HOMEWORK

Problem 1. Consider the Korteweg–de Vries (KdV) equation

$$u_t + u_{xxx} + (u^2)_x = 0.$$

(a) Find the dispersion relation for the associated linear equation. What are the group and phase velocities?
(b) Verify that smooth solutions to KdV conserve the L^2_x-norm.

Problem 2. Let A denote a real symmetric, positive-definite, $d \times d$ matrix. Show that

$$(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-x\cdot Ax}\, dx = 2^{-\frac{d}{2}} [\det A]^{-\frac{1}{2}} e^{-\frac{\xi\cdot A^{-1}\xi}{4}}.$$

Problem 3. For $f \in L^2(\mathbb{R}^d)$ show that

$$\big[(4\pi\sigma^2)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4\sigma^2}}\big] * f \to f \quad \text{as} \quad \sigma \to 0$$

both almost everywhere and in the L^2_x topology.

Problem 4 (Sharpness of local smoothing)**.** Let $d \geq 1$. Show that

$$\sup_{f\in\mathcal{S}(\mathbb{R}^d)\setminus\{0\}} \frac{\int_{\mathbb{R}} \int_{|x|\leq 1} |e^{it\Delta} f(x)|^2\, dx\, dt}{\|\langle\nabla\rangle^{-\frac{1}{2}-\varepsilon} f\|_2^2} = \infty$$

for any $\varepsilon > 0$.
Hint: Compute the LHS for functions of the form $f(x) = e^{-\frac{|x|^2}{4} + ix\cdot\xi_0}$.

Problem 5 (Local smoothing in 1D)**.** Let $f \in \mathcal{S}(\mathbb{R})$. Show that

$$\sup_{x\in\mathbb{R}} \||\nabla|^{\frac{1}{2}} e^{it\partial_x^2} f\|_{L^2_t} \lesssim \|f\|_{L^2(\mathbb{R})}.$$

Hint: Let $\phi \in \mathcal{S}(\mathbb{R})$ with $\phi(0) = 1$ and $\operatorname{supp}(\hat{\phi}) \subset (-1, 1)$. Observe that

$$\text{LHS} \sim \lim_{\varepsilon\to 0} \Big\|\phi(\varepsilon t) \int_{\mathbb{R}} e^{-ix\xi} |\xi|^{\frac{1}{2}} e^{-it\xi^2} \hat{f}(\xi)\, d\xi \Big\|_{L^2_t}$$

and use Plancherel.

Problem 6 (Fraunhofer formula in L^2)**.** Let $f \in L^2(\mathbb{R}^d)$. Show that

$$\lim_{|t|\to\infty} \Big\|[e^{it\Delta} f](x) - (2it)^{-\frac{d}{2}} e^{i\frac{|x|^2}{4t}} \hat{f}\big(\tfrac{x}{2t}\big)\Big\|_{L^2_x} = 0.$$

Using this show that if $f \in \mathcal{S}(\mathbb{R}^d)$, then

$$\|e^{it\Delta} f\|_{L^p_x} \sim |t|^{-\frac{d}{2}+\frac{d}{p}} \quad \text{for any} \quad 2 \leq p \leq \infty,$$

where the implicit constants are allowed to depend on d, p, and f.

Problem 7 (Pointwise Fraunhofer formula)**.** Let $f \in \mathcal{S}(\mathbb{R})$. Show that

$$[e^{it\Delta} f](x) = (2it)^{-\frac{1}{2}} e^{i\frac{|x|^2}{4t}} \hat{f}\big(\tfrac{x}{2t}\big) + t^{-\frac{1}{2}-\beta} O(\|\langle x\rangle f\|_{L^2_x})$$

for any $\beta < \frac{1}{4}$.
Hint: Observe that

$$e^{it\Delta} = M(t)D(t)\mathcal{F}M(t) = M(t)D(t)\mathcal{F} + M(t)D(t)\mathcal{F}[M(t) - 1],$$

where $[M(t)f](x) = e^{i\frac{|x|^2}{4t}} f(x)$, $[D(t)f](x) = (2it)^{-\frac{1}{2}} f\big(\frac{x}{2t}\big)$, and $\mathcal{F}$ denotes the Fourier transform.

Problem 8. Find the scaling symmetry for KdV. What is the scaling critical regularity for KdV?

