LECTURES ON GEODESICS IN THE SPACE OF KAEHLER METRICS AND HELE-SHAW FLOWS: LECTURE 1

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CAVEAT LECTOR: these notes are a guide to the four lectures I am giving on July 6-10, 2015 at Northwestern University. The sources of the results of the first lectures are [D1, M, S, RZ] (see also [Ch, Dar, DL] for further background and more recent results). The notes are sometimes taken almost verbatim from the sources, and I may have forgotten to indicate that in some places.

1. Main results proved/discussed Lecture 1

- The Mabuchi-Semmes-Donaldson Riemannian metric on the space of Kaehler metrics in a fixed class.
- Geodesics in the space of Kaehler metrics. The Riemannian Connection.
- Reformulation of geodesic equation as an HCMA. Null foliation.
- Null leaves as complexified Hamiltonian orbits. Moser maps. Formal solution.
- Solvability: the HRMA in $1+1$ dimension.

2. Geodesics

Let M be a complex manifold. We use the following standard notation: $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. We often find it convenient to use the real operators $d = \partial + \bar{\partial}$, $d^c := \frac{i}{4\pi}(\bar{\partial} - \partial)$ and $dd^c = \frac{i}{2\pi}\partial\bar{\partial}$.

Let $L \to M$ be a holomorphic line bundle. The Chern form of a Hermitian metric h on L is defined by

$$c_1(h) = \omega_h := -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\|e_L\|_h^2\,, \tag{1}$$

where e_L denotes a local holomorphic frame (= nonvanishing section) of L over an open set $U \subset M$, and $\|e_L\|_h = h(e_L, e_L)^{1/2}$ denotes the h-norm of e_L. We say that (L, h) is positive if the (real) 2-form ω_h is a positive $(1,1)$ form, i.e., defines a Kähler metric. A complex valued 2-form is of type $(1,1)$ precisely if it satisfies $\omega(Jv, Jw) = \omega(v, w)$. The Kaehler form is real and of type $(1,1)$.

We write $\|e_L(z)\|_h^2 = e^{-\varphi}$ or locally $h = e^{-\varphi}$, and then refer to φ as the Kähler potential of ω_h in U. In this notation,

$$\omega_h = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi = dd^c\varphi. \tag{2}$$

Date: July 9, 2015.

If we fix a Hermitian metric h_0 and let $h = e^{-\varphi} h_0$, and put $\omega_0 = \omega_{h_0}$, then

$$\omega_h = \omega_0 + dd^c \varphi. \tag{3}$$

The metric h induces Hermitian metrics h^k on $L^k = L \otimes \cdots \otimes L$ given by $\|s^{\otimes k}\|_{h_N} = \|s\|_h^k$.

2.1. **Background on geodesics.** Let (M, J, ω) be a compact Kaehler manifold of dimension n. Here, $J : TM \to TM$ is the complex structure tensor, ω is a symplectic form of type $(1,1)$, and the associated Riemannian metric is $g_J(X, Y) = \omega(JX, Y)$.

The space of Kaehler metrics in the cohomology class of ω is the infinite dimensional space,

$$\mathcal{H}_\omega = \{\varphi \in C^\infty(M) \, : \, \omega_\varphi := \omega + i\partial\bar{\partial}\varphi > 0\}. \tag{4}$$

We will often assume that $[\omega] \in H^2(M, \mathbb{Z})$ is an integral class, so that there exists a Hermitian holomorphic line bundle $(L, h) \to (M, \omega)$ whose curvature form $\partial\bar{\partial} \log h = \omega$. Then $e^{-\varphi} = h$ may be interpreted as a Hermitian metric on L and $\mathcal{H}_\omega$ may be interpreted as the Hermitian metrics on L having positive curvature.

There are several natural Riemannian metrics on $\mathcal{H}_\omega$. The one which has received the most attention is the Mabuchi-Semmes-Donaldson metric [M, S, D1], defined by

$$g_M(\zeta, \eta)_\varphi := \int_M \zeta \eta \, d\mu_\varphi, \quad \varphi \in \mathcal{H}_\omega, \quad \zeta, \eta \in T_\varphi \mathcal{H}_\omega \simeq C^\infty(M),$$

where

$$d\mu_\varphi = \frac{\omega_\varphi^n}{n!}$$

is the volume form associated to ω_φ.

Remark: There are other natural metrics on $\mathcal{H}_\omega$. For interest, Calabi's metric is

$$\langle \psi_1, \psi_2 \rangle_\varphi := \int_M (\Delta_\varphi \psi_1)(\Delta_\varphi \psi_2) dV_\varphi.$$

It turns out to have constant curvature $+1$. Another metric is

$$\langle \psi_1, \psi_2 \rangle_\varphi := \int_M (\nabla_\varphi \psi_1) \cdot (\nabla_\varphi \psi_2) dV_\varphi,$$

where the inner product is that of ω_φ.

2.2. **Equation for geodesics.** In this section we follow [D1] almost verbatim.

