int_M \psi_t\omega'^n + nA \\ \sup_M \psi_t \leq \int_M \psi_t\omega'^n + C \end{cases}$$

where C is a uniform constant. We also have $R'_{i\bar{j}}(t) = (1-t)g'_{i\bar{j}} + tg'_{i\bar{j}}(t) \geq tg'_{i\bar{j}}(t)$, and since $\omega' = \omega'_t - \sqrt{-1}\partial\overline{\partial}\psi_t > 0$ we have $n - \triangle_t\psi_t > 0$ so that the Green formula for (M, ω'_t) gives us

$$\begin{cases} \psi_t(x) = \frac{1}{V}\int_M \psi_t \omega_t'^n + \int_M (-\triangle_t\psi_t)(G_t+A')\omega_t'^n \geq \frac{1}{V}\int_M \psi_t\omega_t'^n - nA' \\ \sup_M(-\psi_t) \leq -\frac{1}{V}\int_M \psi_t\omega_t'^n + nA' \end{cases}$$

but now A' is NOT uniform anymore. In fact by Bonnet-Myers theorem $\mathrm{diam}_{\omega'_t}(M)$ is bounded above by a constant times $\frac{1}{\sqrt{t}}$, so that A' is bounded above by $\frac{C}{t}$. It follows that for $t \geq t_0 > 0$ we have a uniform bound

$$\sup_M \psi_t - \inf_M \psi_t \leq C + \frac{1}{V}\int_M \psi_t(\omega'^n - \omega_t'^n) = C + I_{\omega'}(\psi_t).$$

From the definitions of $I_{\omega'}$ and $J_{\omega'}$ it is immediate to get

$$\frac{n+1}{n}J_{\omega'} \leq I_{\omega'} \leq (n+1)J_{\omega'},$$

$$\frac{1}{n+1}I_{\omega'} \leq I_{\omega'} - J_{\omega'} \leq \frac{n}{n+1}I_{\omega'},$$

so the oscillation of ψ_t is controlled by $I_{\omega'} - J_{\omega'}$. But now we show that this is increasing in t so that it is uniformly bounded above by its value at time $t = 1$. Going back to (5.11) we get

$$\frac{d}{dt}\left(I_{\omega'}(\psi_t) - J_{\omega'}(\psi_t)\right) = \frac{1}{V}\int_M (\triangle_t\dot{\psi} + t\dot{\psi})\triangle_t\dot{\psi}\omega_t'^n. \tag{5.15}$$

Recall that $\lambda_1(t)$, the first nonzero eigenvalue of $\triangle_t$, satisfies $\lambda_1(t) \geq t$. Now let $f_i(t)$ be an L^2-orthonormal basis of eigenfunctions of $\triangle_t$ where $f_0(t) = 1$ for all t,

$$\triangle_t f_i(t) + \lambda_i(t)f_i(t) = 0.$$

Express $\dot{\psi} = \sum_{i=0}^{\infty} c_i(t)f_i(t)$, with $c_i(t) \in \mathbb{R}$ and compute

$$\frac{d}{dt}\left(I_{\omega'}(\psi_t) - J_{\omega'}(\psi_t)\right) = \sum_{i=1}^{\infty} c_i(t)^2(\lambda_i(t) - t)\lambda_i(t) \geq 0,$$

because $\lambda_i(t) \geq \lambda_1(t) \geq t$. So we have a bound on the oscillation of ψ_t if t is away from zero. In fact this gives us a bound on $\|\psi_t\|_\infty$ simply because we have, integrating $(*_t)$

$$\int_M e^{h_{\omega'} - t\psi_t} \omega'^n = V,$$

but also

$$\int_M e^{h_{\omega'}} \omega'^n = V.$$

Supposing that ψ_t is never 0 we get a contradiction between these two last equations. Hence ψ_t attains the value 0 somewhere, so

$$\|\psi_t\|_\infty \leq \sup_M \psi_t - \inf_M \psi_t.$$

Finally we deal with the case $t = 0$. Since $\|\psi_t\|_\infty \leq \frac{C}{t}$ for some uniform $C > 0$, we get

$$\|t\psi_t\|_\infty \leq C,$$

so using $(*_t)$ we get a uniform bound

$$\|\omega' + \sqrt{-1}\partial\bar{\partial}\psi_t\|_\infty \leq C,$$

and by Yau's estimates on the Calabi Conjecture [23],[19], we have a uniform bound

$$\|\psi_t\|_\infty \leq C.$$

Hence E is closed.

In the general case when M has nontrivial holomorphic vector fields, Bando and Mabuchi can still construct the family of solutions ψ_t, if the starting ϕ belongs to $P(M,\omega) \cap \Lambda_1^\perp$ (see [5], [19]). For such ϕ we then get that $F_\omega(\phi) \geq 0$ exactly as above. $\square$

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