Homework 2

1

For upper bound we use $\overline{Mdim}(K_A)) > Hdim(K_A)$: If A has r 1's in it then, we can cover K_A by r^n squares of side $\frac{1}{b^n}$.

$$\overline{Mdim}(K_A)) \leq \frac{logr^n}{log1(1/b^n)} = log_b(r) \tag{0.1}$$

For lower bound we use mass distribution principle: we consider measure μ with $\frac{1}{r^n}$ support at each of the squares r^n. Then for $b^{-n} \leq \varepsilon \leq b^{n-1}$ we have $r^{-n} \leq \varepsilon^{log_b r} \leq r^{n-1}$. So since for such ε a $B(x, \varepsilon)$ can hit at most r squares we have

$$\mu(B(x, \varepsilon)) \leq r \cdot r^{-n} \leq r \cdot \varepsilon^{log_b r}.$$

Alternative proof: Using map $T_b(x) := bx(mod1)$ we have $T_b K_A = K_A$ and so by Furstenberg's Lemma we have $Hdim(K_A) = Mdim(K_A)$. We already got the upper bound above. For the lower bound, we note that a cover with $|E_i| \leq varepsilon$, s.t. $b^{-n} \leq \varepsilon \leq b^{n-1}$, will hit at least $r^n/2$ squares.

2

a

The set A_S hits exactly

$$2^{\sum_k^n 1_S(k)},$$

(because up to $x_1, .., x_n$ we have up to n-level dyadic intervals) and so

$$\frac{logN(A_S, 2^{-n})}{log2^n} = \frac{1}{n}\sum_k^n 1_S(k).$$

b

The key idea is to come up with an S s.t. $S \cap \{1, ..., a_n\} = S \cap \{1, ..., a_{n+1} - 1\}$.

Let

$$S := \bigcup_{n\geq 0}\{2^{2n}, ..., 2^{2n+1}-1\}$$

then for upper density

$$\frac{S\cap\{1,...,2^{2n+1}-1\}}{2^{2n+1}-1} = \frac{1+2^2+...+2^{2n}}{2^{2n+1}-1} = \frac{2}{3}$$

for lower density

$$\frac{S\cap\{1,...,2^{2n+2}-1\}}{2^{2n+2}-1} = \frac{1+2^2+...+2^{2n}}{2^{2n+2}-1} = \frac{1}{3}.$$

3

a

Since $K = \bigcup T_i(K)$, each $B(x,\varepsilon)$ intersects some $T_k(K)$. Then

$$T_k(K) = T_k(\bigcup T_i(K)) = \bigcup T_k(T_i(K))$$

so one of them intersects $B(x, \varepsilon/2)$ and so on and so forth.

b,c

The open set condition is that there exists open $V \subset X$ s.t. $f_j(V) \subset V$ and $f_j(V)\cap f_i(V) = \varnothing$.

Let $V = \bigcup T_{\sigma_j}^{-1}(B_j)$, where $T_{\sigma_j} = T^{j1}\circ ... \circ T^{jl}K \subset B_j$, then V satisfies the OST. Let K_1 be the attractor for that family T_{σ_j}, i.e.

$$K_1 = \bigcup T_{\sigma_j}(K_1)$$

and so we can cover K_1 by N disjoint balls of radius $\leq max(r_{j_1}\cdots r_{j_l}) \leq C\varepsilon$ and thus

$$Mdim(K_1) \geq \frac{logN}{logC\varepsilon}$$

then by thm in class

$$Hdim(K) \geq Hdim(K_1) = Mdim(K_1) \geq \frac{logN}{logC + log\varepsilon}.$$

So part (c) follows from $Hdim \leq Mdim$.