The geodesic equation is the Euler-Lagrange equation for the energy functional

$$E(\varphi_t) = \int_0^1 \int_M \dot{\varphi}_t^2 d\mu_{\varphi_t} dt$$

where $\varphi_t : [0, 1] \to \mathcal{H}_\omega$ is a path with fixed endpoints.

Proposition **2.1.** *The geodesic equation is*

$$\ddot{\varphi} - \tfrac{1}{2} |\nabla \dot{\varphi}_t|^2_{\omega_{\varphi_t}} = 0$$

Remark: The equation implies that $\dot{\varphi}_t(z)$ is increasing as t increases for all $z \in M$. in particular $\varphi_t(z)$ is strictly convex in t.

Proof. A variation of the path is a one-parameter of paths $\varphi_{t,\varepsilon}$ with fixed endpoints. We consider paths of the form, $\varphi_t + \varepsilon\psi_t$. The first variation of the volume form is the term of order ε in

$$d\mu_{\varphi+\varepsilon\psi} = \frac{(\omega_\varphi + \varepsilon dd^c\psi)^n}{n!} = d\mu_\varphi + \frac{\varepsilon n}{n!} dd^c\psi \wedge \omega_\varphi^{n-1} + O(\varepsilon^2) = d\mu_\varphi + \frac{\varepsilon}{2}\Delta_\varphi \psi d\mu_\varphi + O(\varepsilon^2),$$

and so

$$\begin{aligned} \tfrac{1}{\varepsilon}(E(\varphi_t + \varepsilon\psi_t) - E(\varphi_t)) &= \tfrac{1}{\varepsilon}\left(\int_0^1 \int_M (\dot\varphi + \varepsilon\dot\psi)^2 d\mu_{\varphi+\varepsilon\psi} - \int_0^1 \int_M (\dot\varphi)^2 d\mu_\varphi\right) \\ &= 2\int_0^1 \int_M (\dot\varphi\dot\psi) d\mu_\varphi + \int_M \int_0^1 \dot\varphi_t^2 \Delta_\varphi \psi d\mu_\varphi + O(\varepsilon). \end{aligned}$$

We integrate the time derivative on $\dot\psi_t$ by parts in the first term onto $\dot\varphi_t d\mu_{\varphi_t}$ and $\Delta_\varphi\psi$ in the second term onto $\dot\varphi_t^2$ to get

$$\delta E_{\varphi_t}(\psi) = 2\int_0^1 \int_M \left(-\frac{d}{dt}(\dot\varphi d\mu_{\varphi_t}) + \tfrac{1}{2}(\Delta_\varphi \dot\varphi_t^2)\right)\psi d\mu_\varphi.$$

Hence the geodesic equation is

$$-2\frac{d}{dt}(\dot\varphi d\mu_{\varphi_t}) + \tfrac{1}{2}(\Delta_\varphi \dot\varphi_t^2) = 0.$$

Since

$$\frac{d}{dt} d\mu_{\varphi_t} = \tfrac{1}{2}(\Delta_{\varphi_t}\dot\varphi_t) d\mu_{\varphi_t},$$

the geodesic equation simplifies to

$$-2\ddot\varphi_t - \dot\varphi_t(\Delta_{\varphi_t}\dot\varphi_t) + \tfrac{1}{2}(\Delta_\varphi \dot\varphi_t^2) = -2\ddot\varphi_t + |\nabla\dot\varphi_t|^2_{\omega_{\varphi_t}} = 0. \tag{5}$$

□

The equation

$$\ddot\varphi - \tfrac{1}{2}|\nabla\dot\varphi_t|^2_{\omega_{\varphi_t}} = 0$$

at first seems like a Hamilton-Jacobi equaiton,

$$\frac{\partial}{\partial t}\dot\varphi_t + H_t(x, d\dot\varphi_t(x)) = 0$$

with a time-dependent Hamiltonian $H(x,\xi)$ on the cotangent bundle T^*M. However, H_t depends on the solution ω_{φ_t} and this feedback effect makes it much more complicated.

2.3. **Levi-Civita connection.** If φ_t is a path in $\mathcal{H}_\omega$ and $\psi(t)$ is a vector field along the path φ_t, i.e. a function on $M \times [0,1]$, then the covariant derivative of ψ along the path φ_t is defined by

$$D_t\psi(z) := \frac{\partial\psi(z,t)}{\partial t} - \tfrac{1}{2}\langle \nabla_z\psi(z,t), \nabla_z\dot\varphi_t(z)\rangle_{\omega_{\varphi_t}(z)}. \tag{6}$$

Here, ∇ at time t is the gradient with respect to the metric $\omega_t := \omega_{\varphi_t}$. We often denote it by ∇_t, with the risk that it might be confused with differentiating in t; it is the connection acting in the z variable with a metric depending on t.

Remark: φ_t being known, this is a linear transport equation for ψ_t. Below we will see how to solve it by the method of characteristics, i.e. by using the flow of the time-dependent vector field $\nabla_t\dot\varphi_t$.

