Amer. Math. Soc. Transl.
(2) Vol. 196, 1999

Symmetric Spaces, Kähler Geometry and Hamiltonian Dynamics

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Preface

This article was written in the Spring of 1997. Some while after completing it I learnt of previous work by other authors which overlaps substantially. The "symmetric space" structure which is the main topic of the article was discovered by S. Semmes:

Complex Monge–Ampère and symplectic manifolds, Amer. J. Math. **114** (1992), 495–550.

I am grateful to J.-P. Bourguignon for informing me about this reference. Bourguignon tells me that he and Mabuchi were developing similar ideas at about the same time.

The "moment map" point of view which is mentioned in the last part of the paper was discovered by A. Fujiki:

The moduli spaces and Kähler metrics of polarised algebraic varieties, Sugaku **42** (1990), 231–243; English transl., Sugaku Expositions **5** (1992) 173–191.

I hope that it may still be worth making the present article available in its original form, although it would of course have been written rather differently if I had been aware of these earlier papers.

§1. Introduction

A Riemannian manifold is called a locally symmetric space if its curvature tensor is covariant constant; $\nabla R = 0$. Such symmetric spaces were classified, up to coverings, by E. Cartan, and the theory is tightly bound up with the classification of semi-simple Lie groups [11]. One of the memorable features of the theory is that the (irreducible) symmetric spaces occur in pairs of "compact" and "non-compact" type. The compact type have positive sectional curvature and the non-compact type have negative sectional curvature. In particular, let G be a compact Lie group with a bi-invariant Riemannian metric. This is an example of a symmetric space of compact type; the curvature tensor is given by the formula

$$R(X,Y)Z = \tfrac{1}{4}[[X,Y],Z], \tag{1}$$

and the sectional curvature by

$$K(X,Y) = \tfrac{1}{4}|[X,Y]|^2. \tag{2}$$

(Here X, Y, Z are tangent vectors which, by left or right translation, can be taken to lie in the Lie algebra $\mathfrak{g}$ of G.) The non-compact dual H of G is obtained as follows. The Lie group G has a *complexification* G^c—a complex Lie group containing G as a real subgroup and with Lie algebra $\mathfrak{g}^c = \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$—and H is the homogeneous space $H = G^c/G$ with the metric induced from the invariant pseudo-Riemannian metric on G^c. The tangent space of H at the identity coset is $\mathfrak{g}^c/\mathfrak{g} \cong \mathfrak{g}$, and the curvature of H is given by the formulas

$$R(X,Y)Z = -\tfrac{1}{4}[[X,Y],Z], \tag{3}$$

$$K(X,Y) = -\tfrac{1}{4}|[X,Y]|^2. \tag{4}$$

Cartan's classification applies of course to *finite dimensional* symmetric spaces. The object of this paper is to point out that some of the same ideas appear naturally in an *infinite dimensional* situation. Let (M^{2n}, ω) be a compact symplectic manifold, so that the space $C^\infty(M)$ of functions on M is a Lie algebra under the Poisson bracket $\{\ ,\ \}$, defined by

$$\{f,g\}\omega^n = df \wedge dg \wedge \omega^{n-1}. \tag{5}$$

There is also an invariant L^2 inner product on $C^\infty(M)$:

$$\langle f, g\rangle = \frac{1}{n!}\int_M fg\ \omega^n. \tag{6}$$

We can write $C^\infty(M) = C_0^\infty(M) \oplus \mathbf{R}$, where $C_0^\infty(M)$ is the space of functions of integral zero, the L^2-orthogonal complement of the constants. The subalgebra $C_0^\infty(M)$ is the Lie algebra of an infinite dimensional Lie group $\mathcal{G}_0$. If $H^1(M;\mathbf{R}) = 0$ this is just the identity component of the *symplectomorphism group* SDiff of (M,ω). In general $\mathcal{G}_0$ is the group of "exact" symplectomorphisms, see **[16]** for example. It is sometimes convenient to suppose that $[\omega] \in H^2(M;\mathbf{R})$ is an integral class and to fix a complex Hermitian line bundle $L \to M$ with a unitary connection whose curvature is $-2\pi i\omega$. Then the full space $C^\infty(M)$ is the Lie algebra of a group $\mathcal{G}$: the group of connection-preserving Hermitian bundle maps from L to L (which necessarily cover exact symplectomorphisms of M).

Now the groups $\mathcal{G}, \mathcal{G}_0$ have bi-invariant metrics defined by the L^2 inner product on their Lie algebras, so it is no surprise that they furnish examples of infinite dimensional symmetric spaces. The two cases are scarcely different, since there is a local Riemannian (and Lie group) isomorphism

$$\mathcal{G} \approx \mathcal{G}_0 \times S^1. \tag{7}$$

It is clear that the formal part of the familiar theory goes over to these infinite dimensional spaces, so the curvature tensor of $\mathcal{G}$ is given by substituting the Poisson bracket into (1). Similarly, the geodesics in $\mathcal{G}$ are the translates of 1-parameter subgroups, the stuff of Hamiltonian dynamics. What is less obvious is that, if M is a Kähler manifold, there are "negatively curved duals" of these spaces, as we shall now describe.

Suppose that V is a compact complex n-manifold which admits a Kähler metric ω_0. The "$\overline{\partial}\partial$-lemma" asserts that any other Kähler metric cohomologous to ω_0 can be expressed via a Kähler potential; define $\mathcal{H}$ to be the space of Kähler potentials

$$\mathcal{H} = \{\phi \in C^\infty(V) : \omega_\phi = \omega_0 + i\overline{\partial}\partial\phi > 0\}. \tag{8}$$

This space has a better description in the case when $[\omega_0]$ is an integral class, so there is a corresponding holomorphic line bundle $L \to V$. Then we can identify $\mathcal{H}$ with the space of Hermitian metrics on L having positive curvature (or, more precisely, the compatible unitary connection has positive curvature). For if h_0 is a metric with curvature $-2\pi i\omega_0$, the curvature of $e^{2\pi\phi}h_0$ is $-2\pi i(\omega_0 + i\overline{\partial}\partial\phi)$ [**10**, §0.5]. The advantage of this second description is that it avoids the apparent dependence of (8) on the base point ω_0. The real numbers act on $\mathcal{H}$, by addition of constants, and we define $\mathcal{H}_0 = \mathcal{H}/\mathbf{R}$, which can be viewed as the *space of Kähler metrics on* V, in the given cohomology class.

Each Kähler potential $\phi \in \mathcal{H}$ gives a measure $d\mu_\phi = \frac{1}{n!}\omega_\phi^n$ on V. We define a Riemannian metric on the infinite dimensional manifold $\mathcal{H}$ using the L^2-norms furnished by these measures—a tangent vector $\delta\phi$ to $\mathcal{H}$ at a point $\phi \in \mathcal{H}$ is just a function on V, and we set

$$\|\delta\phi\|_\phi^2 = \int_V (\delta\phi)^2 d\mu_\phi. \tag{9}$$

Our main result is

THEOREM 1. *The Riemannian manifold $\mathcal{H}$ is an infinite dimensional symmetric space; it admits a Levi-Civita connection whose curvature is covariant constant. At a point $\phi \in \mathcal{H}$ the curvature is given by*

$$R_\phi(\delta_1\phi, \delta_2\phi)\delta_3\phi = -\tfrac{1}{4}\{\{\delta_1\phi, \delta_2\phi\}_\phi, \delta_3\phi\}_\phi,$$

where $\{\ ,\ \}_\phi$ is the Poisson bracket on $C^\infty(V)$ of the symplectic form ω_ϕ.

(Recall that in infinite dimensions the usual argument gives the uniqueness of a Levi–Civita [i.e. torsion-free, metric-compatible] connection, but not the existence in general.) The formula for the curvature of $\mathcal{H}$ entails that the sectional curvature is non-positive, given by

$$K_\phi(\delta_1\phi, \delta_2\phi) = -\tfrac{1}{4}\|\{\delta_1\phi, \delta_2\phi\}_\phi\|_\phi, \tag{10}$$

and, comparing with (1)–(4), these formulas certainly suggest that $\mathcal{H}$ should be regarded as the negatively curved dual of the group $\mathcal{G}$.

In the first half of this paper we dicuss the proof of this theorem. In the second half we explore the geometry of the space $\mathcal{H}$, in particular the geodesic equation. We state a number of natural conjectures, or questions, which we hope may be interesting to investigate further, and we outline the relevance of these ideas to well-established problems in Kahler geometry. In particular we will see that plausible results on the existence of geodesics would give a proof of the uniqueness of constant scalar curvature Kahler metrics.

