1) Let $M$ be a set with two compatible $n$-dimensional atlases $\mathcal{A}$ and $\mathcal{B}$. Show that these atlases induce the same topology on $M$.

**Solution.** Let $(U, \varphi)$ be a chart in $\mathcal{A}$ and write $\mathcal{B} = \{(V_\alpha, \psi_\alpha)\}$. We need to show that $U$ is open in the topology determined by $\mathcal{B}$, and that $\varphi$ is a homeomorphism with its image (again using the $\mathcal{B}$ topology on $M$). Since $\varphi$ is bijective, it is enough to show that $\varphi$ and its inverse are continuous (using the $\mathcal{B}$ topology on $M$).

Since by assumption $(U, \varphi)$ is a chart compatible with each $(V_\alpha, \psi_\alpha)$, we have that $\varphi(U \cap V_\alpha)$ and $\psi_\alpha(U \cap V_\alpha)$ are open in $\mathbb{R}^n$, and $\varphi \circ \psi^{-1}_\alpha : \psi_\alpha(U \cap V_\alpha) \to \varphi(U \cap V_\alpha)$ is a diffeomorphism.

Since $\psi_\alpha$ is a homeomorphism with its image (using the $\mathcal{B}$ topology on $M$), it follows that $U \cap V_\alpha$ is open in the $\mathcal{B}$ topology, and hence so is $U = \bigcup_\alpha U \cap V_\alpha$.

Finally, on each $U \cap V_\alpha$ we can write $\varphi = (\varphi \circ \psi^{-1}_\alpha) \circ \psi_\alpha$, which is a composition of homeomorphism (in the $\mathcal{B}$ topology on $M$), and so is continuous, and the same for its inverse $\varphi^{-1} = \psi^{-1}_\alpha \circ (\psi_\alpha \circ \varphi^{-1})$ on $\varphi(U \cap V_\alpha)$, which is continuous too.

2) Show that there is no structure of differentiable manifold on $[0, +\infty)$ which induces the standard Euclidean topology.

**Solution.** Suppose we had such a structure. Since $(0, +\infty)$ is an open subset of $[0, +\infty)$, it inherits a differentiable structure. By the invariance of domain (as mentioned in class), we must have that the dimension of the differentiable structure on $(0, +\infty)$ (and so also on $[0, +\infty)$) is 1.

Pick a 1-chart $(U, \varphi)$ for the manifold $[0, +\infty)$ centered at 0, so $U$ is an open set in $[0, +\infty)$ which contains 0 and $\varphi : U \to \varphi(U) \subset \mathbb{R}$ is a homeomorphism with its image, an open subset of $\mathbb{R}$, and $\varphi(0) = 0$. Since by assumption the manifold topology coincides with the Euclidean one, we may shrink $U$ and assume that it equals $[0, \varepsilon)$ for some $\varepsilon > 0$. Of course, its homeomorphic image under $\varphi$ is a connected open neighborhood of 0 $\in \mathbb{R}$. But this can’t happen, since if we remove 0 from $[0, \varepsilon)$ this remains connected, while removing 0 from a connected open subset of $\mathbb{R}$ makes it disconnected.

3) Show that there is no atlas on the circle $S^1$ which induces the standard topology and is composed of only one chart.
Solution. If we had only one chart, then clearly the domain of the chart must be \( U = S^1 \). By invariance of domain (as mentioned in class), the atlas must be 1-dimensional. Then \( \varphi : U \to \varphi(U) \subset \mathbb{R} \) is a homeomorphism with its image, an open subset of \( \mathbb{R} \). But since \( U = S^1 \) is compact, the image \( \varphi(U) \) must be compact too. Since \( \mathbb{R} \) contains no compact open subset, we have a contradiction.