The Christoffel symbol

$$\Gamma : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

is

$$\Gamma(\psi_1, \psi_2) = -\tfrac{1}{2}\langle \nabla \psi_1, \nabla \psi_2 \rangle_{\omega_\varphi}.$$

The connection is compatible with the metric, i.e.

$$\tfrac{d}{dt} ||\psi_t||^2_{\varphi_t} \quad = \quad 2\langle D_t \psi, \psi \rangle_{\omega_{\varphi_t}}. \tag{7}$$

Indeed,

$$\begin{aligned} \tfrac{d}{dt} ||\psi_t||^2_{\varphi_t} &= \int_M 2\psi \tfrac{d\psi}{dt} + \tfrac{1}{2}\psi^2 \Delta \dot{\varphi} d\mu_\varphi \\ &= \int_M 2\dot{\psi}\psi - \tfrac{1}{2}\langle \nabla \psi^2, \nabla \dot{\varphi} \rangle d\mu_\varphi \\ &= 2\int_M (\dot{\psi} - \tfrac{1}{2}\langle \nabla \psi, \nabla \dot{\varphi} \rangle) \psi d\mu_\varphi = 2\langle D_t \psi, \psi \rangle_{\varphi_t} \end{aligned}$$

proving (7).

2.4. **Moser flow.** Let φ_t be a path starting at the background metric, and define

$$X_t := -\tfrac{1}{2}\nabla_{\omega_{\varphi_t}} \dot{\varphi}_t. \tag{8}$$

Here, $\nabla_{\omega_{\varphi_t}}$ is the gradient with respect to the metric ω_{φ_t}. For fixed t,

$$L_{X_t}\omega_{\varphi_t} = d\iota_{X_t}\omega_{\varphi_t}.$$

Let $X^{\omega_{\varphi_t}}_{\dot{\varphi}_t}$ denote the Hamilton vector field of the Hamiltonian $\dot{\varphi}_t$ with respect to the symplectic form, ω_{φ_t}. The gradient and the Hamilton vector fields are related by,

$$\nabla_{\omega_{\varphi_t}} \dot{\varphi}_t = J X^{\omega_{\varphi_t}}_{\dot{\varphi}_t}.$$

This is because the metric and symplectic form are J-related. Then

$$\iota_{X_t}\omega_{\varphi_t} = \omega_{\varphi_t}(J X^{\omega_{\varphi_t}}_{\dot{\varphi}_t}, \cdot).$$

Now, by definition of the Hamilton vector field,

$$\omega_{\varphi_t}(X^{\omega_{\varphi_t}}_{\dot{\varphi}_t}, \cdot) = d\dot{\varphi}_t.$$

By definition, $d^c\dot{\varphi}_t = J d\dot{\varphi}_t$, where for a 1-form α, $J\alpha(X) = \alpha(JX)$. Thus,

$$d^c\dot{\varphi}_t(Y) = \omega_{\varphi_t}(X^{\omega_{\varphi_t}}_{\dot{\varphi}_t}, JY) = -\omega_{\varphi_t}(J X^{\omega_{\varphi_t}}_{\dot{\varphi}_t}, Y),$$

since ω_{φ_t} and J are compatible. It follows that

$$\iota_{X_t}\omega_{\varphi_t} = -d^c\dot{\varphi}_t$$

and so

$$L_{X_t}\omega_{\varphi_t} = -dd^c\dot{\varphi}_t.$$

Now let

$$f_t : M \to M, \;\; \frac{d}{dt} f_t(x) = X_t(f_t(x)) \tag{9}$$

be the one parameter family of diffeomorphisms integrating X_t with $f_0 = id$. Then

$$\frac{d}{dt} f_t^* \omega_t = f_t^* L_{X_t}\omega_t + \dot{\omega}_t = -dd^c\dot{\varphi}_t + dd^c\dot{\varphi}_t = 0. \tag{10}$$

Hence,

$$f_t^*\omega_t = \omega_0, \tag{11}$$

i.e. $f_t : (M, \omega_0) \to (M, \omega_t)$ is a symplectic diffeomorphism. In particular, if f_t is invertible, then the pullback operator

$$f_t^{-1*} : L^2(M, \omega_0) \to L^2(M, \omega_t)$$

is unitary:

$$\int_M \psi(f_t^{-1}(x)) d\mu_{\varphi_t} = \int_M \psi(x) d\mu_0. \tag{12}$$

Thus, $f_t^{-1*} : T_{\varphi_0}\mathcal{H}_\omega \to T_{\varphi_t}\mathcal{H}_\omega$ is an isometry.

2.5. **Parallel translation.** We now consider the equation for parallel translation of a vector field ψ_t along a curve φ_t, i.e. $\frac{D}{Dt}\psi_t = 0$. By (6), and from the definition of X_t, the condition that ψ_t be parallel is that

$$\frac{\partial \psi(z,t)}{\partial t} - \tfrac{1}{2}\langle \nabla_z \psi(z,t), X_t\rangle_{\omega_{\varphi_t}(z)} = 0. \tag{13}$$

By the definition of f_t this is

$$\frac{d}{dt}\psi_t(f_t(z)) = 0. \tag{14}$$

Thus, $\psi_t(f_t(z)) = F(z)$ for some smooth F or $\psi_t(z) = F(f_t^{-1}(z))$.