§2. First proof

We will now give a direct proof of Theorem 1. The first step is to find the geodesic equation in $\mathcal{H}$. This is the Euler–Lagrange equation, $\delta E = 0$, for the energy functional

$$E = \int_0^1 \int_V \dot{\phi}^2 d\mu_\phi dt \tag{11}$$

on 1-parameter families $\phi(t)$ in $\mathcal{H}$, with fixed end points. The first variation of the measure is given by

$$d\mu_{\phi+\psi} = (1 + \tfrac{1}{2}\Delta\psi)\ d\mu_\phi + O(\psi^2),$$

where $\Delta = \Delta_\phi$ is the Laplacian on V of the metric ω_ϕ. So the first variation of E under a small variation $\psi(t)$ is

$$\delta E = \int_0^1 \int_V 2\dot\phi\dot\psi + \tfrac{1}{2}\dot\phi^2\Delta\psi\ d\mu_\phi = \textstyle\int_0^1 \int_V \psi(-2\frac{d}{dt}(\dot\phi d\mu_\phi) + \frac{1}{2}\Delta(\dot\phi^2)d\mu_\phi),$$

and the geodesic equation is

$$\frac{d}{dt}(\dot\phi d\mu_\phi) = \tfrac{1}{4}\Delta(\dot\phi^2)d\mu_\phi.$$

The t-derivative of $d\mu_\phi$ is $\frac{1}{2}\Delta\dot\phi\ d\mu_\phi$, and

$$\Delta(\dot\phi^2) = 2\dot\phi\Delta\dot\phi - 2|\nabla\dot\phi|^2_\phi,$$

so the geodesic equation reduces to

$$\ddot\phi = -\tfrac{1}{2}|\nabla\dot\phi|^2_\phi. \tag{12}$$

This geodesic equation shows us how to define a connection on the tangent bundle of $\mathcal{H}$. The notation is simplest if one thinks of such a connection as a way of differentiating vector fields along paths. Thus if $\phi(t)$ is any path in $\mathcal{H}$ and $\psi(t)$ is a field of tangent vectors along the path (that is, a function on $V \times [0,1]$), we define the covariant derivative along the path to be

$$D_t\psi = \frac{\partial\psi}{\partial t} + \tfrac{1}{2}(\nabla\psi, \nabla\dot\phi)_\phi. \tag{13}$$

This connection is torsion-free because in the canonical "coordinate chart", which represents $\mathcal{H}$ as an open subset of $C^\infty(V)$, the "Christoffel symbol"

$$\Gamma : C^\infty(V) \times C^\infty(V) \to C^\infty(V)$$

at ϕ is just

$$\Gamma(\psi_1, \psi_2) = \tfrac{1}{2}(\nabla\psi_1, \nabla\psi_2)_\phi,$$

which is symmetric in ψ_1, ψ_2. The connection is metric-compatible because

$$\begin{aligned}\frac{d}{dt}\|\psi\|^2_\phi = \frac{d}{dt}\int_V \psi^2 d\mu_\phi &= \int_V 2\frac{\partial\psi}{\partial t}\psi + \tfrac{1}{2}\psi^2\Delta(\dot\phi)d\mu_\phi \\ &= \int_V 2\frac{\partial\psi}{\partial t}\psi + \tfrac{1}{2}(\nabla(\psi^2), \nabla\dot\phi)\ d\mu_\phi \\ &= 2\int_V (\frac{\partial\psi}{\partial t} + \tfrac{1}{2}(\nabla\psi, \nabla\dot\phi)\ )\psi d\mu_\phi = 2\langle D_t\psi, \psi\rangle\end{aligned} \tag{14}$$

At this stage we could go on to compute the curvature of $\mathcal{H}$. But before doing this we will pause to explain the really significant property of the connection (13), which gets to the heart of the link betweeen Kähler and symplectic geometries (and which gives, by the way, more insight into the manipulations leading to (14)). By Moser's theorem [**17**], the symplectic manifolds (V, ω_ϕ) are all symplectically equivalent. The proof in this case is somewhat easier than the general situation considered by Moser.

Let $\phi(t)$ be a path starting at 0 in $\mathcal{H}$ and consider the t-dependent vector field

$$X_t = \tfrac{1}{2}\nabla_{\omega(t)}\dot{\phi},$$

where $\nabla_{\omega(t)}$ is the gradient operator defined by the metric $\omega(t) = \omega_0 + i\overline{\partial}\partial\phi(t)$. Then for fixed t the Lie derivative of $\omega(t)$ along X_t is

$$L_{X_t}(\omega(t)) = d(X_t \perp \omega(t))$$

But $X_t \perp \omega(t) = \frac{1}{2}Id\dot{\phi}(t)$, where I is the action of the complex structure on 1-forms, so

$$L_{X_t}\omega(t) = \tfrac{1}{2}(dId)\dot{\phi} = -i\overline{\partial}\partial\dot{\phi}.$$

On the other hand, the t-derivative of $\omega(t)$ is obviously $i\overline{\partial}\partial\dot{\phi}$. So if $f_t : V \to V$ is the 1-parameter family of diffeomorphisms obtained by integrating X_t (with $f_0 = 1_V$), we have

$$\frac{d}{dt}f_t^*\Big(\omega(t)\big) = f_t^*(L_{X_t}\omega(t) + \frac{d\omega(t)}{dt}\Big) = 0,$$

i.e. the diffeomorphism f_t gives the desired symplectomorphism from (V, ω_0) to $(V, \omega(t))$. Now let $\mathcal{Y} \subset \mathcal{H} \times \mathrm{Diff}(V)$ be the set of pairs (ϕ, f) such that $f^*(\omega_\phi) = \omega_0$. This is a principle bundle over $\mathcal{H}$ with structure group the group $\mathrm{SDiff}(V)$ of symplectomorphisms of (V, ω_0). Then the discussion above shows that our connection on the tangent space of $\mathcal{H}$ is induced from an SDiff connection on $\mathcal{Y} \to \mathcal{H}$ via the action of SDiff on the vector space $C^\infty(V)$; that is, we have a connection-preserving bundle isomorphism

$$T\mathcal{H} = \mathcal{Y} \times_{\mathrm{SDiff}} C^\infty(V).$$

In this framework the 1-parameter family of diffeomorphisms f_t above appears as the horizontal lift of the path $\phi(t)$ to $\mathcal{Y}$. The metric-preserving property (14) is just the fact that the action of SDiff on $C^\infty(V)$ preserves the L^2 norm.

If $H^1(V) \neq 0$, the group SDiff is not quite the same as the group $\mathcal{G}_0$ of exact symplectomorphisms considered in Section 1 (with $(M, \omega) = (V, \omega_0)$), but one can easily adjust the definitions to get a principle $\mathcal{G}_0$ bundle over $\mathcal{H}$ which descends to a $\mathcal{G}_0$ bundle $\mathcal{X}_0$ over $\mathcal{H}_0$. As usual, the discussion is cleaner in the case when $[\omega_0]$ is integral, so we have a holomorphic line bundle $L \to V$ and we can regard $\mathcal{H}$ as the space of metrics of positive curvature on L. Then we let $\mathcal{X}$ be the set of pairs $(h, \tilde{f})$, where h is a metric on L and $\tilde{f}$ is a connection-preserving Hermitian bundle map from (L, h_0) to (L, h), which necessarily covers a diffeomorphism f of V with $f^*(\omega_\phi) = \omega_0$. (More precisely, we should take $\mathcal{X}$ to be the connected component of $(h_0, 1_L)$ in this space.) Then $\mathcal{X}$ is a principle $\mathcal{G}$ bundle over $\mathcal{H}$, and our Levi–Civita connection is induced from a $\mathcal{G}$-connection on $\mathcal{X}$.

We now go back to compute the curvature tensor of $\mathcal{H}$. To do this we consider a 2-parameter family $\phi(s,t)$ in $\mathcal{H}$, and a vector field $\psi(s,t)$ along $\phi(s,t)$. We denote s and t derivatives by suffixes ϕ_s, etc., where convenient. The curvature is given by the commutator

$$R(\phi_s, \phi_t)\psi = (D_s D_t - D_t D_s)\psi. \tag{15}$$

Expanding out, this is

$$\begin{aligned} R(\phi_s, \phi_t)\psi = \tfrac{1}{2}\big((\nabla\phi_s, \nabla\psi_t) - (\nabla\phi_t, \nabla\psi_s) + \tfrac{\partial}{\partial s}(\nabla\phi_t, \nabla\psi) - \tfrac{\partial}{\partial t}(\nabla\phi_s, \nabla\psi)\big) \\ + \tfrac{1}{4}\left((\nabla\phi_s, \nabla(\nabla\phi_t, \nabla\psi)) - (\nabla\phi_t, \nabla(\nabla\phi_s, \nabla\psi))\right). \end{aligned} \tag{16}$$