4

Suppose $a < dim(A), b < dim(B)$ then by Frostman

$$\mu_A(D) \leq c|D|^a$$
$$\mu_B(F) \leq c|F|^b \Rightarrow$$
$$\mu_A \times \mu_B(D \times F) \leq c|D|^{a+b}.$$

For the other upper inequality, let $\alpha \geq dim(A)$ and $\beta \geq Mdim(\beta)$. Then consider cover A_j, B_j and for each B_j cover $\{B_{jk}\}$ with $|B_{jk}| = |A_j|$ so that there are $L_j \leq |A_j|^{-\beta}$. Consider

$$\sum_j \sum_{k=1}^{L_j} |A_j \times B_{jk}|^{a+b} \leq c \sum_j L_j |A_j|^{a+b} \leq c \sum_j |A_j|^a < \infty.$$

5

For the kernel and a=d-2

$$\Delta k_a(x) = c\frac{(d-a)(a-2)}{|x|^{d-a+2}}$$
$$= c\frac{2(d-4)}{|x|^4}.$$

With logarithm we split measure into

$$U_\mu(x) = U_{\mu_1}(x) + U_{\mu_2}(x)$$

where U_{μ_2} is supported outside $|z| \leq r$.

6

Truncated kernel

$$k_a^r(x) := 1_{|x|>r} k_a(x) + 1_{|x|\le r} c.$$

then

$$U_\mu(x) = lim_{r\to 0} \int k_a(x-y)^r d\mu(y)$$

so

$$\begin{aligned} lim_{x\to x_0} \int k(x-y)d\mu(y) &\ge lim_{x\to x_0} \int k^r(x-y)d\mu(y) \\ &= \int k^r(x_0-y)d\mu(y) \\ &\to U_\mu(x_0). \end{aligned}$$

7

a

Weak convergence follows for $f(x,y) = p(x)q(y)$, where p,q are polynomials

$$\int f d\mu_n d\mu_n \to \int f d\mu d\mu$$

then we approximate continuous f on compact support by polynomials. So again we truncate and obtain

$$\lim_n \int k^r(x_0-y)d\mu_n(y)d\mu_n(x) = \int k^r(x_0-y)d\mu(y)d\mu(x).$$

Then

$$\lim_n \int k(x_0-y)d\mu_n(y)d\mu_n(x) \ge \int k^r(x_0-y)d\mu(y)d\mu(x) \to I(\mu).$$

b

Let μ_n be a minimizing sequence

$$\lim_{n\to\infty} I(\mu_n) = V_\alpha(K)$$

and so by Prokhorov's thm they have a converging subsequence $\mu_n \to \lambda_K$. Then by (a)

$$V_\alpha(K) \leq I(\lambda_K) \leq \lim_{n\to\infty} I(\mu_n) = V_\alpha(K).$$

c

$$\begin{aligned}
&V_\alpha(K) \leq \|\delta\nu + (1-\delta)\lambda_K\| \\
&= \delta^2 \|\nu\|^2 + 2\delta(1-\delta)\langle \nu, \lambda_K\rangle + (1-\delta)^2 \|\lambda_K\|^2 \Rightarrow \\
&\langle \nu, \lambda_K\rangle \geq \frac{1-(1-\delta)^2}{2\delta(1-\delta)} V_\alpha(K) - \delta^2 \|\nu\|^2 \to V_\alpha(K).
\end{aligned}$$

d

If $C(K) = 0$ then $V(K) = \infty$ and so $\nu|_K \equiv 0$. Conversely, if $\mu = 0$ then $V(K) = \infty$.

If we $\nu(\{x : U_{\lambda_K}(x) \leq V(K)\})$ then by integrating:

$$\int_E U_{\lambda_K}(x) d\nu \leq V(K), \tag{0.2}$$

which contradicts (c).

e

If there was $x_0 \in S(\lambda_K)$ s.t.

$$U_{\lambda_K}(x) \geq V(K) \Rightarrow$$

by lower-continuity in neighbourhood O we obtain contradiction

$$\begin{aligned}
V(K) &= \int_O U_{\lambda_K}(x) d\lambda + \int_{S(\lambda_K)\setminus O} U_{\lambda_K}(x) d\lambda \\
&> V(K).
\end{aligned}$$