By the calculations above, parallel translation preserves norms and inner products of vectors.

Since the metric and connection are compatible, we can construct a normal frame along a curve (in particular, a geodesic) φ_t by finding an orthonormal basis of $L^2(M, d\mu_0)$ and then transporting it as above.

2.6. **Curvature.** Donaldson [D1] proves:

THEOREM **2.2.** *The Riemannian curvature tensor of g_M at φ is given by*

$$R_\varphi(\psi_1, \psi_2)\psi_3 = \frac{1}{4}\{\{\psi_1, \psi_2\}, \psi_3\}_{\omega_\varphi},$$

i.e. by repeated Poisson bracket. The sectional curvatures are given by

$$K_\varphi(\psi_1, \psi_2) = -\frac{1}{4}||\{\psi_1, \psi_2\}_{\omega_\varphi}||_{\omega_\varphi}.$$

Remark: Donaldson computes the sign incorrectly in [D1] but it is computed correctly by Mabuchi in Theorem 4.3 of [M] and (1.8) of Semmes [S].

Recall that

$$K_\varphi(\psi_1, \psi_2) = \langle R_\varphi(\psi_1, \psi_2)\psi_2, \psi_1\rangle, \text{ if } \psi_1, \psi_2 \text{ are orthonormal.}$$

Let $\mathrm{ad}(\psi_1)\psi_2 = \{\psi_2, \psi_1\}$. Then (ψ) is a skew-adjoint operator on $L^2(M, dV_\omega)$ where ω is the symplectic form and $dV_\omega = \frac{\omega^n}{n!}$ is the associated volume form. Indeed, $\mathrm{ad}(\psi_1)\psi_2 = \mathrm{X}^\omega_{\psi_1}(\psi_2)$, and $L_{X^\omega_{\psi_1}}\omega = 0$. Hence

$$\begin{aligned}\langle R_\varphi(\psi_1,\psi_2)\psi_2,\varphi_1\rangle &= -\tfrac{1}{4}\int_M \left(L_{X^\omega_{\psi_2}}\{\psi_1,\psi_2\}\right)\psi_1 dV_\omega = \tfrac{1}{4}\int_M \{\psi_1,\psi_2\}(L_{X^\omega_{\psi_2}}\psi_1)dV_\omega \\ &= = \tfrac{1}{4}\int_M \{\psi_1,\psi_2\}\{\psi_1,\psi_2\}dV_\omega.\end{aligned}$$

Better, let us write X_A for the Hamilton vector field of A with respect to ω. Also write $X(f)$ for $df(X)$. We use,

$$[X_A, X_B] = X_{A,B},\ \ X_B(A) = \{B,A\} = -\{A,B\} = -X_A(B).$$

Then

$$\begin{aligned} g_\varphi(R_\varphi(A,B)B,A) = \int_M ([X_A,X_B](B))\,AdV_\omega &= \int_M \left(X_{\{A,B\}}B\right)AdV_\omega = -\int_M (X_B\{A,B\})AdV_\omega \\ &= \int_M \{A,B\})X_B A dV_\omega = \int_M \{A,B\})\{B,A\}dV_\omega \\ &= -\int_M |\{A,B\})|^2 dV_\omega. \end{aligned} \tag{15}$$

- The sectional curvatures are all ≤ 0, i.e. $(\mathcal{H}_\omega, g_M)$ is non-positively curved.

- The curvature tensor R_φ depends only on the Poisson bracket at φ and is therefore covariant constant. This is because parallel translation is compatible with the moving symplectic structures.

One may define the Poisson bracket $\{f,g\}$ by the formula, $df\wedge dg\wedge\omega^{n-1} = \{f,g\}_\omega\,\omega^n$. If $\psi_{1,t},\psi_{2,t},\psi_{3,t}$ are parallel along φ_t then the Poisson bracket of any two and the further Poisson bracket with the third are also parallel. It follows that the curvature is parallel.

2.7. $\mathcal{H}_{\omega_0}$ **as a symmetric space.** A locally symmetric space is a Riemannian manifold (X,g) such that $\nabla R = 0$. They have the form G/K where G is a Lie group and G is endowed with a bi-invariant metric. In the non-compact case, K is the maximal subcompact subgroup of G. The tangent space $T_{gK}X$ can mapped to the tangent space $T_K X$ at the origin by the derivative $dL_{g^{-1}}$, resp. $dR_{g^{-1}}$ of left or right translation. The curvature tensor is

$$R(X,Y)Z = \frac{1}{4}[[X,Y],Z].$$

If G is compact, its non-compact dual is $G_{\mathbb{C}}/G$. The infinite dimensional analogue is to define $\mathcal{G} = SDiff(M,\omega_0)$ (or better, the Hamiltonian subgroup of exact symplectic diffeomorphisms.