Here we have simply written $(\ ,\ )$ for the inner product on cotangent vectors in V defined by $\omega_{\phi(s,t)}$. Thus, if a, b are fixed cotangent vectors, then

$$\frac{\partial}{\partial s}(a,b) = ((i\bar{\partial}\partial\phi_s)^\sharp, a\otimes b),$$

where $(i\bar{\partial}\partial\phi_s)^\sharp$ denotes the symmetric 2-tensor on V defined by the real $(1,1)$-form $i\bar{\partial}\partial\phi_s$ in the usual way. So we have

$$\frac{\partial}{\partial s}(\nabla\phi_t, \nabla\psi) = (\nabla\phi_{st}, \nabla\psi) + (\nabla\phi_t, \nabla\psi_s) + (i\bar{\partial}\partial\phi_s)^\sharp(\nabla\phi_t\otimes\nabla\psi),$$

and similarly for $\frac{\partial}{\partial t}(\nabla\phi_s, \psi)$. Then (16) reduces to

$$\begin{aligned} R(\phi_s,\phi_t)\psi = \tfrac{1}{2}\left((i\bar{\partial}\partial\phi_s)^\sharp(\nabla\phi_t\otimes\nabla\psi) - (i\bar{\partial}\partial\phi_t)^\sharp(\nabla\phi_s\otimes\nabla\psi)\right) \\ + \tfrac{1}{4}\left((\nabla\phi_s, \nabla(\nabla\phi_t,\nabla\psi)) - (\nabla\phi_t, \nabla(\nabla\phi_s,\nabla\psi))\right). \end{aligned} \tag{17}$$

Now the second expression on the right hand side of (17) can be written as

$$\tfrac{1}{4}(\nabla\nabla\phi_s)(\nabla\phi_s\otimes\nabla\psi) - (\nabla\nabla\phi_t)(\nabla\phi_s\otimes\phi_t),$$

so

$$4R(\phi_s,\phi_t)\psi = P(\phi_s)(\nabla\phi_t\otimes\nabla\psi) - P(\phi_t)(\nabla\phi_s\otimes\nabla\psi), \tag{18}$$

where P is the differential operator, from functions to symmetric 2-tensors,

$$P(f) = 2(i\bar{\partial}\partial f)^\sharp - \nabla\nabla f. \tag{19}$$

On the other hand, we can write the Poisson bracket $\{\ ,\ \}$ using the complex structure structure I:

$$\{f,g\} = (\nabla f, I\nabla g) = -(I\nabla f, \nabla g).$$

So

$$\{\{\phi_s,\phi_t\},\psi\} = (\nabla(\phi_s, I\nabla\phi_t), I\nabla\psi) = Q(\phi_s)(\nabla\phi_t\otimes\nabla\psi) - Q(\phi_t)(\nabla\phi_s\otimes\nabla\psi),$$

where

$$Q(f) = (I\otimes I)(\nabla\nabla f).$$

The calculation of the curvature tensor is therefore completed by showing that the operators P and Q are the same. First, as a matter of linear algebra, the map $\frac{1}{2}(1 + I\otimes I)$ from $s^2(T^*V)$ to itself is the standard projection to the $(1,1)$ part. So the assertion is that for a Kahler manifold the operator $i\bar{\partial}\partial$, which is defined by the complex structure, can be obtained as a projection of the second covariant derivative $\nabla\nabla$, which uses the Levi–Civita connection. This can be seen easily, for example by working in an osculating coordinate system as in ([**10**, p.108]). So we have shown that $R(\phi_s,\phi_t)\psi = -\frac{1}{4}\{\{\phi_s,\phi_t\},\psi\}$, as desired.

The expression for the curvature tensor in terms of Poisson brackets shows that R is invariant under the action of the symplectomorphism group. Since the connection on $T\mathcal{H}$ is induced from an SDiff-connection, it follows that R is covariant constant, and hence $\mathcal{H}$ is indeed an infinite-dimensional symmetric space.

§3. The decomposition of $\mathcal{H}$

Here we clarify the relationship between the spaces $\mathcal{H}$ and $\mathcal{H}_0 = \mathcal{H}/\mathbf{R}$. There is obviously a decomposition of the tangent space:

$$(T\mathcal{H})_\phi = \left\{\psi : \int_V \psi d\mu_\phi = 0\right\} \oplus \mathbf{R}. \tag{20}$$

We claim that this corresponds to a Riemannian decomposition

$$\mathcal{H} = \mathcal{H}_0 \times \mathbf{R}. \tag{21}$$

At one level, this follows from the picture developed above, since the tangent space decomposition (20) is invariant under the symplectomorphism group, and is therefore covariant constant with respect to our torsion-free connection. However it is interesting to see this more explicitly, partly because we see the appearance of a functional I on the space of Kähler potentials, which is well known in the literature, see [**1**], [**21**] for example. The decomposition (20) give a 1-form α on $\mathcal{H}$ with

$$\alpha_\phi(\psi) = \int_V \psi d\mu_\phi, \tag{22}$$

and the point is that this 1-form is *closed.* Indeed

$$\alpha_{\phi+\tilde{\psi}}(\psi) - \alpha_\phi(\psi) = \frac{1}{2}\int_V \psi\Delta\tilde{\psi} + O(\tilde{\psi}^2), \tag{23}$$

so

$$(d\alpha)_\phi(\psi,\tilde{\psi}) = \frac{1}{2}\int_V \psi\Delta\tilde{\psi} - \tilde{\psi}\Delta\psi = 0 \tag{24}$$

since the Laplacian is self-adjoint. This means that there is a function $I : \mathcal{H} \to \mathbf{R}$ with $I(0) = 0$ and $dI = \alpha$, and it is this function which gives rise to the Riemannian decomposition (21). We call a Kähler potential ϕ *normalised* if $I(\phi) = 0$. Then any Kähler metric has a unique normalised potential, and the restriction of our metric on $\mathcal{H}$ to $I^{-1}(0)$ endows the space $\mathcal{H}_0$ of Kähler metrics with a Riemannian structure; this is independent of the choice of base point ω_0 and clearly makes $\mathcal{H}_0$ into a symmetric space. The functional I can be written more exlpicitly by integrating α along lines in $\mathcal{H}$ to give the formula

$$I(\phi) = \sum_{p=0}^{n} \frac{1}{(p+1)!(n-p)!}\int_V \omega_0^{n-p}(i\bar{\partial}\partial\phi)^p\ \phi. \tag{25}$$

§4. Second proof

We will now outline another proof of Theorem 1 which avoids detailed calculations and shows more clearly the analogy with ordinary symmetric spaces G^c/G.

The tangent bundle of a Lie group, finite or infinite dimensional, is trivialised by left-invariant vector fields, which are, of course, closed under Lie bracket. Conversely, suppose we have a manifold Z and a trivialisation $TZ = Z \times U$, so for each element u of the vector space U we have a vector field X_u on Z. Suppose that this collection of vector fields is closed under Lie bracket: this means that U becomes a Lie algebra with a bracket $[\ ,\ ]_U$ such that $X_{[u,v]_U} = [X_u, X_v]$. If Z is finite dimensional, one of Lie's basic integration theorems tells us that this structure arises from a Lie group structure on Z, but this integration theorem fails in infinite dimensions, as we shall see. Let us call a space Z with this structure an "infinitesimal Lie group". Now suppose that there is a free action of a genuine Lie group S on Z which induces a Lie algebra injection $\mathrm{Lie}(S) \subset U$. Then we can form the orbit space Z/S, which—leaving aside global topological questions—will obviously share some of the familiar properties of homogeneous spaces. In particular, suppose that Z is a "complexification" of S, i.e. that Z has a formally integrable complex structure, the Lie algebra U is complex, the vector fields X_u are holomorphic and U is the complexification of $\mathrm{Lie}(S)$. One way this can occur is when the Lie group S acts freely on a complex manifold T, preserving the complex structure, and Z is a submanifold of T with the property that for each z in Z

$$TZ_z = TO(z)_z \oplus ITO(z)_z, \tag{26}$$

where $O(z)$ is the S-orbit of z in T. In this case we just define U to be $\mathrm{Lie}(S) \otimes \mathbf{C}$ and construct the vector fields X_u by taking complex-linear combinations of the vector fields defining the infinitesimal action of S. (We only need T to have a formally integrable almost complex structure—this implies that the X_u are closed under Lie bracket.) Then we have

PROPOSITION 2. *If S is a Lie group with a bi-invariant metric and Z is an infinitesimal Lie group which is a complexification of S, then Z/S has the structure of a Riemannian symmetric space, with holonomy group S (acting via the adjoint representation) and with curvature* $R(u_1, u_2)u_3 = -\frac{1}{4}[[u_1, u_2], u_3]$.

To see this one just needs to see that the standard theory in finite dimensions only involves infinitesimal calculations—with vector fields and Lie algebras—and so applies equally well under our hypotheses. Of course the part of the standard theory which does *not* go over is the existence of a transitive isometry group of the Riemannian manifold Z/S.

To bring the discussion above to bear on our problem we want to exhibit the principle $\mathcal{G}$ bundle $\mathcal{X} \to \mathcal{H}$ as an infinitesimal complexification, and this can be done in two ways, each involving the action of $\mathcal{G}$ on a complex manifold as described above. In fact it will be easier to work with the smaller space $\mathcal{H}_0$, which is essentially the same by the discussion in Section 3 above.

