1) Let $F : M \to N$ be a smooth map between manifolds (we insist here that $M$ be connected). If $dF_p = 0$ for all $p \in M$, show that $F$ is constant.

2) Let $\text{Mat}(n, \mathbb{R})$ be the space of $n \times n$ matrices with real entries, which as usual is identified with $\mathbb{R}^{n^2}$, and $\text{Sym}(n)$ the subspace of symmetric matrices.

   (a) Show that we can identify $\text{Sym}(n)$ with $\mathbb{R}^{\frac{n(n+1)}{2}}$

   (b) Let $F : \text{Mat}(n, \mathbb{R}) \to \text{Sym}(n)$ be the map given by $F(A) = A^t A$. Show that $dF_A(B)$, its differential at a matrix $A$ applied to a matrix $B$, is given by $A^t B + B^t A$

   (c) Let $O(n) = \{ A \in \text{Mat}(n, \mathbb{R}) \mid A^t A = \text{Id} \}$, and show that for any $A \in O(n)$, the differential of $F$ at $A$ is surjective

   (d) Deduce that $O(n)$ is a smooth manifold of dimension $\frac{n(n-1)}{2}$.

3) Continuing from problem 2, show that $SO(n) = \{ A \in O(n) \mid \text{det } A = 1 \}$, is a smooth manifold, and find its dimension.

4) Let $F : M \to N$ be a smooth map between manifolds, and let $X, Y \in \mathcal{T}(N)$ be two vector fields on $N$ with the property that there are vector fields $\tilde{X}, \tilde{Y} \in \mathcal{T}(M)$ with $dF(\tilde{X}) = X, dF(\tilde{Y}) = Y$. Show that

   $dF([\tilde{X}, \tilde{Y}]) = [X, Y]$. 