The tangent space $T_{\omega_0}\mathcal{H}_{\omega_0}\simeq C^\infty(M)$ is a Lie algebra under Poisson bracket $\{f,g\}_0$ and also has an inner product. The inner product is invariant under $SDiff(M,\omega_0)$ since the volume form is invariant. The Mabuchi et al inner product is the analogue of the bi-invariant metric. In this picture, a 'point' of $\mathcal{H}_{\omega_0}$ is thought of as a coset $f\circ\chi$ where $\chi\in SDiff(M,\omega_0)$ and $f\in\mathcal{G}_{\mathbb{C}}$. There is no genuine $\mathcal{G}_{\mathbb{C}}$ but one may think of it as as pairs (f,ω_φ) so that $f^*\omega_\varphi = \omega_0$.

Moser's theorem on equivalence of symplectic forms underlies this picture.

THEOREM **2.3.** *Let M be compact and let ω_0, ω_1 be two cohomologous symplectic forms, $[\omega_0] = [\omega_1]$. Then (M, ω_0) is symplectomorphic to $(M, \omega_1$: there exists $f \in Diff(M)$ so that $f^*\omega_1 = \omega_0$. In fact, if $\omega_t = t\omega_1 + (1-t)\omega_0$ then there exists a smooth family f_t in $Diff(M)$ so taht $f_t^*\omega_t = \omega_0$.*

One can interpret f_t as the horizontal lift of φ_t to the principal $SDiff(M, \omega_0)$ bundle over $\mathcal{H}_\omega$ defined as follows: Let $\Upsilon \subset \mathcal{H}_\omega \times Diff(M)$ be the set of pairs (φ, f) such that $f^*\omega_\varphi = \omega_0$. The map

$$(f, \omega_\varphi) \in \Upsilon \to \omega_\varphi \in \mathcal{H}_\omega$$

is surjective with fiber $\{f \in Diff(M) : f^*\omega_\varphi = \omega_0\}$. The fiber over ω_0 is $SDiff(M, \omega_0)$, and this group acts on Υ on the right and the orbit of one f is the entire fiber. Moreover, there is a connection-preserving bundle isomorphism,

$$T\mathcal{H}_\omega \simeq \Upsilon \times_{SDiff} C^\infty(M).$$

In other words, $T\mathcal{H}_\omega$ is the quotient of $\Upsilon \times C^\infty(M)$ by the action of $SDiff(M, \omega_0)$ acting by

$$\chi \cdot ((\omega_\varphi, f), \psi) = ((\omega_\varphi, f \circ \chi, \chi^*\psi).$$

Then, f_t is the horizontal lift of φ_t to Υ.

The tangent bundle $T\mathcal{H}_\omega$ is trivial, i.e. $\simeq \mathcal{H}_0 \times C^\infty(M)$, but might best be thought of as an associated vector bundle to a principal bundle of frames. Recall that on a finite dimensional Riemannian manifold (M, g), the principal frame bundle $P(M, g) \to M$ is the bundle whose fiber at x consists of the orthonormal frames $\{e_1, \dots, e_n\}$ at x. Any tangent vector may be expressed as $v = \sum_j a_j e_j$ in this frame. If we change the frame by $g \in O(n)$ we must change the representative vector $(a_1, \dots, a_n)$ by $\rho(g)^{-1}$ where ρ is the standard action of $O(n)$ on $\mathbb{R}^n$. Thus, $TM = P \times_\rho \mathbb{R}^n$ consists of equivalence classes $[\vec{e}, \vec{x}]$ of pairs $(\{e_1, \dots, e_n\}, (a_1, \dots, a_n)\}$ where $[g\vec{e}, \rho(g)^{-1}\vec{a}] = [\vec{e}, \vec{a}]$.

In the infinite dimensional setting, one analogue of the frame bundle is to choose an orthonormal basis $\{\varphi_j\}$ of $L^2(M, dV_0)$ for the Riemannian metric at ω_0. If we pull back under $\chi \in SDiff(M, \omega_0)$ we get another orthonormal basis $\{\chi^*\varphi_j\}$. So the orthornormal bases may be thought of as corresponding to $SDiff(M, \omega_0)$. We do not use vectors in ℓ_2 to represent functions relative to the ONB, however. That is, rather than thinking of G as the unitary group $U(L^2(M, dV_0))$ we think of it as $SDiff_0(M)$, which is a proper subgroup.

We then use $Diff(M)$ to identity tangent spaces at different points $\omega_\varphi \in \mathcal{H}_\omega$. Let $f \in Diff(M)$ so that $f^*\omega_\varphi = \omega_0$. Then $\{\varphi_j \circ f^{-1}\}$ is an orthonormal basis for $T_{\omega_\varphi}\mathcal{H}_\omega$. We then represent a tangent vector at ω_φ by $[f, u]$ with $u \in T_{\omega_0}\mathcal{H}_\omega$ so that $f^*\omega_\varphi = \omega_0$. Then $f^{-1*}u$ is tangent vector at ω_φ. We have the equivalence relation that $[f, u] = [f\chi, \chi^{-1*}u]$ where $\chi \in SDiff(M, \omega_0)$ since $(f\chi)^{-1*}\chi^*u = f^*u$.