The first approach was described in [7]. For simplicity we suppose that the group of holomorphic automorphisms of V is trivial and that $H^1(V) = 0$. We consider the space $\mathcal{J}$ of almost complex structures on V compatible with the symplectic form ω_0. This is the space of sections of a fibre bundle over V with fibre the complex homogeneous space $Sp(2n, \mathbf{R})/U(n)$, and $\mathcal{J}$ inherits a complex structure from that of the fibre. The symplectomorphism group $\mathcal{G}_0$ acts naturally on $\mathcal{J}$, preserving the complex structure. If we have a diffeomorphism $f : V \to V$ and an $\omega_\phi \in \mathcal{H}_0$ with $f^*(\omega_\phi) = \omega_0$, then the pull-back $f^*(I)$ defines a point in $\mathcal{J}$ and this gives an embedding $\lambda : \mathcal{X}_0 \to \mathcal{J}$ which one readily sees, as in [7], has the properties envisaged in the abstract picture above.

For the second approach we can drop the assumption on the holomorphic automorphisms. We consider the space $\mathrm{Maps}(V, V)$, which has a complex structure induced from the complex structure on V (a tangent vector to $\mathrm{Maps}(V, V)$ at a map $g : V \to V$ is a section of the complex vector bundle $g^*(TV)$). Now we define a map $\mu : \mathcal{X}_0 \to \mathrm{Maps}(V, V)$, sending a pair (f, ϕ) to f. This is an embedding, and the image $\mu(\mathcal{X}_0)$ consists of the difffeomorphisms f such that $(f^{-1})^*(\omega_0)$ has type (1,1). Again one sees that this image is a complexification of $\mathcal{G}_0$, where now $\mathcal{G}_0$ acts on Maps(V, V) by composition on the right. Finally one shows that the structure on $\mathcal{X}_0$ defined by the two embeddings λ, μ is the same: this just comes down to the fact that the infinitesimal action of the vector fields on the integrable almost complex structures is given by the $\overline{\partial}$-operator of the tangent bundle of V, which is complex-linear.

In the case when the complex dimension n of V is 1 (and, under our current hypotheses, V must be the Riemann sphere) the condition that $(f^{-1})^*(\omega_0)$ has type $(1,1)$ is vacuous, so in this case $\mathcal{X}_0$ can be identified with the space Diff(V) of diffeomorphisms of V (or, more precisely, the identity component in this space). One needs to beware that the "infinitesimal group" structure on Diff(V) that we are considering here is not the same as the genuine group structure on this space. This corresponds to the fact that there is a non-standard Lie bracket $[\ ,\ ]_I$ on the space of vector fields on the sphere. We express any vector field ξ as $\xi_1 + I\xi_2$, where ξ_1, ξ_2 are Hamiltonian, and then set

$$[\xi_1 + I\xi_2, \eta_1 + I\eta_2]_I = ([\xi_1, \eta_1] - [\xi_2, \eta_2]) + I([\xi_1, \eta_2] + [\xi_2, \eta_1]).$$

§5. The geodesic, WZW, and Monge–Ampère equations

We will now study the geodesic equation (12) in $\mathcal{H}$ in more detail, and interpret the solutions geometrically. Suppose ϕ_t, $t \in [0,1]$, is a path in $\mathcal{H}$. We can view this as a function on $V \times [0,1]$ and in turn as a function on $V \times [0,1] \times S^1$, with trivial dependence on the S^1 factor; that is, we define

$$\Phi(v, t, e^{is}) = \phi_t(v).$$

We regard the cylinder $A = [0,1] \times S^1$ as a Riemann surface with boundary in the standard way—so $t + is$ is a local complex coordinate. Let Ω_0 be the pull-back of ω_0 to $V \times A$ under the projection map, and put $\Omega_\Phi = \Omega_0 + i\overline{\partial}\partial\Phi$, a $(1,1)$-form on $V \times A$. Then we have

PROPOSITION 3. *The path ϕ_t satisfies the geodesic equation (12) if and only if $\Omega_\Phi^{n+1} = 0$ on $V \times A$.*

The proof is left as an exercise for the reader (or use Proposition 4 below).

We now turn to variational problems involving 2-dimensional domains. If R is a compact Riemann surface with boundary, one may consider the *harmonic* maps from R to a finite-dimensional symmetric space $H = G^c/G$. These are the critical points of the energy functional $E(f) = \frac{1}{2}\int_R |\nabla f|^2$, with Euler–Lagrange equation $d^*(Df) = 0$. But there is also a deformation of the harmonic map equation, which makes use of the special geometry of the target space. The tangent space of H at each point is modelled on the Lie algebra of G, so there is a natural Lie bracket $TH \times TH \to TH$ which yields a covariant constant 3-form θ on H. Given a fixed boundary map $\sigma : \partial R \to H$, we can define a functional E_σ^{WZW} on the space of maps from R with boundary value σ as follows. Choose a reference map $f_0 : R \to H$ with boundary value σ, and set

$$E_\sigma^{WZW}(f) = \tfrac{1}{2}\int_R |\nabla f|^2 + \int_Z \theta, \tag{27}$$

where Z is any 3-chain in H with boundary $f(R) - f_0(R)$. This is independent of the choice of Z, since H is simply connected and θ is closed, and the functional depends on f_0 only up to a constant. The Euler–Lagrange equation is, in a local complex coordinate $t + is$ on R,

$$d^*(Df) + [f_t, f_s] = 0. \tag{28}$$

We call this equation the WZW equation because similar ideas appear in the Wess–Zumino–Witten theory in mathematical physics.

Clearly we can take this formal set-up over to the case of maps from R to $\mathcal{H}$, which we can interpret as functions Φ on $V \times R$, such that $\Omega_\Phi = \Omega_0 + i\overline{\partial}\partial\Phi$ is positive on the slices $V \times \{z\}$, $z \in R$.

PROPOSITION 4. *A map from R to $\mathcal{H}$ satisfies the WZW equation if and only if $\Omega_\Phi^{n+1} = 0$.*

To see this we first compute the harmonic map "tension field" $d^*(DF)$. We work in a local complex coordinate $\tau = t + is$ on R. The usual calculation of the Euler–Lagrange equation, just as for the geodesic problem in Section 2, shows that the tension field is

$$\Phi_{ss} + \Phi_{tt} + \tfrac{1}{2}(|\nabla\Phi_s|_\phi^2 + |\nabla\Phi_t|_\phi^2),$$

where ∇ denotes differentiation in the V variable, and $|\ |_\phi$ is the metric on V defined by Ω_Φ for fixed s, t. So the WZW equation is

$$\Phi_{ss} + \Phi_{tt} + \tfrac{1}{2}(|\nabla\Phi_s|_\phi^2 + |\nabla\Phi_t|_\phi^2) + (\nabla\Phi_s, I\nabla\Phi_t)_\phi = 0. \tag{29}$$

It is now just a matter of algebra to show that this equation is precisely $\Omega_\Phi^{n+1} = 0$. We can work at a given point in $V \times R$ and choose local complex coordinates $z_\alpha = x_\alpha + iy_\alpha$ on V such that the metric ω_ϕ is standard at that point, in these coordinates. Then, at this point,

$$\Omega_\Phi^{n+1} \propto 2^{n-1} \det \begin{pmatrix} \Phi_{\tau\overline{\tau}} & . & \Phi_{\tau\overline{z}_\alpha} & . & . \\ . & \frac{1}{2} & 0 & . & 0 \\ . & . & \frac{1}{2} & . & 0 \\ \Phi_{\overline{\tau}z_\alpha} & 0 & . & \frac{1}{2} & . \\ . & 0 & . & . & \frac{1}{2} \end{pmatrix} = \Phi_{\tau\overline{\tau}} - 2\sum_\alpha |\Phi_{\overline{\tau}z_\alpha}|^2.$$

This reduces to the left hand side of (29) when we use the identity

$$\begin{aligned} 8\sum_\alpha |\Phi_{\overline{\tau}z_\alpha}|^2 &= \frac{1}{2}\sum_\alpha (\Phi_{sx_\alpha} + \Phi_{ty_\alpha})^2 + (\Phi_{tx_\alpha} - \Phi_{sy_\alpha})^2 \\ &= \tfrac{1}{2}(|\nabla\Phi_s|^2 + |\nabla\Phi_t|^2) + (\nabla\Phi_s, I\nabla\Phi_t), \end{aligned}$$

and write $\Phi_{\tau\overline{\tau}} = \frac{1}{4}(\Phi_{ss} + \Phi_{tt})$.

The analogue of the WZW functional E^{WZW} for maps to $\mathcal{H}$ is a variant of the functional I introduced in Section 3. Given boundary data $\rho : \partial R \to \mathcal{H}$, we consider the set of functions Φ on $V \times R$ which agree with ρ on the boundary. Then we define the variation of I on this set by

$$\delta I_\rho = \frac{1}{(n+1)!}\int_{V\times R} \delta\Phi\ \Omega_\Phi^{n+1}, \tag{30}$$

where the variation $\delta\Phi$ vanishes on the boundary by hypothesis. This boundary condition means that the same argument as in Section 3 works to show that the formula defines a functional I_ρ, which again has a more explicit expression like (25). This functional I_ρ reduces to the energy functional on paths, by an integration by parts that we leave to the reader, in the case when R is the cylinder and we restrict to S^1-invariant data.

