MICROLOCAL ANALYSIS AND EVOLUTION EQUATIONS: PROBLEM SET
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Exercise 1. Show that

$$[\Delta, \partial_r] = 2\frac{\Delta_\theta}{r^3} - \frac{(n-1)}{r^2}\partial_r.$$

Exercise 2. Show that

$$\partial_r^* = -\partial_r - \frac{n-1}{r}.$$

Exercise 3. Show that

$$[\Delta, \partial_r + \frac{n-1}{2r}] = \frac{2\Delta_\theta}{r^3} + \frac{(n-1)(n-3)}{2r^3}, \tag{0.1}$$

where you should think of both sides as operators from Schwartz functions to tempered distributions. What happens if $n = 3$? If $n = 2$? Be very careful about differentiating negative powers of r in the context of distribution theory...

*Exercise** 4.

(1) Verify that for $u \in \mathcal{S}(\mathbb{R}^n)$ with $n \geq 3$, $|\langle \partial_r u, u\rangle| \lesssim \|u\|^2_{H^{1/2}}$.
Hint: Use the fact that

$$\partial_r = \sum |x|^{-1} x^j \partial_{x^j}.$$

Check that $x/|x|$ is a bounded multiplier on both L^2 and H^1, and hence, by interpolation and duality, on $H^{-1/2}$. An efficient treatment of the interpolation methods you will need can be found in [1]. You will probably also need to use *Hardy's inequality* (see Exercise 5).

(2) Likewise, show that the $\langle r^{-1}u, u\rangle$ term is bounded by a multiple of $\|u\|^2_{H^{1/2}}$ (again, use Exercise 5).

Exercise 5. Prove *Hardy's inequality*: if $u \in H^1(\mathbb{R}^n)$ with $n \geq 3$, then

$$\frac{(n-2)^2}{4}\int \frac{|u|^2}{r^2}\,dx \leq \int |\nabla u|^2\,dx.$$

HINT: In polar coordinates, we have for $u \in \mathcal{S}(\mathbb{R}^n)$

$$\int \frac{|u|^2}{r^2}\, dx = \int_{S^{n-1}} \int_0^\infty |u|^2 r^{n-3}\, dr\, d\theta.$$

Integrate by parts in the r integral, and apply Cauchy-Schwarz.

Exercise 6. Work out the Morawetz estimate for the Schrödinger equation in dimension 3. (This is in many ways the nicest case.)

Exercise 7. This is an exercise to develop background material on the Schrödinger equation. (You may well have seen this material in a PDE class.)

(1) Using the Fourier transform, show that if $\psi_0 \in L^2(\mathbb{R}^n)$, there exists a unique solution $\psi(t, x)$ to the Schrödinger equation with $\psi(0, x) = \psi_0$.
(2) As long as you're at it, use the Fourier transform to derive the explicit form of the solution: show that

$$\psi(t, x) = \psi_0 * K_t$$

where K_t is the "Schrödinger kernel;" give an explicit formula for K_t.
(3) Use your explicit formula for K_t to show that if $\psi_0 \in L^1$ then $\psi(T, x) \in L^\infty(\mathbb{R}^n)$ for any $T \neq 0$.
(4) Show using the first part, i.e. by thinking about the solution operator as a Fourier multiplier, that if $\psi_0 \in H^s$ then $\psi(t, x) \in L^\infty(\mathbb{R}_t; H^s)$, hence give another proof that H^s regularity is conserved.
(5) Likewise, show that the Schrödinger evolution in $\mathbb{R}^n$ takes Schwartz functions to Schwartz functions.
(6) *Rigorously* justify the Morawetz inequality if $\psi_0 \in \mathcal{S}(\mathbb{R}^n)$. Then use a density argument to rigorously justify it for $\psi_0 \in H^{1/2}(\mathbb{R}^n)$.

*Exercise** 8. Write a positive commutator version of the proof of finite propagation speed, using smooth cutoffs instead of integrations by parts. (An account of energy estimates with smooth temporal cutoffs, in the general setting of Lorentzian manifolds, can be found in [2, Section 3].)

*Exercise** 9. Derive (part of) the Morawetz estimate for the wave equation: Let u solve

$$\Box u = 0, (u, \partial_t u)|_{t=0} = (f, g)$$

on $\mathbb{R}^n$, with $n \geq 4$. Show that

$$\left\| r^{-3/2} u \right\|_{L^2_{\mathrm{loc}}(\mathbb{R}^{n+1})} \lesssim \|f\|_{H^1}^2 + \|g\|_{L^2}^2;$$

this is analogous to the weight part of the Morawetz estimate we derived for the Schrödinger equation. There is in fact no need for the local L^2 norm—the global spacetime estimate works too: prove this estimate, and use it to draw a conclusion about the long-time decay of a solution to the wave equation with Cauchy data in $\mathcal{C}_c^\infty(\mathbb{R}^n) \oplus \mathcal{C}_c^\infty(\mathbb{R}^n)$.

HINT: consider $\langle [\Box, \chi(t)(\partial_r + (n-1)/(2r))]u, u\rangle_{\mathbb{R}^{n+1}}$.