2.8. **Interpretation of the geodesic equation as an HCMA.** This initial value problem is a special case of the Cauchy problem for the homogeneous complex/real Monge–Ampère equation (HCMA/HRMA). Let (M, J, ω) be a compact closed connected Kaehler manifold of complex dimension n. The IVP for geodesics is equivalent to the following Cauchy problem for the HCMA on $S_T \times M$, the product of the manifold with a strip $S_T = [0, T] \times \mathbb{R}$,

$$\begin{aligned} (\pi_2^\star\omega + \sqrt{-1}\partial\bar\partial\varphi)^{n+1} = 0, \quad (\omega + i\partial_M\bar\partial_M\varphi)^n \neq 0, \text{ on } S_T \times M, \\ \varphi(0, t, \cdot\,) = \varphi_0(\,\cdot\,), \quad \partial_s\varphi(0, t, \cdot\,) = \dot\varphi_0(\,\cdot\,), \text{ on } \{0\} \times \mathbb{R} \times M. \end{aligned} \tag{16}$$

where $\pi_2 : S_T \times M \to M$ is the projection, and where φ is is required to be $\pi_2^\star\omega$-plurisubharmonic (psh) on $S_T \times M$.

Repeat:

$$\begin{cases} (\pi_2^\star\omega + \sqrt{-1}\partial\bar\partial\varphi)^{n+1} = 0 & \text{on } S_T \times M, \\ (\pi_2^\star\omega + \sqrt{-1}\partial\bar\partial\varphi)^{n} \neq 0 & \text{on } S_T \times M, \\ \varphi(0,t,\,\cdot\,) = \varphi_0(\,\cdot\,) & \text{on } \{0\}\times\mathbb{R}\times M, \\ \dfrac{\partial\varphi}{\partial s}(0,t,\,\cdot\,) = \dot\varphi_0(\,\cdot\,) & \text{on } \{0\}\times\mathbb{R}\times M. \end{cases} \tag{17}$$

Indeed, multiply the geodesic equation by $\det g'$ where g' is the metric in the z variables, and use the Shur complementarity formula:

$$\det \begin{pmatrix} g' & \nabla\dot\varphi_t \\ \nabla^t\dot\varphi_t & \ddot\varphi_t \end{pmatrix} = \det g' \left(\ddot\varphi - |\nabla\dot\varphi_t|_{(g')^{-1}}\right).$$

2.9. **Donaldson's formal solution.** Semmes and Donaldson [S, D1] give a formal solution of the HCMA: $\dot\varphi_0$ be a smooth function on M, considered as a tangent vector in $T_{\varphi_0}\mathcal{H}_\omega$. Let $X_{\dot\varphi_0}^{\omega_{\varphi_0}} \equiv X_{\dot\varphi_0}$ denote the Hamiltonian vector field associated to $\dot\varphi_0$ and (M, ω_{φ_0}) and let $\exp tX_{\dot\varphi_0}$ denote the associated Hamiltonian flow. Then let $\exp -\sqrt{-1}sX_{\dot\varphi_0}$ "be" its analytic continuation in time to the Hamiltonian flow at "imaginary" time $\sqrt{-1}s$. Then "define" the *classical analytic continuation potential* φ_s with initial data $(\varphi_0, \dot\varphi_0)$ by

$$((\exp -\sqrt{-1}sX_{\dot\varphi_0})^{-1})^\star\omega_0 - \omega_0 = \sqrt{-1}\partial\bar\partial\varphi_s. \tag{18}$$

Then φ_s "is" the solution of the initial value problem. Note that this is equivalent to (10), i.e.

$$\omega_0 = f_s^*(\omega_0 + \sqrt{-1}\partial\bar\partial\varphi_s) = f^*\omega_s. \tag{19}$$

We use quotes since there is no obvious reason why $\exp tX_{\dot\varphi_0}$, a rather arbitrary smooth Hamiltonian flow, should admit an analytic continuation in t for any length of time, nor why $\exp -\sqrt{-1}sX_{\dot\varphi_0}$ should be invertible in case such an analytic continuation exists. When the analytic continuation does exist, e.g., if ω_{φ_0} and $\dot\varphi_0$ are real analytic, then φ_s solves the initial value problem for the Monge–Ampère equation for s in some (usually) small time interval [M, S, D1].