The equation $\Omega_\Phi^{n+1} = 0$ is a degenerate Monge–Ampère equation. A solution gives rise to a rather concrete geometric structure on $V \times R$, as is well-known in the literature in this area [**13**]. If $\Omega_\Phi^{n+1} = 0$ and Ω_Φ is strictly positive on the V-slices, then the null space $N \subset T(V \times R)$ of Ω_Φ at each point has real dimension 2. On the one hand this field of subspaces forms a foliation, since Ω_Φ is closed; on the other hand the subspaces are complex lines, since Ω_Φ has type $(1,1)$. So a solution of the WZW equations gives rise to a foliation of $V \times R$ whose leaves are complex curves transverse to the V-slices. This conclusion fits in tidily with a corresponding result in the finite-dimensional case. Suppose first that R is simply connected. Then a map $f : R \to G^c/G$ is a solution of the WZW equation if and only if it has a lift to a *holomorphic* map from R to the complex Lie group G^c. In the Kähler geometry case a foliation of $V \times R$, transverse to the V-slices, can be viewed as a map from R to $\mathrm{Diff}(V)$, and the condition that the leaves are complex curves is precisely saying that this map is holomorphic, with respect to the complex structure on $\mathrm{Diff}(V) \subset \mathrm{Maps}(V, V)$ considered in Section 4. The condition that Ω_Φ has type (1,1) then tells us that the image of the map lies in the subspace $\mathcal{Y} \subset \mathrm{Diff}(V)$, which is our analogue of the complex group G^c. In the case when R is not simply connected we can carry out this analysis on the universal cover; a solution of the finite-dimensional WZW equation on R gives an equivariant holomorphic map from $\tilde{R}$ to G^c, with respect to a representation $\pi_1(R) \to G$, which in turn can be viewed as a flat G-connection over R. In the Kähler geometry case we get a corresponding statement, where the representation $\pi_1(R) \to \mathrm{SDiff}(V)$ is the monodromy of the foliation. (One can also obtain an interpretation of the WZW equations in terms of holomorphic maps into $Sp(2n, \mathbf{R})/U(n)$, using the other point of view in Section 4.)

Let us now spell out what this means in the case of a geodesic ϕ_t, $t \in [0, 1]$, in $\mathcal{H}$, starting at $\phi = 0$, say. We regard the time derivative $\dot{\phi}_0$ as a Hamiltonian on the symplectic manifold (V, ω_0), so it defines a Hamiltonian flow $g_s : V \to V$ in the usual way. For each point $v \in V$ let $\gamma_v : \mathbf{R} \to V$ be its trajectory under the flow: $\gamma_v(s) = g_s(v)$. Then the projection of the appropriate leaf of the foliation gives a *holomorphic* extension $\Gamma_v : [0, 1] \times \mathbf{R} \to V$, with $\Gamma_v(0, s) = \gamma_v(s)$, such that the restriction of the Γ_v to $\{t\} \times \mathbf{R}$ is a trajectory of the Hamiltonian flow of $\dot{\phi}_t$ on the symplectic manifold (V, ω_{ϕ_t}). Moreover the map $f_t : V \to V$ defined by $f_t(v) = \Gamma_v(0, t)$ takes the Hamiltonian system $(V, \omega_0, \dot{\phi}_0)$ to the system $(V, \omega_{\phi_t}, \dot{\phi}_t)$. So we can think of the study of geodesics in $\mathcal{H}$ as a kind of "analytic continuation" of Hamiltonian dynamics, in which the time parameter is made complex, and the complex curves Γ_v give holomorphic cobordisms between the trajectories of the family of Hamiltonian systems.

§6. Examples of geodesics

(i) Suppose that a compact, connected, Lie group G acts on V preserving the complex structure and a Kähler metric ω_0. Then the complexification G^c acts holomorphically on V, and we get a map from the finite-dimensional space G^c/G to $\mathcal{H}_0$, taking the coset of $\alpha \in G$ to $\alpha^*(\omega_0)$. There is no loss in supposing that the action of G is infinitesimally effective, so the L^2 norm on Hamiltonians defines an invariant metric on G, and hence a metric on G^c/G, making it a symmetric space.

(Of course for a simple group this structure is unique up to scale.) Then G^c/G is isometrically embedded as a totally geodesic submanifold in $\mathcal{H}_0$, and any geodesic in G^c/G gives a geodesic in $\mathcal{H}$. We leave the simple verification to the reader.

(ii) Let h be a function on the standard sphere $S^2 \subset \mathbf{R}^3$ with $h(x,y,z) = z$ near to the poles $p_\pm = (0,0,\pm 1)$, and with no critical points apart from these poles. Fix an identification of the tangent space to S^2 at p_- with $\mathbf{C}$. For each $r \in (-1,1)$ the set $U_r = h^{-1}(r,1]$ is, differentiably, a disc with smooth boundary in S^2. By the Riemann mapping theorem there is a unique conformal equivalence $\alpha_r : D \to U_r$ from the unit disc $D \subset \mathbf{C}$, normalised so that $\alpha_r(0) = p_-$ and so that the derivative of α_r at 0 is real and positive with respect to the identification above. The map α_r extends smoothly to $\overline{\alpha}_r : \overline{D} \to \overline{U}_r$. In particular, rotation of the disc defines an action of the circle on the curve $\partial U_r = h^{-1}(r) \subset S^2$. When r is close to ± 1 this is just the restriction of the usual rotation of the sphere about the z-axis, since $h = z$ near $p_\pm$. It is easy to see that there is an area form ω_0, unique up to scale, such that the circle action on each curve $h^{-1}(r)$ is the Hamiltonian flow of the function h with respect to ω_0.

Now let $\overline{\beta}_r$ be the holomorphic map from the cylinder $[0,\infty) \times S^1$ to S^2 defined by composing $\overline{\alpha}_r$ with the standard identification $[0,\infty) \times S^1 \to \overline{D} \setminus \{0\}$. For each point $v \in S^2$ there is, by construction, a unique $e^{is(v)} \in S^1$ such that $\overline{\beta}_{h(v)}(0, e^{is(v)}) = v$. For $t > 0$, define a smooth map $f_t : S^2 \to S^2$ by

$$f_t(v) = \overline{\beta}_{h(v)}(t, e^{is(v)}).$$

The Schwartz lemma implies that f_t is a bijection, and one may also show that the derivative of f is everywhere invertible, so there is a smooth inverse f_t^{-1}. Now we set

$$\omega_t = (f_t^{-1})^*(\omega_0)$$

and write $\omega_t = \omega_0 + i\overline{\partial}\partial\phi_t$ for normalised potentials ϕ_t. Then ϕ_t is an infinite geodesic ray in $\mathcal{H}_0(S^2)$, starting at ω_0 and with initial tangent vector $\dot{\phi} = h$ at $t = 0$. (It seems very likely that the same construction works for any function h on S^2 with two non-degenerate critical points.)

One can argue similarly if k is another function on S^2 with critical points only at $p_\pm$, and with $h > k$ on $S^2 \setminus \{p_+, p_-\}$. Suppose for simplicity that

$$k(x,y,z) = \begin{cases} -1 + \rho(z+1) & \text{near } p_-, \\ 1 + \sigma(z-1) & \text{near } p_+, \end{cases}$$

with $0 < \rho < 1 < \sigma$. Then one can use the conformal equivalences between the annular regions

$$A_r = \{v \in S^2 \ : k(v) < r < h(v)\},$$

and standard annuli $\{w \in \mathbf{C} : 1 < |w| < R\}$, for suitable $R(r)$, to construct symplectic forms ω, ω' on S^2 and a geodesic from ω to ω' in $\mathcal{H}_0$ with tangent vector h at ω and k at ω'. Here the Hamiltonian system (S^2, ω, h) (which is isomorphic to (S^2, ω', k)) will not, in general, be periodic, since the modulus R of the annulus varies with r.

