LECTURES ON GEODESICS IN THE SPACE OF KAEHLER METRICS, LECTURE 4: QUANTIZATION

STEVE ZELDITCH

Another facet of geodesics is GAT or "geometric approximation theory". It is very difficult to work directly with infinite dimensional geometry and GAT is a special method in projective Ḱahler geometry to make fiinite dimensional approximations of the infinite dimensional locally symmetric space $\mathcal{H}_\omega$ by genuine finite dimensional symmetric spaces of type $G_{\mathbb{C}}/G$ with $G = SU(N)$. The compact group $SU(N)$ is analogous to $SDiff(M, \omega_0)$ and $G_{\mathbb{C}} = SL(N, \mathbb{C})$ is analogous to $\Upsilon = \{(f, \omega_\varphi) : f^*\omega_\varphi = \omega_0\}$.

The approximating spaces are known as Bergman metric spaces of degree k and denoted by $\mathcal{B}_k$. They are submanifolds $\mathcal{B}_k \subset \mathcal{H}_\omega$ but not totally geodesic ones. There is also a canonical map $\mathrm{Hilb_k} : \mathcal{H}_\omega \to \mathcal{B}_\mathrm{k}$. The composite map $\mathcal{B}_k \subset \mathcal{H}_o \to \mathcal{B}_k$ is a complicated map denoted by $T_k : \mathcal{B}_k \to \mathcal{B}_k$.

Geodesics of $\mathcal{B}_k$ are induced by geodesics of $SL(N, \mathbb{C})/SU(N)$, which are given by the action of one parameter subgroups e^{tA}. Hence Monge-Amp'ere geodesics of $\mathcal{H}_\omega$ are limits, in some sense, of curves of Bergman Kaehler metrics induced by one parameter subgroups. This was first explored in [PS2], in which a kind of almost everywhere convergence was proved, then uniform convergence was proved in [B, B2]. In the special case of toric Kaehler manifolds, C^2 convergence was proved in [SZ].

We assume throughout that (M, ω_0) is a projective Kaehler manifold. Thus there exists a quantizing line bundle $L \to M$ and a Hermitian metric h on L with curvature form ω_0.

The ideas in these notes are due to Yau, Tian, Donaldson and many others, but the source of the results is not usually indicated. We refer to [PS3] for the historical and further mathematical background. Much of the notes is copied-pasted from prior articles of the author, again without attribution.

1. Bergman metric spaces

Bergman metrics of degree k are special Kähler metrics induced by holomorphic embeddings

$$\iota_s(z) = [s_1, \dots, s_{N_k}] : M \to (\mathbb{CP}^{N_k - 1}, \omega_{FS})$$

of M into complex projective space.

Let $H^0(M, L^k)$ denote the space of holomorphic sections of the kth power $L^k \to M$ of L and let $d_k + 1 = \dim H^0(M, L^k)$. We let $\mathcal{B}H^0(M, L^k)$ denote the manifold of all bases $\underline{s} = \{s_0, \dots, s_{d_k}\}$ of $H^0(M, L^k)$. Given a basis, we define the Kodaira embedding

$$\iota_{\underline{s}} : M \to \mathbb{CP}^{d_k}, \;\; z \to [s_0(z), \dots, s_{d_k}(z)]. \tag{1}$$

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We then define a Bergman metric (or equivalently, Fubini-Study) metric of height k to be a metric of the form

$$h_{\underline{s}} := (\iota_{\underline{s}}^* h_{FS})^{1/k} = \frac{h_0}{\left(\sum_{j=0}^{d_k} |s_j(z)|^2_{h_0^k}\right)^{1/k}}, \tag{2}$$

where h_{FS} is the Fubini-Study Hermitian metric on $\mathcal{O}(1) \to \mathbb{CP}^{d_k}$. The space of all Bergman metrics of degree k is then,

$$\mathcal{B}_k = \{h_{\underline{s}}, \ \ \underline{s} \in \mathcal{B}H^0(M, L^k)\}. \tag{3}$$

We use the same notation for the associated space of potentials φ such that $h_{\underline{s}} = e^{-\varphi} h_0$ and for the associated Kähler metrics ω_φ.