Problem 9 (Solitons)**.** (a) If $u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{C}$ given by $u(t,x) = e^{i\omega t} Q(x)$ is a solution to

$$iu_t + \Delta u = -|u|^p u \quad \text{with } p > 0,$$

what ODE must Q satisfy?
(b) Show that any solution of the ODE you found above conserves a quantity of the form

$$\frac{1}{2}[Q'(x)]^2 + V(Q(x))$$

for a suitable choice of V.
(c) Using part (b), show that Q satisfies a separable ODE. Show that for ω of a suitable sign, this separable ODE admits explicit solutions $Q > 0$ that satisfy

$$\lim_{|x|\to\infty} \big[|Q(x)| + |Q'(x)|\big] = 0.$$

Hint: Perform the change of variables $F = Q^{-\frac{p}{2}}$.

Problem 10. Consider the initial-value problem

$$iu_t + u_{xx} = |u|^2 u \quad \text{with} \quad u(0,x) = u_0(x), \tag{1}$$

where $u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{C}$. During lecture, we will show that for $u_0 \in L^2(\mathbb{R})$, there exist $T = T(\|u_0\|_2)$ and a unique solution $u \in C_t L^2_x \cap L^8_t L^4_x([0,T] \times \mathbb{R})$ to (1).
(a) Show that under the stronger hypothesis $u_0 \in H^1(\mathbb{R})$, the unique local solution constructed during lecture satisfies

$$\|\nabla u\|_{L^8_t L^4_x([0,T]\times\mathbb{R})} + \|\nabla u\|_{L^\infty_t L^2_x([0,T]\times\mathbb{R})} \lesssim \|\nabla u_0\|_2. \tag{2}$$

Hint: Construct a local solution via contraction mapping in a smaller ball containing functions that also satisfy (2) and then use the uniqueness of solutions in the larger ball.
(b) Prove conservation of the energy

$$E(u(t)) = \int_{\mathbb{R}} \tfrac{1}{2}|\nabla u(t,x)|^2 + \tfrac{1}{4}|u(t,x)|^4 \, dx$$

for solutions with $u_0 \in H^1(\mathbb{R})$.
Hint: First show that smooth data leads to smooth solutions and prove conservation of energy for smooth solutions. Then approximate H^1_x data by smooth data.
(c) Show that under the hypothesis $u_0 \in H^1(\mathbb{R})$, uniqueness holds in the more general class $u \in C_t H^1_x([0,T] \times \mathbb{R})$.

Problem 11. Assume $\|u_0\|_\Sigma := \|u_0\|_{H^1(\mathbb{R})} + \|xu_0\|_{L^2(\mathbb{R})} \le \varepsilon$ for ε sufficiently small. Then there exist $T > 1$ and a unique solution $u \in C_t\Sigma_x([0,T]\times\mathbb{R})$ to (1) such that

$$\|\langle t\rangle^{-1} x e^{-it\partial_x^2} u(t)\|_{L_t^\infty L_x^2} + \|u\|_{L_t^\infty H_x^1} \lesssim \varepsilon,$$

where all spacetime norms are over $[0,T]\times\mathbb{R}$.
Hint: Note that $e^{it\partial_x^2} x e^{-it\partial_x^2} = x + 2it\partial_x$.

Problem 12 (Pseudoconformal energy)**.** Consider the initial-value problem

$$iu_t + u_{xx} = |u|^2 u \quad \text{with} \quad u(0,x) = u_0(x) \in \mathcal{S}(\mathbb{R}).$$

Let $H(t) = e^{it\partial_{xx}} x e^{-it\partial_{xx}} = x + 2it\partial_x = 2ite^{i\frac{|x|^2}{4t}}\partial_x e^{-i\frac{|x|^2}{4t}}$.
(a) Prove conservation of the pseudoconformal energy

$$h(t) = \|H(t)u(t)\|_{L_x^2}^2 + 2t^2\|u(t)\|_{L_x^4}^4.$$

Note that $h(t) = 8t^2 E(e^{-i\frac{|x|^2}{4t}} u(t))$, where $E(\cdot)$ is as defined in Problem 10(b).
(b) Conclude that

$$\|u(t)\|_{L_x^4} \lesssim_{u_0} \langle t\rangle^{-\frac{1}{4}}.$$