(iii) Consider the complex n-torus $T = \mathbf{C}^n/\mathbf{Z}n + i\mathbf{Z}n$, and let ω_0 be the standard flat metric. The group $\Gamma = (S^1)^n$ acts on T^n via translations in the Lagrangian subspace $i\mathbf{R}^n \subset \mathbf{C}^n$, and this induces an isometric action of Γ on the space $\mathcal{H}_0$ of Kahler metics on T; so the set $\mathcal{H}_0^\Gamma$ of invariant metrics is totally geodesic in $\mathcal{H}_0$. The Poisson bracket of Γ-invariant functions vanishes, and it follows that the induced metric on $\mathcal{H}_0^\Gamma$ is *flat*, in the sense that its Riemann curvature vanishes. One might expect (though this is probably not automatically true in infinite dimensions) that $\mathcal{H}_0^\Gamma$ is isometric to an open subset in a pre-Hilbert space, and we will now verify that this is the case. For simplicity we will take $n = 1$ and write evrything out explicitly—in terms of our general picture the point is that the group of exact symplectomorphisms of T which commute with Γ is an abelian group—just the space of maps from $\mathbf{R}^n/\mathbf{Z}^n$ to $(S^1)^n$—and it has a *bona fide* complexification $\mathrm{Maps}(\mathbf{R}^n/\mathbf{Z}^n, (\mathbf{C}^*)^n)$. The analysis below arises from writing out the equivalence between $\mathcal{H}_0^\Gamma$ and

$$\frac{\mathrm{Maps}(\mathbf{R}^n/\mathbf{Z}^n, (\mathbf{C}^*)^n)}{\mathrm{Maps}(\mathbf{R}^n/\mathbf{Z}^n, (S^1)^n)}$$

which we obtain from the set-up in Section 4.

Elements of $\mathcal{H}_0^\Gamma$ (when $n = 1$) can be defined by circle-invariant Kähler potentials. Explicitly, $\mathcal{H}_0^\Gamma$ can be viewed as the set of functions $\phi : \mathbf{R}/\mathbf{Z} \to \mathbf{R}$ with $1 - \phi_{xx} > 0$, normalised so that

$$\int_{\mathbf{R}/\mathbf{Z}} \phi + \tfrac{1}{2}\phi_x^2 \, dx = 0,$$

and with metric

$$g_{\mathcal{H}_0^\Gamma}(\delta\phi) = \int_{\mathbf{R}/\mathbf{Z}} (\delta\phi)^2(1 - \phi_{xx})dx.$$

Let U be the set of functions ψ on $\mathbf{R}/\mathbf{Z}$ with $1 + \psi_{\xi\xi} > 0$, normalised so that

$$\int_{\mathbf{R}/bZ} \psi d\xi = 0$$

and endowed with the flat metric

$$g_U(\delta\psi) = \int_{\mathbf{R}/\mathbf{Z}} (\delta\psi)^2 d\xi.$$

(It is convenient to use a different symbol, ξ, for the coordinate on the circle in the definition of U.) We will exhibit an isometry

$$\alpha : (\mathcal{H}_0^\Gamma, g_{\mathcal{H}_0^\Gamma}) \to (U, g_U).$$

For $\phi \in \mathcal{H}_0^\Gamma$ define $\xi = \xi(x)$ by

$$\xi(x) = x - \phi_x(x) \in \mathbf{R}/\mathbf{Z}.$$

Then $\frac{d\xi}{dx} = 1 - \phi_{xx} > 0$, so the map $\xi(x)$ is a diffeomorphism of the circle (it is a covering map of degree 1), with inverse $x = x(\xi)$.

Now let

$$A(\xi) = \left(\frac{\phi_{xx}}{1 - \phi_{xx}}\right)_{x(\xi)}.$$

Then

$$\int_{\mathbf{R}/\mathbf{Z}} A(\xi)d\xi = \int_{\mathbf{R}/\mathbf{Z}} \left(\frac{\phi_{xx}}{1 - \phi_{xx}}\right)(1 - \phi_{xx})dx = 0,$$

so there is a unique function $\psi(\xi)$ with $\int \psi d\xi = 0$ and $\psi_{\xi\xi} = A(\xi)$. Then

$$1 + \psi_{\xi\xi} = \frac{1}{1 - \phi_{xx}} > 0,$$

so ψ is in U. We define $\alpha(\phi) = \psi$. Now suppose we make a small variation $\tilde{\phi} = \phi + \eta$, so $\tilde{\xi}(x) = \xi(x) - \eta_x$. It follows that

$$\tilde{x}(\xi) = x(\xi) + \left(\frac{\eta_x}{1 - \phi_{xx}}\right)_{x(\xi)} + O(\eta^2)$$

and

$$\tilde{A}(\xi) = A(\xi) + \frac{\eta_x \phi_{xxx}}{(1 - \phi_{xx})^3} + \frac{\eta_{xx}}{(1 - \phi_{xx})^2} + O(\eta^2).$$

Now set $\tau(\xi) = \eta(x(\xi))$. The chain rule gives

$$\tau_{\xi\xi} = \frac{1}{1 - \phi_{xx}} \frac{d}{dx}\left(\frac{\eta_x}{1 - \phi_{xx}}\right) = \frac{\eta_x \phi_{xxx}}{(1 - \phi_{xx})^3} + \frac{\eta_{xx}}{(1 - \phi_{xx})^2}.$$

So $\tilde{A}(\xi) = A(\xi) + \tau_{\xi\xi} + O(\eta^2)$. Furthermore,

$$\int_{\mathbf{RZ}} \tau(\xi) d\xi = \int_{\mathbf{R}/\mathbf{Z}} \eta \, (1 - \phi_{xx}) dx = O(\eta^2),$$

since ϕ and $\tilde{\phi}$ are normalised. In sum, the derivative of α maps η to τ and

$$\int \tau^2 d\xi = \int \eta^2 \frac{d\xi}{dx} dx = \int \eta^2 (1 - \phi_{xx}) dx,$$

so α is an isometry. Finally we can write down the inverse map $\alpha^{-1} : U \to \mathcal{H}_0^\Gamma$. Given $\psi \in U$, we set $x(\xi) = \xi + \psi_\xi$ and $B(x) = \left(\frac{\psi_{xi\xi}}{1+\psi_{\xi\xi}}\right)_{x(\xi)}$. Then $\int B(x) dx = \int \psi_{\xi\xi} d\xi = 0$, so $B(x) = \phi_{xx}$ for a unique normalised ϕ, and

$$1 - \phi_{xx} = \frac{1}{1 + \psi_{\xi\xi}} > 0.$$

Thus ψ lies in $\mathcal{H}_0^\Gamma$. It is easy to check that this construction is inverse to the previous one, hence completing the proof.

Notice that this transformation takes the geodesic equation in the invariant case, namely

$$\phi_{tt} = -\tfrac{1}{2}\phi_{xt}^2 / (1 - \phi_{xx}), \tag{31}$$

to the trivial equation $\psi_{tt} = 0$ in U; that is, the equation (31) (and more generally the *real* homogeneous Monge–Ampère equation in any dimension) can be solved explicitly, a fact which is perhaps not obvious.

§7. Existence and uniqueness questions

A Riemannian symmetric space G^c/G, like any complete simply connected Riemannian manifold of non-positive curvature, has the property that any two points can be joined by a unique geodesic. Likewise, if R is a compact Riemann surface-with-boundary and ρ is a smooth map from ∂R to G^c/G, then there is a unique solution of the WZW equation on R with boundary value ρ. (This is proved in [8] for the case when G is a unitary group, and the argument extends easily to the general case. When R is the disc the result is equivalent to a factorisation theorem for loop groups, and the solution can be given explicitly by following the argument in [**19**].) Moreover one can show that this solution gives the absolute minimum of the WZW functional E^{WZW}.

The geodesic result can be seen as a particular case of the 2-dimensional theory, by taking circle-invariant maps of the cylinder. Following through the analogy developed in this paper, and expressing things in terms of the Monge–Ampère equation, we are led to the following:

CONJECTURE/QUESTION 5. *Let R be a compact Riemann surface with boundary and $\rho : V \times \partial R \to \mathbf{R}$ a function such that $\omega_0 + i\overline{\partial}\partial\rho$ is a strictly positive $(1,1)$ form on each slice $V \times \{z\}$ for each fixed $z \in \partial R$. Let S_ρ be the set of functions Φ on $V \times R$ equal to ρ over the boundary and such that $\omega_0 + i\overline{\partial}\partial\Phi$ is strictly positive on every slice $V \times \{w\}, w \in R$. Then there is a unique solution of the Monge–Ampère equation $(\Omega_0 + i\overline{\partial}\partial\Phi)^{n+1} = 0$ in S_ρ, and this solution realises the absolute minimum of the functional I_ρ.*

This question is a version of the Dirichlet problem for the degenerate Monge–Ampère equation, a topic around which there is a substantial literature; see [**1**], [**13**] for example. Note that regularity questions are very important in this theory, since the equation is not elliptic: it may be that it is better to work in some larger class of functions. But the statement above should indicate the general direction of the problem.

An affirmative answer to Question 5 would of course imply the corresponding statements for geodesics. In finite dimensions the other familiar existence result for geodesics is the initial value problem, that is, the definition of the exponential map. It is easy to see that the analogue of these results *fails* for $\mathcal{H}$. This is clear from the point of view of Section 5, since we can construct a Hamiltonian trajectory γ_v which does not extend to a holomorphic strip, even for a short "time" interval. In the case of the S^1 invariant metrics considered in Section 6 (iii) we can solve the initial value proble for a short time, since U is an open set, but not for all time, since U is a proper subset of a pre-Hilbert space. The lack of solutions to the initial value problem in $\mathcal{H}$ can be contrasted with the "dual" space $\mathcal{G}$, where the exponential map *can* be defined (*viz.* Hamiltonian dynamics), but points in $\mathcal{G}$ *cannot* in general be joined by a geodesic (there are exact symplectomorphisms arbitrarily close to the identity which are not obtained by Hamiltonian flows). The existence conjecture (5), if true, would thus lead to a pleasing symmetry between the two cases.