Given a reference basis $\{s_j\}$ one obtains all others by applying an element $A \in GL(N_k, \mathbb{C})$ to it. The new basis $s^A{}_j = \sum A_{jl} s_l$ induces the embedding

$$\iota_{s_A} : M \to \mathbb{CP}^{N_k - 1}, \ \ \iota_{s_A} = A \circ \iota_s,$$

and the associated Bergman metric is,

$$\iota_{s_A}^* \omega_{FS} = \frac{1}{k} i\partial\bar{\partial} \log \sum_{j=1}^{N_k} |s^A{}_j(z)|^2. \tag{4}$$

Since $U(N_k)$ is the isometry group of ω_{FS}, the space of metrics is the quotient symmetric space $\mathcal{P}_{N_k} = GL(N_k, \mathbb{C})/U(N_k)$. With no loss of generality one may restrict to $SL(N_k, \mathbb{C})$ and obtain the quotient $SL(N_k, \mathbb{C})/SU(N_k)$.

1.1. **Quantization of Hermitian metrics as inner products.** We denote by $\mathcal{I}_k$ the space of Hermitian inner products on the finite dimensional vector space $H^0(M, L^k)$. If we fix one reference inner product G_0, then any other may be represented by a positive Hermitian operator relative to G. If we also fix a basis, an inner product is represented by a positive Hermitian matrix. So $\mathcal{I}_k \simeq \mathcal{P}_k$, the positive Hermitian matrices of rank N_k.

As in [D1, D4], we define maps

$$Hilb_k : \mathcal{H} \to \mathcal{I}_k,$$

by the rule that a Hermitian metric $h \in \mathcal{H}$ induces the inner products on $H^0(M, L^k)$,

$$||s||^2_{Hilb_k(h)} = R \int_M |s(z)|^2_{h^k} dV_h, \tag{5}$$

where $dV_h = \frac{\omega_h^m}{m!}$, and where $R = \frac{d_k + 1}{Vol(M, dV_h)}$. Also, h^k denotes the induced metric on L^k.

The sequence $\{\mathrm{Hilb_k}(\varphi)\}$ of inner products induced by $\varphi \in \mathcal{H}_\omega$ is thought of as the quantization of φ.

1.2. **Converting inner products to Bergman metrics.** Further, we define the identifications

$$FS_k : \mathcal{I}_k \simeq \mathcal{B}_k$$

as follows: an inner product $G = \langle \ , \ \rangle$ on $H^0(M, L^k)$ determines a G-orthonormal basis $\underline{s} = \underline{s}_G$ of $H^0(M, L^k)$ and an associated Kodaira embedding (1) and Bergman metric (2). Thus,

$$FS_k(G) = h_{\underline{s}_G}. \tag{6}$$

The right side is independent of the choice of h_0 and the choice of orthonormal basis. As observed in [D1, PS1], $FS_k(G)$ is characterized by the fact that for any G-orthonormal basis $\{s_j\}$ of $H^0(M, L^k)$, we have

$$\sum_{j=0}^{d_k} |s_j(z)|^2_{FS_k(G)} \equiv 1, \quad (\forall z \in M). \tag{7}$$

1.3. **Geometric Approximation theory.** Metrics in $\mathcal{B}_k$ are defined by an algebro-geometric construction. By analogy with the approximation of real numbers by rational numbers, we say that $h \in \mathcal{H}$ (or its curvature form ω_h) has *degree* k if $h \in \mathcal{B}_k$. A basic fact is that the union

$$\mathcal{B} = \bigcup_{k=1}^{\infty} \mathcal{B}_k$$

of Bergman metrics is dense in the C^∞-topology in the space $\mathcal{H}$. Indeed,

$$\frac{FS_k \circ Hilb_k(h)}{h} = 1 + O(k^{-2}), \tag{8}$$

where the remainder is estimated in $C^r(M)$ for any $r > 0$; left side moreover has a complete asymptotic expansion (see [PS2] for precise statements).