2.10. **Null foliation.** The Cauchy data $(\omega_{\varphi_0}, \dot\varphi_0)$ of the IVP determines a Hamiltonian flow $\exp tX_{\dot\varphi_0}^{\omega_{\varphi_0}}$. Semmes and Donaldson observed that the leaves of the foliation are then analytic continuations in time of the orbits

$$\Gamma_z(\sqrt{-1}t) := \exp tX_{\dot\varphi_0}^{\omega_{\varphi_0}}(z) \tag{20}$$

of the Hamiltonian flow [D1, p. 23],[S, 536]. These complexified Hamiltonian flows give rise to the imaginary time maps

$$f_\tau(z) := \Gamma_z(\tau), \tag{21}$$

(that we call *Moser maps*) for each $\tau = s + \sqrt{-1}t \in S_T$ with $s \in [0,T]$ and $t \in \mathbb{R}$, that are symplectic diffeomorphisms between the Hamiltonian system $(M, \omega_{\varphi_0}, \dot\varphi_0)$ and $(M, \omega_{\varphi_s}, \dot\varphi_s)$, for each $s = \mathrm{Re}\,\tau \in [0,T]$, in particular,

$$(f_s^{-1})^\star \omega_{\varphi_0} - \omega_{\varphi_0} = \sqrt{-1}\partial\bar\partial(\varphi_s - \varphi_0), \quad s \in [0,T]. \tag{22}$$

The kernel of a $(1,1)$ form is always a complex space at each point, and the condition that $\omega_0 + dd^c\varphi$ be Kaehler for all t implies that $dd^c\Phi$ can only have a 1-dimensional complex kernel transverse to the fixed t slices. This real 2-dimensional distribution is integrable since $dd^c\Phi$ is closed.

LEMMA **2.4.** $\frac{\partial}{\partial\tau} - X_\tau \in \ker dd^c\Phi$. *More preciseley,*

$$\frac{\partial}{\partial\tau} - \nabla^{1,0}_{g_{\varphi_s}}\dot\varphi_s \in \ker(\pi_2^\star\omega + \sqrt{-1}\partial\bar\partial\varphi)\big|_{(\tau,\Gamma_z(\tau))}.$$

Proof. We have,

$$\frac{df_\tau}{dt} = X^{\omega_{\varphi_s}}_{\dot\varphi_s} \circ f_\tau = -J\nabla_{g_{\varphi_s}}\dot\varphi_s \circ f_\tau, \qquad \frac{df_\tau}{ds} = -\nabla_{g_{\varphi_s}}\dot\varphi_s \circ f_\tau, \quad f_0 = \mathrm{id}, \tag{23}$$

and

$$\iota_{\frac{\partial}{\partial\tau}}(\pi_2^\star\omega + \sqrt{-1}\partial\bar\partial\varphi) = \sqrt{-1}\bar\partial\frac{\partial\varphi}{\partial\tau} = \sqrt{-1}\bar\partial\dot\varphi_s,$$

and

$$\iota_{\nabla_{g_{\varphi_s}}\dot\varphi_s}(\pi_2^\star\omega + \sqrt{-1}\partial\bar\partial\varphi) = \iota_{\nabla_{g_{\varphi_s}}\dot\varphi_s}\omega_{\varphi_s} = d^c\dot\varphi_s = \sqrt{-1}(\bar\partial - \partial)\dot\varphi_s,$$

and we use the convention $\frac{\partial}{\partial\tau} = \frac{1}{2}\frac{\partial}{\partial s} - \frac{\sqrt{-1}}{2}\frac{\partial}{\partial t}$ and $Y^{1,0} = \frac{1}{2}Y - \frac{\sqrt{-1}}{2}JY$.

□

It then follows that

$$f_{s+\sqrt{-1}t} = h_{s+\sqrt{-1}t} \circ f_s, \tag{24}$$

with $h_{s+\sqrt{-1}t}$ a C^1 symplectomorphism of (M, ω_{φ_s}). Also, from (24) and (23)

$$h_{s+\sqrt{-1}t} = \exp tX^{\omega_{\varphi_s}}_{\dot\varphi_s}. \tag{25}$$

We conclude therefore from (24) and (23) that the maps f_τ defined by (**??**) satisfy (21), i.e., for each $z \in M$, induce analytic continuation to the strip of the Hamiltonian orbit $\exp tX^{\omega_{\varphi_0}}_{\dot\varphi_0}.z$. Hence we have shown both that the Cauchy data is T-good and that the Moser maps of (21) are C^1 and admit C^1 inverses for each $s \in [0,T]$. This completes the proof of the "potential down" part of Theorem 4.2.

Remark: We caution $f_{s+\sqrt{-1}t}$ does not satisfy a group law in the complex variable $s+\sqrt{-1}t$ except in the special case where the Hamiltonian flow is a holomorphic one. We do have

$$\exp(t_1+t_2)X^{\omega_{\varphi_s}}_{\dot\varphi_s}(z) = \exp t_1 X^{\omega_{\varphi_s}}_{\dot\varphi_s}(\exp t_2 X^{\omega_{\varphi_s}}_{\dot\varphi_s} z). \tag{26}$$

3. Donaldson Example

Let $\dot\varphi_0 = h$. Let $U_r = h^{-1}(r, 1]$. Identify $\mathbb{C}$ with $T_q\mathbb{CP}^1$ (south pole). Let $\alpha_r : \mathbf{D} \to U_r$ be the unique Riemann map with $\alpha_r(0)$ equal to the south pole. Rotation of the disc defines a circle action on ∂U_r. Define ω_0 so that the Hamiltonian flow of h on each level coincides with the S^1 action. Here, the coordinates $(t, e^{i\theta})$ on the cylinder $[0,\infty] \times S^1$ correspond to $e^{-t}e^{i\theta}$.