The only part of (5) which seems to be easily dealt with is the uniqueness. This can be proved by a maximum principle argument which we give now, although it is a variant of a standard argument in the literature on Monge–Ampère equations.

LEMMA 6. *Suppose $\tilde{\Omega}$ is a $(1,1)$ form on $V \times R$ such that $\tilde{\Omega} \geq 0$ on $V \times R$ and $\tilde{\Omega} > 0$ on each V-slice in $V \times R$. Suppose that $f : V \times R \to \mathbf{R}$ is a smooth function such that $(\tilde{\Omega} + i\overline{\partial}\partial f)^{n+1} = 0$ in $V \times R$. Then the maximum value of f is attained on the boundary.*

To prove this lemma, choose a function $\gamma : R \to \mathbf{R}$ with $i\overline{\partial}\partial\gamma > 0$ on R. Then for all $\epsilon > 0$ the form $\tilde{\Omega}_\epsilon = \tilde{\Omega} + \epsilon i\overline{\partial}\partial\gamma$ (where γ is regarded as a function on $V \times R$) is strictly positive on $V \times R$. Let $f_\epsilon = f - \epsilon\gamma$, so $(\tilde{\Omega}_\epsilon + i\overline{\partial}\partial f_\epsilon)^{n+1} = 0$. At an interior maximum of f_ϵ we have $i\overline{\partial}\partial f_\epsilon \geq 0$, and this would give a contradiction, since we would have $\tilde{\Omega}_\epsilon + i\overline{\partial}\partial f_\epsilon > 0$; so the maximum value of f_ϵ is attained on the boundary. Now take the limit as ϵ tends to 0 to deduce, just as in the proof of the ordinary maximum principle, that the maximum value of f is attained on the boundary.

COROLLARY 7. *If one solution Φ to the boundary value problem in Conjecture/Question* 5 *exists, it is unique.*

For, if Φ' were another solution, we could take $\tilde{\Omega} = \Omega_0 + i\bar{\partial}\partial\Phi$ and $f = \Phi' - \Phi$ in Lemma 6 and deduce that $f \leq 0$. Changing the roles of Φ and Φ', we deduce that $f \geq 0$, so we must have $f = 0$.

In a similar way we get another extremal characterisation of solutions, again following standard lines, cf. [**13**]. Given boundary data ρ, let $\mathcal{S}^+(\rho)$ be the set of functions Ψ on $V \times R$ such that $\Psi = \rho$ on the boundary and $\Omega_0 + i\bar{\partial}\partial\Psi > 0$ in $V \times R$. This set is not empty, for we can first produce a function Ψ_0 equal to ρ on the boundary with $\Omega_0 + i\bar{\partial}\partial\Psi_0$ strictly positive on V-slices, using the fact that $\mathcal{H}$ is contractible, and then find Ψ in $\mathcal{S}^+(\rho)$ in the form $\Psi = \Psi_0 + K\gamma$, where γ is a function on R as in the proof of Lemma 6, but chosen so that in adddition $\gamma = 0$ on ∂R, and the parameter K is made large.

PROPOSITION 8. (i) *Given ρ as in Conjecture/Question* 5 *and any $\Psi \in \mathcal{S}^+(\rho)$, the minimum value of Ψ is attained on the boundary. In particular, for any $p \in V \times R$,*

$$\inf_{\Psi \in \mathcal{S}^+(\rho)} \Psi(p) > -\infty.$$

(ii) *If a solution $\Phi \in \mathcal{S}(\rho)$ of the boundary value problem in Conjecture/Question* 5 *exists, then, for all $p \in V \times R$,*

$$\Phi(p) = \inf_{\Psi \in \mathcal{S}^+(\rho)} \Psi(p).$$

To prove (i) we just use the fact that $i\bar{\partial}\partial\Psi$ is strictly positive in the R-slices to see that Ψ has no interior minimum. Part (ii) follows immediately from Lemma 6, together with fact that, if Φ is a solution and we choose a function γ as above, vanishing on the boundary, then $\Phi + \epsilon\gamma$ lies in $\mathcal{S}^+_\rho$, for any $\epsilon > 0$.

Of course we could try to view the infimum appearing in Proposition 8 as *defining* a "generalised solution" of the Dirichlet problem.

The discussion above focuses on the solutions to the boundary value problem. On the other hand, leaving aside the questions of the existence of these extrema, it is interesting to study the infimum of the functional I_ρ. This is particularly so in the case of the geodesic problem, when the functional can be rewritten as the "energy" of a path. If these infima are strictly positive, for all choices of fixed, distinct, end points, they make $\mathcal{H}$ into a metric space, in the usual fashion. In this connection we will now observe that *assuming* the existence of minimising geodesics we can write down an explicit lower bound for the functional. To do this we observe that for any $\phi \in \mathcal{H}$

$$\int_V \phi \, d\mu_0 \leq I(\phi) \leq \int_V \phi \, d\mu_\phi.$$

These inequalities just express the *convexity* of the functional I; to derive them, differentiate along the path $t(\phi)$. It follows that if ϕ is normalised (i.e. $I(\phi) = 0$) and not identically zero, then it must take both strictly positive and negative values. Now suppose that $\phi_t, t \in [0,1]$, is a geodesic from 0 to ϕ, where ϕ is normalised. Then the length (or energy) of the geodesic is given by

$$L = \int_V \dot{\phi}^2 \, d\mu_{\phi_t},$$

for any $t \in [0,1]$. In particular, taking $t = 0$,

$$\sqrt{L} \geq M^{-1/2} \int_V |\dot{\phi}_0| d\mu_0 > M^{-1/2} \int_{\dot{\phi}_0 > 0} \dot{\phi}_0 d\mu_0,$$

where M is the volume of V (which is of course the same for all metrics in $\mathcal{H}$). Now the geodesic equation obviously gives $\ddot{\phi} \leq 0$, so $\phi \leq \dot{\phi}_0$. It follows that

$$\int_{\dot{\phi}_0 > 0} \dot{\phi} d\mu_0 \geq \int_{\phi > 0} \phi d\mu_0,$$

where the last term is strictly positive by the remarks above, and depends only on ϕ and not on the geodesic. A similar argument gives

$$\sqrt{L} > -M^{-1/2} \int_{\phi < 0} \phi d\mu_\phi.$$

So we are lead to a conjecture whose statement does not involve the existence of geodesics:

CONJECTURE/QUESTION 9. *If $\phi \in \mathcal{H}_0$ is normalised and $\tilde{\phi}_t$, $t \in [0,1]$, is any path from 0 to ϕ in $\mathcal{H}$, then*

$$\int_0^1 \int_V \left(\frac{d\tilde{\phi}}{dt} \right)^2 d\mu_{\tilde{\phi}_t} dt \geq M^{-1} \left(\max \left(\int_{\phi > 0} \phi d\mu_0, - \int_{\phi < 0} \phi d\mu_\phi \right) \right)^2.$$

The restriction to normalised potentials ϕ is not important, since we know that $\mathcal{H}$ splits as a product, and we could immediately write down a corresponding inequality, involving $I(\phi)$, for any $\phi \in \mathcal{H}$.

§8. Extremal Kähler metrics

In this section we will explain how the material we have developed in this paper is related to certain well-known topics in Kähler geometry. Consider first the question, posed by Calabi [**4**] and solved by Yau [**22**], of finding a Kähler metric in $\mathcal{H}_0$ with a prescribed volume form. A volume form gives a linear functional $\lambda : C^\infty(V) \to \mathbf{R}$, and the space $\mathcal{H}$ of Kähler potentials is an open subset of $C^\infty(V)$. By the definition of the functional I, finding a solution to the problem is the same as finding a $\phi \in \mathcal{H}_0$ such that the derivative of I at ϕ is λ; that is, we have to minimise the the linear functional λ over the convex set $\{\phi \in \mathcal{H} : I(\phi) \leq 0\}$. This variational formulation is well known; the point we want to bring out is that the prescribed-volume problem essentially involves the *affine* geometry of $\mathcal{H}$. (The problem discussed in the previous section is a variant of this prescribed-volume problem, in that we have fixed boundary values and the prescribed "volume form" is 0.)

