A FINITENESS THEOREM FOR POLARIZED MANIFOLDS

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There are many previous finiteness theorems about diffeomorphism types in Riemannian geometry. Cheeger's finiteness theorem asserts that given constants D, v, and Λ, there are only finitely many n-dimensional compact differential manifold X admitting Riemannian metric g such that $\mathrm{diam}_g(X) \leqslant D$, $\mathrm{Vol}_g(X) \geqslant v$ and the sectional curvature $|\mathrm{Sec}(g)| \leqslant \Lambda$. This theorem can be proved as a corollary of the Cheeger-Gromov convergence theorem (cf. [5, 11]), which shows that if (X_k, g_k) is a family compact Riemannian manifolds with the above bounds, then a subsequence of (X_k, g_k) converges to a $C^{1,\alpha}$-Riemannian manifold Y in the $C^{1,\alpha}$-sense, and furthermore, X_k is diffoemorphic to Y for $k \gg 1$. In [1], Cheeger's finiteness theorem is generalized to the case where the hypothesis on the sectional curvature bound is replaced by the weaker bounds of Ricci curvature $|\mathrm{Ric}(g)| \leqslant \lambda$ and the $L^{\frac{n}{2}}$-norm of curvature $\|\mathrm{Sec}(g)\|_{L^{\frac{n}{2}}} \leqslant \Lambda$. Furthermore, if $n = 4$ and g is an Einstein metric, then the integral bound of curvature can be replaced by a bound for the Euler characteristic.

We call (X, L) a polarized n-manifold, if X is a compact complex manifold with an ample line bundle L. In [6], a finiteness theorem for polarized manifolds is obtained. More precisely, Theorem 3 of [6] asserts that for any two constants $V > 0$ and $\Lambda > 0$, there are finite many polynomials $P_1, \cdots, P_\ell$ such that if (X, L) is a polarized n-manifold with $c_1(L)^n \leqslant V$ and $-c_1(X) \cdot c_1(L)^{n-1} \leqslant \Lambda$, then one P_i is the Hilbert polynomial of (X, L), i.e. $P_i(\nu) = \chi(X, L^\nu)$. Consequently, polarized n-manifolds with the above bounds have only finitely many possible deformation types and finitely many possible diffeomorphism types.

For any constants $\lambda > 0$ and $D > 0$, denote

$$\mathfrak{N}(n, \lambda, D) = \{(X, L) \mid \exists \omega \in c_1(L) \text{ with } \mathrm{Ric}(\omega) \geqslant -\lambda\omega, \quad \mathrm{diam}_\omega(X) \leqslant D\}.$$

Then

$$c_1(L)^n = n!\mathrm{Vol}_\omega(X) \leqslant V = V(n, \lambda, D)$$

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by the Gromov-Bishop comparison theorem, and

$$-c_1(X)\cdot c_1(L)^{n-1} = -\int_X \mathrm{Ric}(\omega)\wedge\omega^{n-1} \leqslant n\lambda V.$$

The following proposition is a corollary of Theorem 3 in [6]. Here we give an analytic proof.

Proposition 0.1. *Polarized manifolds in $\mathfrak{N}(n,\lambda,D)$ have only finitely many possible Hilbert polynomials, and for any $(X,L)\in\mathfrak{N}(n,\lambda,D)$ we have*

$$|\chi(X,L^\nu)| \leqslant C(n,\lambda,D)\nu^n, \tag{0.1}$$

for all $\nu \geqslant 1$, where $C(n,\lambda,D)$ is a constant depending only on n,λ and D. Furthermore, any $(X,L)\in\mathfrak{N}(n,\lambda,D)$ can be embedded in the same $\mathbb{CP}^N$ with $L^m \cong \mathcal{O}_{\mathbb{CP}^N}(1)|_X$ for integers $m = m(n,\lambda,D) > 0$ and $N = N(n,\lambda,D) > 0$. As a consequence, manifolds in $\mathfrak{N}(n,\lambda,D)$ have only finitely many possible deformation types and finitely many possible diffeomorphism types.

