SNAP 2017 - EIGENVALUES AND EIGENFUNCTIONS PROBLEM SET

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Eigenvalues

(1) Compute the Dirichlet and Neumann spectra of Δ on $[0,a] \times [0,b] \subset \mathbb{R}^2$.

(2) Compute the Dirichlet and Neumann spectra of Δ with $\Omega = \mathbb{D}$, the unit disk.

(3) Prove the translation and scaling properties of Δ in $\mathbb{R}^d$, namely show that $\sigma(\Omega + x_0) = \sigma(\Omega)$ for any $x_0 \in \mathbb{R}^d$ and that $\sigma(\alpha\Omega) = \alpha^{-2}\sigma(\Omega)$, where $\alpha > 0$ is a scaling factor.

(4) Show that the Laplace equation is invariant under orthogonal transformations: if $\Delta u = 0$ and $A : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal matrix, then $\Delta u(Ax) = 0$.

(5) Let $\Omega \subset \mathbb{R}^2$ and V be either $H^1(\Omega)$ or $H^1_0(\Omega)$. Recall the first version of the variational principle:

$$\lambda_n = \inf_{u \in H_{n-1}(V)} \rho(u) = \sup_{u \in \text{span}\{u_1,\dots,u_n\}} \rho(u),$$

where $H_{n-1} = V \cap \text{span}\{u_1, ..., u_{n-1}\}^\perp$. Prove the second version of the variational (or min-max) principle :

$$\lambda_n = \inf_{X \in \phi_n(V)} \sup_{u \in X} \rho(u),$$

where

$$\rho(u) = \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}$$

is the Rayleigh quotient and $\phi_n(V)$ is the set of all n-dimensional linear subspaces of V.

(6) Find a counterexample highlighting the fact that the Neumann Laplacian does not enjoy the domain monotonicity property. More precisely, find two domains Ω_1, Ω_2 such that $\Omega_1 \subset \Omega_2$ but such that $\mu_k(\Omega_1) < \mu_k(\Omega_2)$ for some

$k \in \mathbb{N}$.

(7) Let $\Omega \subset \mathbb{R}^2$ be a bounded open set. Write a two-sided estimate on the first eigenvalue $\lambda_1(\Omega)$ in terms of zeros of Bessel functions, the radius R_- of the biggest disk contained in Ω and R_+ of the smallest disk containing Ω.

(8) The goal of this exercise is to justify rigorously our use of Dirichlet-Neumann bracketing, in which we think of an additional Dirichlet condition as *clamping* a drum and an additional Neumann condition as *cutting* a drum. This interpretation naturally leads to the observation that an additional Dirichlet condition will increase the spectrum whereas adding a Neumann condition will decrease the spectrum. More precisely, let $\Omega \subset \mathbb{R}^n$ be a bounded open set and consider a hypersurface $\Gamma \subset \Omega$ that splits Ω in two disjoint part Ω_1 and Ω_2. Denote by $\sigma_{DN}(\Omega_1 \cup \Gamma \cup \Omega_2)$ the spectrum of Δ with Dirichlet condition on $\partial\Omega$ and Neumann condition on Γ. Then, prove that

$$\sigma_{DN}(\Omega_1 \cup \Gamma \cup \Omega_2) \le \sigma_D(\Omega) \le \sigma_D(\Omega_1 \cup \Omega_2).$$

Remark: recall that the spectrum of a disjoint union is the ordered union of the spectra of the connected components.

(9) Consider the domains Ω_1 and Ω_2 in Figure 1, where the red line denotes a Dirichlet boundary condition and a dotted blue line a Neumann boundary condition.

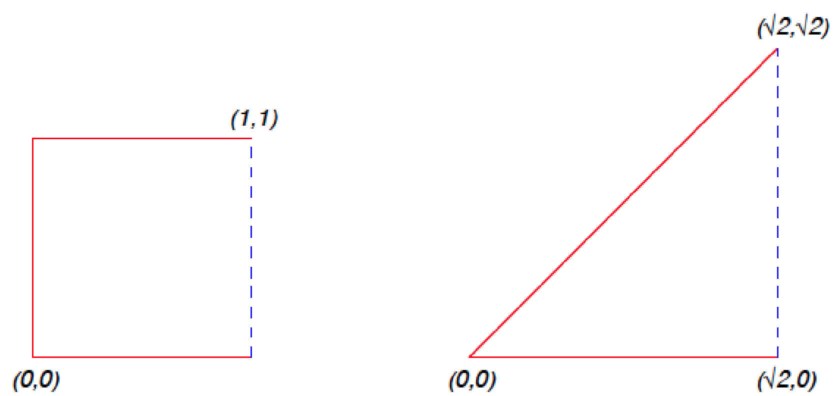

Fig1. Two domains for the mixed boundary conditions problem. (From [LPP])

