PROBLEM SET 2

SNAP 2017 - WEEK 2

This homework covers the relation between B. Motion and the solution of some classical parabolic and elliptic PDEs.

Here, if not said otherwise, B denotes a Brownian motion in $\mathbb{R}^d$.

(1) Let f be a C^2 function such that for all $x \in \mathbb{R}^d$, $t > 0$

$$\mathbb{E}_x e^{\lambda s}|f(B_s)|ds < \infty \text{ and } \mathbb{E}_x e^{\lambda s}|\Delta f(B_s)|ds < \infty$$

(a) Show that

$$X_t = e^{-\lambda t} f(B_t) - \int_0^t e^{-\lambda s}\left(\frac{1}{2}\Delta f(B_s) - \lambda f(B_s)\right) ds$$

is a martingale.

(b) Suppose U is a bounded open set, $\lambda \geq 0$ and $u : U \to \mathbb{R}$ is a bounded solution of

$$\frac{1}{2}\Delta u(x) = \lambda u(x), \qquad \text{for } x \in U,$$

with $\lim_{x \to x_0} u(x) = f(x_0)$ for all $x_0 \in \partial U$. Show that

$$u(x) = \mathbb{E}_x\left[f(B(\tau))e^{-\lambda\tau}\right],$$

where

$$\tau = \inf\{t \geq 0 : B_t \notin G\}.$$

(2) Let B be a complex Brownian motion starting at i.

(a) Show that $e^{i\lambda B_t}$ is a martingale for any $\lambda \in \mathbb{R}$.

(b) Let T be the first time that B hits the real axis. Show that

$$\mathbb{E}e^{i\lambda B} = e^{-\lambda}.$$

(c) Invert the Fourier transform and conclude that $B(T)$ is Cauchy distributed.

(3) (Harmonic measures) Let G be a closed set and $x \in \mathbb{R}^d$. Define a measure on ∂G as

$$\mu_{x,G}(C) = \mathbb{P}_x\left(B_\tau \in C, \tau < \infty\right), \qquad \text{where } \tau = \inf\{t \geq 0 : B_t \in G\}.$$

Show that for any compact set $G \subset \mathbb{R}^d$ and any Borel set $B \subseteq \partial G$ the function $x \mapsto \mu_{x,A}(B)$ is harmonic on G^c.

(4) (Poisson's formula) Show that for any subset C of the unit sphere and ant $x \notin \partial\mathcal{B}(0,1)$, one has

$$\mu_{x,\mathcal{B}(0,1)}(C) = \int_C \frac{|1-|x|^2|}{|x-y|^d} dV(y),$$

where dV is the uniform distribution on the unit sphere.

(5) Let A be a compact set of $\mathbb{R}^2$ such that for $x \in A^c$, $\mathbb{P}_x(\tau_A < \infty) = 1$. Define the (harmonic measure from infinity) μ_A on ∂A by

$$\mu_A(C) = \lim_{x\to\infty} \mathbb{P}_x(B(\tau_A \in C)).$$

Show that if $A \subseteq \mathcal{B}(x,R)$ then

$$\mu_A(C) = \frac{\int \mu_{x,A}(C) dV_R(x)}{\int \mu_{x,A}(A) dV_R(x)},$$

where dV is the uniform distribution on $\mathcal{B}(x,R)$.

(6) Suppose V is a bounded, continuous function. Show that

$$u(t,x) = \mathbb{E}_x\left[\exp\left(\int_0^t V(B_r)dr\right)\right]$$

solves the equation

$$\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + V(x)u(t,x)$$

with initial condition

$$u(0,x) = 1.$$