1.4. **Bergman metrics in terms of positive Hermitian matrices.** We have fixed a reference metric ω_0, and it determines reference Bergman metrics $\omega_0(k)$ by GAT. We choose a basis of sections $\{s_i(z)\} = \{s_1(z), ..., s_{N_k}(z)\}$ of $H^0(M, L^k)$ which is orthonormal with respect to the reference (background) metric h_0^k on L^k and the corresponding Kähler metric $\omega_0 = -\frac{1}{k} i\partial\bar{\partial} \log h_0^k$ on M

$$\frac{1}{V} \int_M \bar{s}_i(z) s_j(z) h_0^k \frac{\omega_0^n}{n!} = \delta_{ij}, \tag{9}$$

where $n = \dim M$. The Bergman kernel of the background metric is the kernel of the orthogonal projection onto $H^0(M, L^k)$ with respect to the inner product above, and is given by

$$B_k(z_1, z_2) = \sum_{j=1}^{N_k} s_j(z_1) \bar{s}_j(z_2) \tag{10}$$

Given a positive Hermitian matrix $P = P_{ij}$ the associated Bergman metric is,

$$\omega_{a\bar{b}}(z) = \frac{1}{k} \partial_a \bar{\partial}_{\bar{b}} \log \bar{s}_i(z) P_{ij} s_j(z). \tag{11}$$

In terms of $A \in GL(N_k, \mathbb{C})$ above, $P = A^\dagger A$. We introduce the Bergman potential as follows

$$\varphi_P = \frac{1}{k} \log \bar{s}_i(z) P_{ij} s_j(z) = \frac{1}{k} \log |\langle e^\Lambda U s(z), U s(z) \rangle|^2. \tag{12}$$

1.5. **Bergman geodesic rays.** In the symmetric space metric of $\mathcal{B}_k$, a geodesic is

$$\varphi_k(t) = \frac{1}{k} \log |e^{tA} Z(z)|^2.$$

If $Z(z) = [s_\alpha(z)]$ and $e^A = U^* e^{D(\lambda)} U$ then

$$\varphi_k(t) = \frac{1}{k} \log \sum_j e^{t\lambda_j} |s_j^U(z)|^2,$$

where in $SL(N_k, \mathbb{C})/SU(N_k)$ the ray starts at the origin and has initial vector (U, Λ). so

$$\dot{\varphi}_k(0) = \frac{1}{k} \frac{\langle (A + A^*) Z(z), Z(z) \rangle}{||Z(z)||^2} = \iota_{\underline{s}}^* h_A.$$

Here, we use that if $iA \in \mathbf{u(N)}$ then $A^* = A$. In general,

$$\dot{\varphi}_k(t) = \frac{1}{k} \frac{\langle A e^{tA} Z(z), e^{tA} Z(z) \rangle}{||e^{tA} Z(z)||^2} = \iota_{\underline{e^{tA} s}}^* h_A.$$

PROPOSITION **1.1.** *We have,*

$$\frac{d}{dt} \iota_{\underline{s}}^* \left((\exp it \Xi_{h_A}^{\omega_{FS}})^* \omega_{FS} \right) = \mathcal{L}_{J \Xi^{\omega_t}_{h_{e^{tA^*} A e^{tA}}}} \iota_{e^{tA} \underline{s}}^* \omega_{FS}.$$

Proof. We apply the previous formula, but to the basis $e^{tA} \underline{s}$. This new embedding gives a new restriction $h_A|_{e^{tA} \underline{s}(M)} = h_{e^{tA^*} A e^{tA}}$.

□

Now that we have defined the spaces $\mathcal{H}$ and $\mathcal{B}_k$, we can compare Monge-Ampère geodesics and Bergman geodesics. Geodesics of $\mathcal{H}$ satisfy the Euler-Lagrange equations for the energy functional and as in the first Lecture are the paths $h_t = e^{-\varphi_t} h_0$ which satisfy the equation

$$\ddot{\varphi} - \frac{1}{2} |\nabla \dot{\varphi}|^2_{\omega_\varphi} = 0, \tag{13}$$

which may be interpreted as a homogeneous complex Monge-Ampère equation on $A \times M$ where A is an annulus. It is not hard to see that Bergman geodesic rays are not the same as Mabuchi geodesic rays, but they are 'sub-solutions', i.e. have positive Monge-Ampère mass in space-time.