Proof. Let $(X,L)\in\mathfrak{N}(n,\lambda,D)$, and $\omega\in c_1(L)$ be a Kähler metric with $\mathrm{Ric}(\omega)\geqslant-\lambda\omega$, and $\mathrm{diam}_\omega(X)\leqslant D$. Fix a Hermitian metric h on L with curvature equal to ω. The Gromov-Bishop comparison theorem gives

$$1\leqslant c_1(L)^n = n!\mathrm{Vol}_\omega(X)\leqslant V = V(n,\lambda,D).$$

We would like to estimate $h^{0,p}(L^\nu)=\dim H^{0,p}(X,L^\nu)$, $0\leqslant p\leqslant n$, for $\nu\geqslant 1$. We denote by $\langle\cdot,\cdot\rangle$ the pointwise inner product on $\Omega^{0,p}(X,L^\nu)$ (smooth L^ν-valued $(0,p)$-forms on X) induced by the metric h^ν on L^ν whose curvature is $-\sqrt{-1}\nu\omega$, and by $|\cdot|$ its corresponding norm. For any $s\in\Omega^{0,p}(X,L^\nu)$ we have

$$\Delta|s|^2 = g^{i\bar{j}}\partial_i\partial_{\bar{j}}|s|^2 = |\nabla s|^2 + |\overline{\nabla}s|^2 + \langle\Delta s, s\rangle + \langle s, \overline{\Delta}s\rangle,$$

where $\Delta s = g^{i\bar{j}}\nabla_i\nabla_{\bar{j}}s$ is the rough Laplacian and $\overline{\Delta}s = g^{i\bar{j}}\nabla_{\bar{j}}\nabla_i s$ its "conjugate". Commuting covariant derivatives we get

$$\overline{\Delta}s = \Delta s - \nu n s - \mathrm{Ric}^\sharp(s),$$

where if $p\geqslant 1$ and we write locally $s = s_{\overline{i_1}\dots\overline{i_p}}d\overline{z}^{i_1}\wedge\cdots\wedge d\overline{z}^{i_p}$ with $s_{\overline{i_1}\dots\overline{i_p}}$ local smooth sections of L^ν, then

$$\mathrm{Ric}^\sharp(s) = \sum_{j=1}^p g^{k\bar{\ell}}R_{k\overline{i_j}}\, s_{\overline{i_1}\dots\bar{\ell}\dots\overline{i_p}}d\overline{z}^{i_1}\wedge\cdots\wedge d\overline{z}^{i_p},$$

while if $p=0$ we let $\mathrm{Ric}^\sharp(s)=0$. This gives

$$\Delta|s|^2 = |\nabla s|^2 + |\overline{\nabla}s|^2 + 2\mathrm{Re}\langle\Delta s, s\rangle - \nu n|s|^2 - \langle s, \mathrm{Ric}^\sharp(s)\rangle.$$

Next, we apply the Bochner-Kodaira identity [9, Theorem 6.2], which for any $s\in\Omega^{0,p}(X,L^\nu)$ gives

$$\Delta_{\bar{\partial}}s = -\Delta s + \nu s + \mathrm{Ric}^\sharp(s),$$

and so if we assume that $\Delta_{\overline{\partial}} s = 0$, we obtain

$$\begin{aligned}\Delta|s|^2 &= |\nabla s|^2 + |\overline{\nabla} s|^2 + 2\langle \mathrm{Ric}^\sharp(s), s\rangle + 2\nu|s|^2 - \nu n|s|^2 - \langle s, \mathrm{Ric}^\sharp(s)\rangle \\ &= |\nabla s|^2 + |\overline{\nabla} s|^2 + \langle \mathrm{Ric}^\sharp(s), s\rangle - \nu(n-2)|s|^2,\end{aligned}$$

noting that $\langle \mathrm{Ric}^\sharp(s), s\rangle = \langle s, \mathrm{Ric}^\sharp(s)\rangle$. Using that

$$\langle \mathrm{Ric}^\sharp(s), s\rangle \geqslant -\lambda p|s|^2,$$

we finally obtain

$$\Delta|s|^2 \geqslant -(\nu(n-2) + \lambda p)|s|^2.$$

A standard Moser iteration argument (see e.g. [2, Lemma 2.4]) applied to this differential inequality gives

$$\sup_X |s|^2 \leqslant A(\nu(n-2)+\lambda p)^n \int_X |s|^2 \frac{\omega^n}{n!} = A(\nu(n-2)+\lambda p)^n \|s\|^2_{L^2}, \tag{0.2}$$

where A depends only on the Sobolev constant of ω and on n. Thus $A = A(n, V, \lambda, D)$ by a result of Croke [3].