The eigenfunctions for Ω_1 are

$$\sin((1/2 + m)\pi x)\sin(n\pi y), \quad m, n \in \mathbb{N},$$

and the eigenfunctions for Ω_2 are

$$\sin\left(\frac{(1/2+k)\pi x}{\sqrt{2}}\right)\sin\left(\frac{(1/2+l)\pi y}{\sqrt{2}}\right) \\ - \sin\left(\frac{(1/2+l)\pi x}{\sqrt{2}}\right)\sin\left(\frac{(1/2+k)\pi y}{\sqrt{2}}\right), \quad k,l \in \mathbb{N}, k > l.$$

Using these formulae, compute the eigenvalues and show that

$$\sigma_{DN}(\Omega_1) = \sigma_{DN}(\Omega_2).$$

Eigenfunctions

(1) Compute the number of nodal domains as well as length of the nodal set of the first 5 Dirichlet eigenfunctions of the Laplacian on the unit disk $\mathbb{D} \subset \mathbb{R}^2$.

(2) Consider an orthonormal basis of eigenfunctions $\{\phi_k\}$ of the Laplacian Δ corresponding to eigenvalues λ_k. Show that for $k \geq 2$, the eigenfunction ϕ_k has at least two nodal domains.

(3) Find the inradius of the nodal domains of the Dirichlet eigenfunctions on a rectangle $[0,a] \times [0,b] \subset \mathbb{R}^2$.

(4) The famous Sogge's L^p bound theorem states that given an eigenfunction ϕ_λ on a n-dimensional compact Riemannian manifold (M,g) and for $2 \leq p \leq \infty$, there holds

$$||\phi_\lambda||_{L^p} = O(\lambda^{\delta(p)}),$$

where

$$\delta(p) = \begin{cases} n(\frac{1}{2}-\frac{1}{p})-\frac{1}{2} & \text{if } \frac{2(n+1)}{n-1} \leq p \leq \infty, \\ \frac{n-1}{2}(\frac{1}{2}-\frac{1}{p}) & \text{if } 2 \leq p \leq \frac{2(n+1)}{(n-1)}. \end{cases}$$

The goal of this exercise is to verify that these bounds are all saturated on the round sphere $(\mathbb{S}^2, g_{\mathbb{S}^2})$. Show that the highest weight spherical harmonics (Gaussian beams) saturate the bounds when

$$2 \leq p \leq \frac{2(n+1)}{(n-1)} = 6.$$

Recall that $L^2(\mathbb{S}^2)$ is the direct sum $\oplus_{k=0}^{\infty}\mathbb{SH}^k$, where $\mathbb{SH}^k$ is the restriction to $\mathbb{S}^2$ of the homogenous harmonic polynomials of degree k in $\mathbb{R}^3$. The dimension of $\mathbb{SH}^k$ is $2k+1$ and every element $u \in \mathbb{SH}^k$ satisfies

$$\Delta_g u = k(k+1)u$$

on the round sphere. Using $(\theta, \phi) \in [0, 2\pi) \times [0, \pi]$ as longitudinal and latitudinal coordinates on $\mathbb{S}^2$, we can write the standard orthonormal basis of $\mathbb{SH}^k$ as $\{Y_m^k\}_{m=-k}^k$ as

$$Y_m^k(\theta, \phi) = c_{k,m} P_k^m \cos(\phi) e^{im\theta}.$$

Here, $c_{k,m}$ is the L^2 normalization factor and P_k^m is the associated Legendre polynomial. Using the Rodrigues formula, we can write the associated Legendre polynomials in the following way

$$P_k^m(x) = \frac{(-1)^m}{2^k k!}(1 - x^2)^{\frac{m}{2}} \frac{d^{k+m}}{dx^{k+m}}(x^2 - 1)^k.$$

Finally, let us remark that the *Gaussian beams* (also called *sectoral harmonics*) correspond to the case $m = \pm k$.

References

[LPP] M. Levitin, L. Parnovksy, I. Polterovich, *Isospectral domains with mixed boundary conditions*, J. Phys. A: Math. Gen. 39 (2006) 2073-2082.

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