Geodesics in $\mathcal{B}_k$ with respect to the symmetric space metric are given by orbits of certain one-parameter subgroups $\sigma_k^t = e^{tA_k}$ of $GL(d_k + 1, \mathbb{C})$. In the identification of $\mathcal{B}_k$ with the symmetric space $\mathcal{I}_k \simeq GL(d_k + 1, \mathbb{C})/U(d_k + 1)$ of inner products, the 1 PS (one parameter subgroup) $e^{tA_k} \in GL(d_k + 1)$ changes an orthonormal basis $\hat{\underline{s}}^{(0)}$ for the initial inner product G_0 to an orthonormal basis $e^{tA_k} \cdot \hat{\underline{s}}^{(0)}$ for G_t where G_t is a geodesic of $\mathcal{I}_k$. Geometrically, a Bergman geodesic may be visualized as the path of metrics on M obtained by holomorphically embedding M using a basis of $H^0(M, L^k)$ and then moving the embedding under the 1 PS subgroup e^{tA_k} of motions of $\mathbb{CP}^{d_k}$. The difficulty is to interpret this simple extrinsic motion in intrinsic terms on M.

2. Phong-Sturm endpoint theorem

GIven $h_0, h_1 \in \mathcal{H}$, let $h(t)$ denote the Monge-Ampère geodesic between them. We then consider the geodesic $G_k(t)$ of $\mathcal{I}_k$ between $G_k(0) = Hilb_k(h_0)$ and $G_k(1) = Hilb_k(h_1)$ or equivalently between $FS_k \circ Hilb_k(h_0)$ and $FS_k \circ Hilb_k(h_1)$. Without loss of generality, we may assume that the change of orthonormal basis (or change of inner product) matrix $\sigma_k = e^{A_k}$ between $Hilb_k(h_0), Hilb_k(h_1)$ is diagonal with entries $e^{\lambda_0}, ..., e^{\lambda_{d_k}}$ for some $\lambda_j \in \mathbb{R}$. Let $\hat{\underline{s}}^{(t)} = e^{tA_k} \cdot \hat{\underline{s}}^{(0)}$ where e^{tA_k} is diagonal with entries $e^{\lambda_j t}$. Define

$$h_k(t) := FS_k \circ G_k(t) = h_{\hat{\underline{s}}^{(t)}} =: h_0 e^{-\varphi_k(t)}. \tag{14}$$

It follows immediately from (7) that

$$\varphi_k(t;z) \; = \; \frac{1}{k} \log \left(\sum_{j=0}^{N} e^{2\lambda_j t} |\hat{s}_j^{(0)}|^2_{h_0^k} \right). \tag{15}$$

We emphasize that $\varphi_k(t;z)$ is the intrinsic $\mathcal{B}_k$ geodesic between the endpoints $FS_k \circ Hilb_k(h_0)$ and $FS_k \circ Hilb_k(h_1)$. It is of course quite distinct from the $Hilb_k$-image of the Monge-Ampère geodesic; the latter is not intrinsic to $\mathcal{B}_k$ and one cannot gain any information on the $\mathcal{H}$-geodesic by studying it.

The main result of Phong-Sturm [PS1] is that the Monge-Ampère geodesic φ_t is approximated by the 1PS Bergman geodesic $\varphi_k(t,z)$ in the following weak C^0 sense:

$$\varphi_t(z) = \lim_{\ell \to \infty} \left[\sup_{k \geq \ell} \varphi_k(t,z) \right]^*, \text{ uniformly as } \ell \to \infty, \tag{16}$$

where u^* is the upper envelope of u, i.e., $u^*(\zeta_0) = \lim_{\varepsilon \to 0} \sup_{|\zeta - \zeta_0| < \varepsilon} u(\zeta)$. In particular, without taking the upper envelope, $\sup_{k \geq \ell} \varphi_k(t,z) \to \varphi(t,z)$ almost everywhere as $\ell \to \infty$. See also [B] for the subsequent proof of an analogous result for the adjoint bundle $L^k \otimes K$ (where K is the canonical bundle) with an error estimate $||\varphi_k(t) - \varphi(t)||_{C^0} = O(\frac{\log k}{k})$.

In [SZ] it is proved that convergence is in C^2 in the case of toric Kaehler manifolds.

Let us summarize the notation for hermitian metrics and geodesics of metrics:

- For any metric h on L, h^k denotes the induced metric on L^k, and for any metric H on L^k, $H^{\frac{1}{k}}$ is the induced metric on L;
- Given $h_0 \in \mathcal{H}$, $h_t = e^{-\varphi_t} h_0$ is the Monge-Ampère geodesic;
- $h_k = FS \circ Hilb_k(h) \in \mathcal{B}_k$ is the natural approximating Bergman metric to h, and $h_k(t) = e^{-\varphi_k(t)} h_0$ is the Bergman geodesic (14).