For each $v \in S^2$ there exists a unique $e^{is(v)} \in S^1$ so that $\alpha_{h(v)}(0, e^{is(v)}) = v$. Put

$$f_t(v) = \alpha_{h(v)}(t, e^{is(v)})$$

in polar coordinates. Thus,

$$\Gamma_v(t,s) = \alpha_{h(v)}(e^{-t}e^{is(v)}e^{is}).$$

The leaf through $v \times \{0\}$ is the image of this map. For fixed t one rotates the circle of radius e^{-t}. For fixed s one has the image of a radial line. Hence $f_t(v)$ takes v up the radial line through v.

Remark: The construction of ω_0 makes the perfect Morse function h an action variable, i.e. all of its orbits are 2π-periodic. Hence (ω_0, J, h) is very similar to a toric Kahler manifold of dimension one. Of course, ω_0 is invariant under the S^1 action defined by $X_h^{\omega_0}$. The orbits are linear in time and therefore have analytic continuations.

4. Necessary conditions for unique solvability ("potential down").

Existence of a C^3 solution gives rise to several necessary conditions on the initial data. The most obvious one is that the Hamilton orbits need to possess unique analytic continuations to a strip S_T.

Definition **4.1.** *We say that the Cauchy problem (17) with smooth initial data $(M, \omega_{\varphi_0}, \dot\varphi_0)$ is T-good if the C^∞ map $\Gamma : \mathbb{R} \times M \to M$, $(t,z) \mapsto \exp tX^{\omega_0}_{\dot\varphi_0}(z)$ admits a (unique) C^∞ extension $\Gamma : S_T \times M \to M$ which is holomorphic on S_T for each $z \in M$.*

In particular, $\lim_{s\to 0}\Gamma_z(s + \sqrt{-1}t) = \Gamma_z(\sqrt{-1}t)$ in the C^∞ sense. Note that the strip is one-sided, i.e., the analytic continuation is only assumed to exist for $s \geq 0$. A two-sided strip would force the Hamilton orbit to be real analytic in t, and so is less general. For instance, in several settings, such as C^∞ torus-invariant Cauchy data on toric varieties, the Hamilton orbits are known to possess analytic continuations. The uniqueness of Γ is automatic.

Theorem **4.2.** *(*[RZ]*, based on Semmes-Donaldson)* (Necessary conditions) *If the Cauchy problem (17) with $\omega_{\varphi_0} \in C^1$ and $\dot\varphi_0 \in C^3(M)$ has a solution in $C^3(S_T \times M) \cap PSH(S_T \times M, \pi_2^\star\omega)$ then the Cauchy data is T-good and the maps f_s defined by* (21) *are C^1 and admit a C^1 inverse for each $s \in [0,T]$. The solution is unique in $C^3(S_T \times M) \cap PSH(S_T \times M, \pi_2^\star\omega)$.*

This result is important in clarifying the nature of the obstructions to solving the HCMA. The T-goodness is a straightforward combination of the Semmes–Donaldson arguments [S, D1]. The uniqueness proof requires a global conservation law type argument. The key difference is that the stripwise equations vary from leaf to leaf, and one has to prove an a priori estimate that ensures that the stripwise elliptic problems are not degenerating. The uniqueness proof is also completely different from the corresponding proof for the Dirichlet problem, where the maximum principle is available.

The obstructions leads to the following ill-posedness:

THEOREM **4.3.** [RZ]
For each $\omega_{\varphi_0} \in \mathcal{H}_\omega$ *there exists a dense set of* $\dot{\varphi}_0 \in C^3(M)$ *for which the IVP for HCMA admits no* C^3 *solution for any* $T > 0$.

Proof. (Sketch) The Cauchy data of solutions for each strip which live up to time T must lie in the range of a certain Dirichlet-to-Neumann operator of the strip. This obstruction to solving HCMA is the basis for the density of bad directions.

Invertibility of f_s is a different type of obstruction related to intersections of characteristics = leaves of the Monge–Ampère foliation. Even when the AC and strip-wise Cauchy problems can all be solved, there does not generally exist a solution of the global HCMA equation.

A further obstruction to solvability is (22) which can be split into two requirements. First, the space-time complex Hamilton orbits need not intersect, i.e., f_s should be smoothly invertible for each $s \in [0, T]$. Second, $(f_s^{-1})^* \omega_{\varphi_0}$ must be of type $(1, 1)$.

□

A related result on the boundary problem is in [Dar, DL]. X. Chen [Ch] proved that if the endpoint data is C^∞ then the geodesic between them is $C^{1,\bar{1}}$ on $\mathbb{R} \times M$ (i.e. mixed $z_i, \bar{z}_j$ second derivatives are bounded).

THEOREM **4.4.** *For any* (M, ω), *there exist pairs* (φ_1, φ_2) *in* $C^3(M) \times C^3(M)$ *for which the solution of the HCMA with endpoints* φ_1, φ_2 *is not* C^3. *The set of such pairs has non-empty interior.*

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