Exercise 10. Check that the Poisson bracket satisfies the Jacobi identity, i.e., endows $\mathcal{C}^\infty(T^*X)$ with the structure of a Lie algebra.

Exercise 11. Check (by actually performing a change of coordinates) that if $f, g \in \mathcal{C}^\infty(T^*X)$, then $\{f, g\}$ is well-defined, independent of coordinates.

*Exercise** 12. Let $p(x, \xi) = \sum g^{ij}(x)\xi_i\xi_j$. Show that the projection to X of the integral curves of the vector field H_p are geodesics for the metric $g_{ij}(x)$ (with g^{ij} the inverse, i.e., the induced metric on covectors rather than vectors). Hint: you have a couple of options. One is to compute! The other is to remember or read about the Legendre transform in classical mechanics to relate Hamilton's equations of motion to the variational problem for the action functional

$$I(x(t)) = \int g_{ij}(x(t))\dot{x}_i(t)\dot{x}_j(t)\,dt.$$

Exercise 13. Suppose $PQ - I \in \Psi^{-\infty}(X)$ (Q is a right parametrix) and $Q'P - I \in \Psi^{-\infty}(X)$ (Q' is a left parametrix). Show that in fact $QP - I \in \Psi^{-\infty}$, i.e. that Q is automatically a *two-sided* parametrix.

Exercise 14. Show that an elliptic pseudodifferential operator on a compact manifold is Fredholm. (HINT: You can show, for instance, that the kernel is finite dimensional by observing that the existence of a parametrix implies that the identity operator on the kernel is equal to a smoothing operator, which is compact.)

*Exercise** 15.

(1) Let X be a compact manifold. Show that if $P \in \Psi^m(X)$ is elliptic, and has an actual inverse operator P^{-1} as a map from smooth functions to smooth functions, then $P^{-1} \in \Psi^{-m}(X)$. (HINT: Show that the parametrix differs from the inverse by an operator in $\Psi^{-\infty}(X)$—remember that an operator is in $\Psi^{-\infty}(X)$ if and only if it maps distributions to smooth functions.)

(2) More generally, show that if $P \in \Psi^m(X)$ is elliptic, then there exists a generalized inverse of P, inverting P on its range, mapping to the orthocomplement of the kernel, and annihilating the orthocomplement of the range, that lies in $\Psi^{-m}(X)$.

*Exercise** 16. Let X be compact, and P an elliptic operator on X, as above, with positive order. Using the spectral theorem for compact, self-adjoint operators, show that if $P^* = P$, then there is an orthonormal basis for $L^2(X)$ of eigenfunctions of P, with eigenvalues tending to $+\infty$. Show that the eigenfunctions are in $\mathcal{C}^\infty(X)$. (Hint: show that there exists a basis of such eigenfunctions for the generalized inverse Q and then see what you can say about P.)

Exercise 17. Let X be compact.

(1) Show that the principal symbol of Δ, the Laplace-Beltrami operator on a compact Riemannian manifold, is just
$$|\xi|_g^2 \equiv \sum g^{ij}(x)\xi_i\xi_j,$$
the metric induced on the cotangent bundle.

(2) Using the previous exercise, conclude that there exists an orthonormal basis for $L^2(X)$ of eigenfunctions of Δ, with eigenvalues tending toward $+\infty$.

Exercise 18. Work out the Hörmander "square root trick" on a compact manifold X as follows.

(1) Show that if $P \in \Psi^0(X)$ is self-adjoint, with positive principal symbol, then P has an approximate square root, i.e. there exists $Q \in \Psi^0(X)$ such that $Q^* = Q$ and $P - Q^2 \in \Psi^{-\infty}(X)$. (Hint: Use an iterative construction, as in the proof of existence of elliptic parametrices.)

(2) Show that operators in $\Psi^{-\infty}(X)$ are L^2-bounded.

(3) Show that an operator $A \in \Psi^0(X)$ is L^2-bounded. (Hint: Take an approximate square root of $\lambda I - A^*A$ for $\lambda \gg 0$.)

Exercise 19. If $P \in \Psi^m(X)$ is elliptic at (x_0, ξ_0), show that there exists a *microlocal elliptic parametrix* $Q \in \Psi^{-m}(X)$ such that
$$(x_0, \xi_0) \notin \mathrm{WF}'(PQ - I) \cup \mathrm{WF}'(QP - I).$$

(In other words, you should think of Q as inverting P *microlocally near* (x_0, ξ_0).)

(Hint: If B is a microlocal cutoff, microsupported sufficiently close to (x_0, ξ_0) and microlocally the identity in a smaller neighborhood, then show
$$W = BP + \lambda\,\mathrm{Op}(\langle\xi\rangle^m)(\mathrm{Id} - B)$$
is globally elliptic provided $\lambda \in \mathbb{C}$ is chosen appropriately. Now, using the existence of an elliptic parametrix for W, prove the theorem.)

References

[1] Taylor, M. E., *Partial differential equations. I. Basic theory* Applied Mathematical Sciences, 115. Springer-Verlag, New York, 1996.

[2] A. Vasy, *The wave equation on asymptotically Anti-de Sitter spaces,* Anal. PDE, to appear.

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