PROBLEM SET 1

SNAP 2017 - WEEK 2

This homework covers a few properties of Brownian Motion.

(1) Let B be a standard Brownian motion on $[0,1]$. Prove that the process X_t, $0 \leq t \leq 1$, defined by
$$X_t = B_{1-t} - B_1$$
is also a standard BM on $[0,1]$. Fix $a > 0$ and do the same for the process
$$Y_t = a^{-1/2} B_{at}.$$

(2) Compute the first two moments of the random variable
$$X(\omega) = \int_0^1 B_s(\omega) ds.$$

(3) Prove that almost surely, for all $0 < a < b < \infty$, 1-dimensional Brownian motion is not monotone on the interval $[a,b]$.
Hint: Split the interval into several pieces and use independent increments.

(4) (a) If B is a standard BM in $\mathbb{R}^d$, prove that for every $x \in \mathbb{R}^d$ with $\|x\| = 1$, the process
$$X_t = \langle x, B_t \rangle$$
is a 1-dimensional standard BM.
(b) Try to come up with an example to show that the converse is not true.

(5) Prove that
$$Z_t = t^{-1/2} \log \int_0^t \exp(B_s) ds$$
converges in law as t goes to infinity to $S_1 = \sup_{s \leq 1} B_s$.

(6) Show that with probability one Brownian motion is not differentiable at a point.