APPENDIX

EXTENSION OF SEMI-FLAT FORMS ACROSS SINGULAR FIBERS

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The purpose of this appendix is to show that the natural semi-flat forms that one constructs on elliptic $K3$ surfaces with only (reduced and) irreducible singular fibers, which are defined only on the complement of the singular fibers, extend as closed positive currents to the whole total space. We also show that there are such elliptic $K3$ surfaces which are non-isotrivial, and also admit another elliptic fibration. Combining these two results with the argument in [1, §3] (which rules out the existence of closed positive currents in our class on such $K3$ surfaces) shows that on such $K3$ surfaces the semi-flat form (away from the singular fibers) cannot be semipositive definite, see [1, Theorem 3.1].

To start, we put ourselves in a slightly more general setting, as follows. Let (X^n, ω_X) be a compact Kähler manifold, Y a compact Riemann surface, and $f : X \to Y$ a surjective holomorphic map with connected fibers. Let Y^0 be the locus of regular values for f, whose complement in Y is a finite set, and $X^0 = f^{-1}(Y^0)$, which is Zariski open in X, so that $f : X^0 \to Y^0$ is a proper holomorphic submersion. We will call the fibers over points in $Y \backslash Y^0$ the singular fibers of f.

Suppose that for every $y \in Y^0$ we have a smooth function ρ_y on the fiber $X_y = f^{-1}(y)$ which satisfies

$$\omega_X|_{X_y} + i\partial\overline{\partial}\rho_y \geq 0, \quad \int_{X_y} \rho_y (\omega_X|_{X_y})^n = 0. \tag{1}$$

Proposition 0.1. *If all the singular fibers of f are reduced and irreducible, then there is a constant C such that*

$$\sup_{X_y} \rho_y \leq C,$$

holds for all $y \in Y^0$.

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Proof. Let $\omega_y = \omega_X|_{X_y}$, and g_y be its Riemannian metric, where in the following we fix any $y \in Y^0$. Thanks to (1), on X_y we have

$$\Delta_{g_y} \rho_y \geq -n+1. \tag{2}$$

We have that $\mathrm{Vol}(X_y, g_y) = c$, a constant independent of y, and that the Sobolev constant of (X_y, g_y) has a uniform upper bound independent of y thanks to the Michael-Simon Sobolev inequality [5], see the details e.g. in [8, Lemma 3.2]. Furthermore, $\mathrm{diam}(X_y, g_y) \leq C$, a constant independent of y, thanks to [8, Lemma 3.3].

So far we have not used the assumptions that all singular fibers are reduced and irreducible. This is used now to prove that the Poincaré constant of (X_y, g_y) also has a uniform upper bound independent of y, as shown by Yoshikawa [10] (see also the shorter exposition in [6, Proposition 3.2]).

At this point we can use a classical argument of Cheng-Li [2], which is clearly explained in [7, Chapter 3, Appendix A, pp.137-140], to deduce that the Green's function $G_y(x, x')$ of (X_y, g_y), normalized by

$$\int_{X_y} G_y(x, x')\omega_y(x') = 0,$$

satisfies the bound

$$G_y(x, x') \geq -A, \tag{3}$$

for all $y \in Y^0$ and for all $x, x' \in X_y$, with a uniform constant A. The point of that argument is that A only depends on the constant in the Sobolev-Poincaré inequality, that here as we said we control uniformly, on the dimension and on bounds for the volume and diameter, which we all have.

We can now apply Green's formula on X_y. Choose a point $x \in X_y$ such that $\rho_y(x) = \sup_{X_y} \rho_y$, and then, using that ρ_y has average zero, together with (2) and (3), we obtain

$$\begin{aligned}
\rho_y(x) &= -\int_{X_y} \Delta_{g_y} \rho_y(x') G_y(x, x')\omega_y(x') \\
&= -\int_{X_y} \Delta_{g_y} \rho_y(x')(G_y(x, x') + A)\omega_y(x') \\
&\leq (n-1)\int_{X_y} (G_y(x, x') + A)\omega_y(x') \\
&\leq (n-1)A\mathrm{Vol}(X_y, g_y).
\end{aligned}$$

□

We now specialize to the setting where X is a $K3$ surface, $Y = \mathbb{P}^1$ and $f : X \to \mathbb{P}^1$ is an elliptic fibration. We further assume that ρ_y is chosen so that $\omega_X|_{X_y} + i\partial\overline{\partial}\rho_y > 0$ is the unique flat metric on X_y cohomologous to $\omega_X|_{X_y}$ (and we still assume that ρ_y has fiberwise average zero). In this case

ρ_y varies smoothly in $y \in Y^0$, and so it defines a smooth function ρ on X^0. Thanks to Proposition 0.1, we conclude that

$$\sup_{X^0} \rho \leq C.$$

This, together with the Grauert-Remmert extension theorem [3], immediately gives:

Corollary 0.2. *In this setting, if we have that $\omega_X + i\partial\overline{\partial}\rho \geq 0$ on X^0, then this extends to a closed positive current on all of X, in the class $[\omega_X]$.*

This proves the desired extension property. Lastly, as we mentioned at the beginning, to apply this result in [1, §3] we need the following examples:

Proposition 0.3. *There exists a complex projective $K3$ surface X which admits two elliptic fibrations, one of which is non-isotrivial and has only reduced and irreducible singular fibers.*

Proof. Let $X \subset \mathbb{P}^2 \times \mathbb{P}^1$ be a general hypersurface of degree $(3,2)$. It is known that X has Picard number 2 [9, Section 5.8]. The projection to the $\mathbb{P}^1$ factor gives an elliptic fibration on X, which is clearly not isotrivial provided X is general.

To obtain the other fibration we compose the first fibration with the automorphism σ of X obtained as follows. Projecting X to the $\mathbb{P}^2$ factor shows that X is a double cover of $\mathbb{P}^2$ ramified along a sextic, and the covering involution of this cover is the σ that we want.

Explicitly, if we let $L = \mathcal{O}_{\mathbb{P}^2}(1)|_X$, $M = \mathcal{O}_{\mathbb{P}^1}(1)|_X$, the the first elliptic fibration is defined by $|M|$ and the second elliptic fibration by $|3L - M|$ (since $\sigma^* M = 3L - M$).

Lastly, we show that every elliptic fibration on X has only reduced and irreducible singular fibers. Given an elliptic fibration $f : X \to \mathbb{P}^1$, let $j : J \to \mathbb{P}^1$ be its Jacobian family [4, Section 11.4]. Then J is also an elliptic $K3$ surface, every fiber of j is isomorphic to the corresponding fiber of f, J has the same Picard number as X, but j always has a section. We can then apply the Shioda-Tate formula [4, Corollary 11.3.4] to j to obtain

$$2 = \rho(J) = 2 + \sum_{t \in \mathbb{P}^1} (r_t - 1) + \text{rank MW}(j),$$

where r_t is the number of irreducible components of the fiber J_t and $\text{MW}(j)$ is the Mordell-Weil group of j. In particular we conclude that $r_t = 1$ for all t, i.e. all fibers of j (and therefore all fibers of f) are irreducible. Lastly, all fibers of f are reduced by [4, Proposition 3.1.6 (iii)]. □

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