2.1. **The initial value problem.** In [RZ] the initial value problem for geodesic rays is studied from the quantization viewpoint. Consider the Hilbert spaces of sections $L^2(M, L^N), N \in \mathbb{N}$, associated to powers of a Hermitian line bundle (L, h_0) polarizing (M, ω_{φ_0}), and the corresponding orthogonal projection operators $\Pi_N \equiv \Pi_{N,\varphi_0} : L^2(M, L^N) \to H^0(M, L^N)$, onto the Hilbert subspaces $H^0(M, L^N)$ of holomorphic sections.

Consider the self-adjoint zeroth-order Hermitian Toeplitz operators $\Pi_N \dot{\varphi}_0 \Pi_N$, where $\dot{\varphi}_0$ denotes the operator of multiplication by $\dot{\varphi}_0$. Define the associated one-parameter subgroups of unitary operators on $H^0(M, L^N)$

$$U_N(t) := \Pi_N e^{\sqrt{-1} t N \Pi_N \dot{\varphi}_0 \Pi_N} \Pi_N. \tag{17}$$

There is no obstruction to analytically continuing the quantization: each $U_N(t)$ admits an analytic continuation in time t and induces the imaginary time subgroup

$$U_N(-\sqrt{-1}s) : H^0(M, L^N) \to H^0(M, L^N), \tag{18}$$

with $U_N(-\sqrt{-1}s) \in GL(H^0(M, L^N), \mathbb{C})$. Set

$$\varphi_N(s, z) := \frac{1}{N} \log U_N(-\sqrt{-1}s, z, z). \tag{19}$$

Definition **2.1.** *Call $\varphi_\infty(s, z) := \lim_{l\to\infty}(\sup_{N\geq l} \varphi_N)_{\text{reg}}(s, z)$ the quantum analytic continuation potential, where u_{reg} denotes the upper semicontinuous regularization of u.*

The limit φ_∞ is constructed out of the quantized potentials φ_N similarly to the geodesic rays constructed by Phong–Sturm, by using upper envelopes.

Denote $S_T := [0, T] \times \mathbb{R}$. The IVP for geodesics is equivalent to the following Cauchy problem for the homogeneous complex Monge–Ampère equation:

$$\begin{aligned} (\pi_2^\star\omega + \sqrt{-1}\partial\bar\partial\varphi)^{n+1} = 0 \qquad & \text{on } S_T \times M, \\ \varphi(0, s, \cdot\,) = \varphi_0(\,\cdot\,), \quad \partial_s\varphi(0, s, \cdot\,) = \dot\varphi_0(\,\cdot\,) \ & \text{on } \{0\} \times \mathbb{R} \times M. \end{aligned} \tag{20}$$

Definition **2.2.** *The smooth lifespan (respectively, lifespan) of the Cauchy problem* (20) *is the supremum over all $T \geq 0$ such that* (20) *admits a smooth (respectively $\pi_2^\star\omega$-psh) solution. We denote the smooth lifespan (respectively, lifespan) for the Cauchy data $(\omega_{\varphi_0}, \dot\varphi_0)$ by $T^\infty_{\text{span}} \equiv T^\infty_{\text{span}}(\omega_{\varphi_0}, \dot\varphi_0)$ (respectively, $T_{\text{span}} \equiv T_{\text{span}}(\omega_{\varphi_0}, \dot\varphi_0)$).*

Definition **2.3.** *The quantum lifespan T^Q_{span} of the Cauchy problem* (20) *is the supremum over all $T \geq 0$ such that φ_∞ of Definition 2.1 solves the HCMA* (20).

We pose the following conjecture, which would give a general method to solve the Cauchy problem for the HCMA to the extent possible.

Conjecture **2.4.** *The quantum analytic continuation potential φ_∞ solves the HCMA* (20) *for as long as it admits a solution. In other words, $T^Q_{\text{span}} = T_{\text{span}}$.*

The conjecture is proved by toric Kaehler manifolds in [RZ]. It is also true for real analytic metrics.

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Department of Mathematics, Northwestern University, Evanston, IL 60208, USA
E-mail address, S. Zelditch: zelditch@math.northwestern.edu