Now we use the arguments in Lemma 11 and Theorem 12 of the paper of Li [7]. By the Hodge Theorem, we have an isomorphism $H^{0,p}(X, L^\nu) \cong \mathcal{H}^{0,p}(X, L^\nu)$, the space of $\Delta_{\overline{\partial}}$-harmonic forms in $\Omega^{0,p}(X, L^\nu)$. Let

$$\rho = \sum |s_i|^2$$

for an orthonormal basis s_i of $\mathcal{H}^{0,p}(X, L^\nu)$. The function ρ is easily seen to be independent of the choice of orthonormal basis. Let $x \in X$ such that

$$\rho(x) = \sup_X \rho > 0.$$

Then

$$E_0 = \{s \in \mathcal{H}^{0,p}(X, L^\nu) | s(x) = 0\},$$

is a proper linear subspace of $\mathcal{H}^{0,p}(X, L^\nu)$, with orthogonal complement $E_0^\perp$. We claim that $\dim E_0^\perp \leqslant \binom{n}{p}$. If $s_1, \cdots, s_r$, $r > \binom{n}{p}$, is an orthonormal basis of $E_0^\perp$, then there are a_i, $i = 1, \cdots, r$, such that $\sum a_i s_i(x) = 0$. Thus $\sum a_i s_i \in E_0$, which is a contradiction.

Let $s_1, \cdots, s_r \in \mathcal{H}^{0,p}(X, L^\nu)$ be an orthonormal basis of $E_0^\perp$, which we can complete to an orthonormal basis of $\mathcal{H}^{0,p}(X, L^\nu)$ with an orthonormal basis $s_{r+1}, \cdots, s_N$ of E_0. We have

$$\begin{aligned}h^{0,p}(L^\nu) = \int_X \rho \frac{\omega^n}{n!} \leqslant V \sup_X \rho = V \sup_X \left(\sum_{i=1}^r |s_i|^2\right) & \\ \leqslant \binom{n}{p} V \sup_i \|s_i\|^2_{L^\infty} & \\ \leqslant \binom{n}{p} V A(\nu(n-2)+\lambda p)^n, &\end{aligned}$$

using (0.2), and thus for any $\nu \geqslant 1$ we have

$$|\chi(X, L^\nu)| = \left|\sum_p (-1)^p h^{0,p}(L^\nu)\right| \leqslant \sum_p \binom{n}{p} VA(\nu(n-2)+\lambda p)^n \leqslant C(n,\lambda,D)\nu^n,$$

thus proving (0.1). Since the Hilbert polynomial P of (X, L) is given by

$$P(\nu) = \chi(X, L^\nu) = \int_X e^{\nu c_1(L)}\mathrm{Todd}_X = a_0\nu^n + a_1\nu^{n-1} + \cdots + a_n \in \mathbb{Z},$$

it follows that we have only finitely many possible values for $a_0, \cdots, a_n$ by taking sufficiently many values of ν and solving the linear equations. Hence (X, L) has only finitely many possible Hilbert polynomials.

Now, by Matsusaka's Big Theorem (cf. [8]), there is an $m_0 > 0$ depending only on P such that for any $m \geqslant m_0$, L^m is very ample, and $H^i(X, L^m) = \{0\}$, $i > 0$. By choosing a basis Σ of $H^0(X, L^m)$, we have an embedding $\Phi_\Sigma : X \hookrightarrow \mathbb{CP}^N$ such that $L^m = \Phi_\Sigma^* \mathcal{O}_{\mathbb{CP}^N}(1)$. We regard $\Phi_\Sigma(X)$ as a point in the Hilbert scheme $\mathcal{H}ilb_N^{P_m}$ parametrizing the subshemes of $\mathbb{CP}^N$ with Hilbert polynomial $P_m(\nu) = P(m\nu)$, where $N = h^0(X, L^m) - 1$. Finally, $\Phi_\Sigma(X)$ belongs to finitely many possible components of finitely many possible Hilbert schemes, and thus $\mathfrak{N}(n, \lambda, D)$ has only finitely many possible deformation and diffeomorphism types. $\square$

Note that for any polarized manifold (X, L), the volume of (X, ω) as in the definition of $\mathfrak{N}(n, \lambda, D)$ is bounded below uniformly away from zero. We remark that a similar diffeomorphism finiteness result fails for the family of closed Riemannian manifolds (M, g) of real dimension m with $\mathrm{Ric}(g) \geqslant -\lambda g, \mathrm{diam}_g(X) \leqslant D, \mathrm{vol}_g(X) \geqslant v > 0$. Indeed Perelman [10] constructed Riemannian metrics on $\sharp_k \mathbb{CP}^2$ for all $k \geqslant 1$, which have positive Ricci curvature, unit diameter and volume bounded uniformly away from zero.

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