There is another problem, again going back to Calabi [**5**], which is tightly bound up with the geometry of $\mathcal{H}$ as a symmetric space which we have discussed in this paper. This is the problem of finding an "extremal Kähler metric" in $\mathcal{H}$, which, by definition, means a metric of *constant scalar curvature*. This problem includes the renowned question of the existence of Kähler–Einstein metrics, in the case when $c_1(V)$ is a multiple of $[\omega_0]$. The link with the symmetric-space geometry of $\mathcal{H}$ arises from the fact that the scalar curvature can be viewed as a *moment map* for the action of $\mathcal{G}$ on the symplectic manifold $\mathcal{J}$. This means that, save for the absence of a genuine complexified group $\mathcal{G}^c$, the search for critical Kahler metrics can be fitted into a general pattern of problems involving "Kahler quotients", see [**9**, Chapter 6] or [**12**], for example. This is explained in [**7**], so we will not reproduce the discussion here. The main point is that one could hope that this approach may shed light on Yau's conjecture [**23**] relating the existence of critical Kähler metrics to stability in the sense of Hilbert schemes and geometric invariant theory. (In [**7**] this was referred to a Tian's conjecture, but Tian has kindly pointed out to the author that it is due to Yau. The original conjecture is stated only for the Kähler–Einstein case, but it seems natural to extend it to general critical metrics. Such an extension has been suggested by LeBrun, particularly in connection with ruled surfaces, where Burns and de Bartolomeis [**3**] and LeBrun [**14**] have shown that the existence of critical metrics is tied up with the stability of the corresponding vector bundles.)

To bring this down to earth, consider the question of the uniqueness of critical metrics. For any metric ϕ in $\mathcal{H}$ let $\mathcal{D}_\phi$ be the second-order differential operator from functions on V to sections of $s^2_{\mathbf{C}}(TV)$, given by

$$\mathcal{D}_\phi f = \overline{\partial}_{TV}((df)^\sharp),$$

where $(df)^\sharp$ is the vector field dual to the 1-form df under the metric ω_ϕ and $\overline{\partial}_{TV}$ is the $\overline{\partial}$-operator on the tangent bundle of V. (This operator is the "complex Hessian": in flat space it is given in local coordinates by $\mathcal{D}(f) = \frac{\partial^2 f}{\partial \overline{z}_\alpha \partial \overline{z}_\beta}$; in general $\mathcal{D}_\phi f$ is the component of the Riemannian Hessian $\nabla\nabla f$ complementary to $\overline{\partial}\partial f$.) Write $S(\phi)$ for the scalar curvature of a metric ω_ϕ.

PROPOSITION 10. *If ϕ_t is a geodesic in $\mathcal{H}$, then*

$$\frac{d}{dt}\int_V \dot\phi S(\phi_t) d\mu_{\phi_t} = \int_V |\mathcal{D}_{\phi_t}\dot\phi|^2 d\mu_{\phi_t}.$$

This equation can been seen most easily in the $\mathcal{J}$-description, with a fixed symplectic form and varying complex structure. Then we write $\mathcal{D}_J$ for the operator above formed using a complex structure J. The reason these operators are important is that they define the infinitesimal action of the symplectomorphism group on $\mathcal{J}$ (or, more precisely, on the subset of integrable structures, see the discussion in [**7**]). A geodesic in $\mathcal{H}$ goes over to a path $J(t)$ such that

$$\frac{dJ(t)}{dt} = I\mathcal{D}_{J(t)} H,$$

for some *fixed* function H, independent of t. So

$$\frac{d}{dt}\int_V HS d\mu = \int_V H\frac{dS}{dt} d\mu.$$

Now the moment map identity established in [**7**] states that for any variation δJ in $\mathcal{J}$, and any function h,

$$\int_V (\delta J, I\mathcal{D}_J h) d\mu = \int_V h\delta S \, d\mu,$$

so

$$\int_V H\frac{dS}{dt} d\mu = \int_V (I\mathcal{D}_J H, I\mathcal{D}_J H) d\mu = \|\mathcal{D}_J H\|^2,$$

and transforming back to the other description, with a fixed complex structure, this gives the equation stated in Proposition 10.

COROLLARY 11. *If ϕ_0, ϕ_1 are two critical Kähler metrics in $\mathcal{H}_0$ which can be joined by a geodesic, then there is a holomorphic automorphism α of V such that $\alpha^*(\omega_{\phi_0}) = \omega_{\phi_1}$.*

This follows from Proposition 10 because if ϕ_t is the geodesic the derivative of $\int_V S\dot{\phi} d\mu$ vanishes at the endpoints by hypothesis, and so we must have $\mathcal{D}_{\phi_t}\dot{\phi} = 0$ for all t. But this means that $\phi_t = \alpha_t^*\phi_0$, where $\alpha_t = \exp(t\xi)$ for a holomorphic vector field ξ on V, i.e. we are in the situation considered in example (i) of Section 6. Another way of expressing this corollary is that if the geodesic existence conjecture (5) is true then we can deduce the essential uniqueness of critical Kähler metrics. In the Kähler–Einstein case this uniqueness has been proved by Bando and Mabuchi [**2**], using a different method, but the question for general critical metrics seems to be open.

The significance of Proposition 10 becomes clearer if we recall that Mabuchi [**15**] has shown that one can define a functional, the "K-energy", on $\mathcal{H}$ by specifying its first variation to be

$$\delta K = \int_V \delta\phi \ S(\phi) \ d\mu_\phi.$$

Thus the critical Kähler metrics are the critical points of K on $\mathcal{H}_0$. (This definition is in a similar vein to that of the functional I: one has to check that the second variation is symmetric in its two arguments to see that K is well-defined. It fits into the general pattern of Kähler quotient theory, in that there is a functional defined in a similar way, using the moment map, for any Kähler quotient.) Then Corollary 11 is the statement that K is *convex* along geodesics in $\mathcal{H}$, a fact which again fits into the general pattern, compare [**9**], Chapter 6, for example.

We give two other illustrations of the use of this convexity of the Mabuchi fuctional.

First, we consider certain modifications of the geodesic equation given by the motion of a "particle" in $\mathcal{H}$ moving in the potential $-\lambda K$, where $\lambda \geq 0$ is a real parameter. The equation of motion is

$$\ddot{\phi} = -\tfrac{1}{2}|\dot{\phi}|_\phi^2 + \lambda S(\phi),$$

where S is the scalar curvature. Along such a path we have

$$\ddot{J} = \|\mathcal{D}_\phi\dot{\phi}\|^2 + \lambda\|S\|^2 \geq 0,$$

so these paths would do equally well for proving the uniqueness (Corollary 11). It seems reasonable to hope that any two points can be joined by a path of this kind, and the equation for $\lambda > 0$ has the virtue of being *elliptic* on $V \times [0,1]$.

Second, this point of view is fruitful in understanding the Calabi equation [**5**]

$$\dot{\phi} = -S(\phi) + S_0,$$

where S_0 is the average of the scalar curvature, which is a topological invariant of $(V, [\omega])$. This is the gradient flow equation of the functional K on $\mathcal{H}_0$, and on the other hand it generates the gradient flow of the functional $\|S\|^2$ on $\mathcal{J}$: the formal structure is the same as for the Hermitian Yang–Mills gradient flow discussed in [**9**], for example. Calabi's inequality

$$\frac{d}{dt}\|S - S_0\|^2 = -\|\mathcal{D}S\|^2$$

follows immediately from the moment map identity. (Chruściel [6] has proved that when V is a Riemann surface the initial value problem for this Calabi equation has a solution for all positive time, which converges to the constant curvature metric as $t \to \infty$.)

Turning to the existence of extremal Kähler metrics: one of the cornerstones of the finite dimensional Kähler quotient theory, in its algebro-geometric formulation as geometric invariant theory, is the "Hilbert criterion". This states that the stability of a complex orbit can be detected by looking at complex one-parameter subgroups $\mathbf{C}^* \subset G^c$. If one takes these ideas over to the case of $\mathcal{J}$, we do not have a genuine complex group $\mathcal{G}^c$, but the geodesics in $\mathcal{H}$ furnish a substitute for the 1-parameter subgroups. In this way one can formulate a statement which is the analogue of the Hilbert criterion for stability (where stability is interpreted in terms of zeros of the moment map) in our infinite dimensional situation:

CONJECTURE/QUESTION 12. *The following are equivalent:*
(1) *There is no critical Kähler metric in* $\mathcal{H}_0$.
(2) *There is an infinite geodesic ray* $\phi_t, t \in [0, \infty)$, *in* $\mathcal{H}$, *such that*

$$\int_V S\dot{\phi} d\mu_\phi < 0$$

for all $t \in [0, \infty)$.
(3) *For any point* $\phi \in \mathcal{H}_0$ *there is a geodesic ray as in* (2) *starting at* ϕ.

Notice that, by Proposition 10, it is sufficient that the inequality in part (2) of this conjecture should hold for *large* t; that is, the condition has to do with the asymptotics of the geodesic ray. A further step in this programme would be to find conditions for the existence of these rays in terms of the complex geometry of V, perhaps making contact with other work, by Tian [**20**], [**21**], Nadel [**18**], and others, on obstructions to the existence of Kähler–Einstein metrics.

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