Homework 1

Question 1

By Hölder

$$\begin{aligned}\sum |f(E_i)|^\beta &\leq C^\beta \sum_i |E_i|^{\beta\alpha} \\ &\leq C^\beta H^{\alpha\beta}(X) + C^\beta \varepsilon\end{aligned}$$

and so

$$\begin{aligned}\inf\{\beta : H^\beta(X) = 0\} &\leq \inf\{\beta : H^{\alpha\beta}(Y) = 0\} \\ &= \frac{1}{\alpha} \inf\{m : H^m(Y) = 0\}.\end{aligned}$$

Question 2

$$\begin{aligned}f_0(x) &= x \\ f_{n+1}(x) &:= \begin{cases} 0.5 f_n(3x) & [0, 1/3] \\ 0.5 & [1/3, 2/3] \\ 0.5 + 0.5 f_n(3x-2) & [2/3, 1]. \end{cases}\end{aligned}$$

Therefore, we obtain by recursion

$$\begin{aligned}|f_n(x) - f_n(y)| \leq& \frac{1}{2}|f_{n-1}(3x) - f_{n-1}(3y)| \\ &\leq (\frac{3}{2})^n |x-y|. \Rightarrow\end{aligned}$$

Therefore, the convergence rate is:

$$|f(x) - f_n(x)| \leq 2^{-n}\frac{1}{3}$$

Thus, since points in the Cantor set satisfy: $\frac{1}{3^{n+1}} \leq |x-y| \leq \frac{1}{3^n}$ we obtain

$$\begin{aligned}|f(x) - f(y)| &\leq \frac{2}{3}2^{-n} + |f_n(x) - f_n(y)| \\ &\leq \frac{2}{3}2^{-n} + (\frac{3}{2})^n |x-y|\end{aligned}$$

$$\leq c|x-y|^{log_3(2)}.$$

where we used that

$$\frac{1}{2^{n+1}} \leq |x-y|^{log_3(2)} \leq \frac{1}{2^n}.$$

The other direction $dim(C) \leq log_3(2)$ is easier:

for all $\beta > log_3(2)$

$$\sum |I_i|^\beta \leq e^{k(log2 - \beta log3)} \to 0.$$

Question 3

For the Minkowski

$$M_n(E) := \#\{Q_n : Q_n \cap E\}$$

$$M(E, \varepsilon) := min\{k : \exists E_i \ covering \ E \ with \ |E_i| \leq \varepsilon\}$$

Each E_i with $|E_i| \leq \sqrt{d}2^{-n}$ can intersect at most 2^d cubes and so

$$M_n(E) \geq M(E, \sqrt{d}2^{-n}) \geq \frac{1}{2^d} M_n(E).$$

For Hausdorff dimension

$$H^s_{E_i} \leq H^s_{Q_i}$$

since the covering is arbitrary for the lhs. On the other hand, consider cubes $\{Q_i\}$ that contain $\{E_i\}$ and with diameter $2|E_i|$, then

$$\sum |Q_i|^s \leq 2^s \sum |E_i|^s$$

and so

$$H^s_{E_i} \leq H^s_{Q_i} \leq 2^s H^s_{E_i}.$$

Question 4

We have

$$0 < \frac{logl_n}{n} < log2 \Rightarrow$$
$$\frac{logl_n}{n} \to L \Rightarrow b := e^{-L}.$$

So for large n, $l_n \approx b^{-n}$. We have upper bound $dim(C) \leq log_b 2^d$ from

$$\sum |I_i|^s \leq 2^{d*n}(\sqrt{d}l_n)^s \leq d^{s/2} e^{n(\log 2^d + s\frac{\log l_n}{n})}.$$

We have lower bound $dim(C) \geq log_b 2^d$ by considering μ assigning weight 2^{-nd} to each cell at the nth step. Then $B(x,\varepsilon)$ intersects at most 2^d cells for $l_{n+1} \leq \varepsilon \leq l_n$ and so

$$\mu(B(x,\varepsilon)) \leq 2^d 2^{-nd} \leq c\varepsilon^{log_b 2^d} + o(1/n).$$

$H_h > 0$

For $H_h > 0$ choosing h s.t. $c := lim 2^{nd} h(l_n) > 0$ then $2^{-nd} \approx h(l_n)c$ and so.

$$\mu(B(x,\varepsilon)) \leq 2^d 2^{-nd} \leq h(l_n)c \leq ch(r). \tag{0.1}$$

$m_h < \infty$

For $m_h < \infty$ choosing h s.t. $lim 2^{nd} h(l_n) < \infty$. Since

$$m_h^\varepsilon = inf \sum h(\varepsilon_j) : K \subset \bigcup K_j, |K_j| \leq \varepsilon_j \tag{0.2}$$

we upper bound by cover with $|K_j| = l_n$ and thus

$$m_h^\varepsilon \leq N(l_n)h(l_n) \leq cN(l_n)2^{-nd} < \infty, \tag{0.3}$$

since for $B(x,\varepsilon)$ intersects at most 2^d cells for $l_{n+1} \leq \varepsilon \leq l_n$.

Question 5

One one hand cover F by radius 2δ balls

$$vol(F_\delta) \leq c(2\delta)^d N(F).$$

on the the other hand, by radius δ disjoint balls

$$c(\delta)^d N(F) \leq vol(F_\delta).$$

6

a

$$limsup \sum r_j^\beta = limsup \sum r_j^\alpha r_j^{\beta-\alpha}$$
$$\leq lim \sup \varepsilon^{\beta-\alpha} P^\alpha \to 0.$$

b

By taking disjoint B_j with $|B_j| = \varepsilon$ then for $N_\varepsilon(K)$ the largest number of such balls we obtain

$$N_\varepsilon(K)\varepsilon^s \leq P_\delta^s < \infty$$

and so $\overline{Mdim}(K) \leq s$.

c

Let $\delta := 2^{-M-2}$ and consider disjoint balls B_j with $|B_j| \leq \delta$ and s.t.

$$\sum |B_j|^\alpha > P_\alpha(K)/4.$$

Next let n_k be the number of balls with $2^{-k-1} \leq |B_j| \leq 2^{-k}$ then

$$\sum n_k 2^{-(k+1)\alpha} > P_\alpha(K)/4.$$

d

Let $t \leq dim_P(K)$. Note that there exists k_* s.t.

$$n_k > (P_\alpha(K)/4^{\alpha+1})2^{kt}(1 - 2^{(t-\alpha)})$$

otherwise

$$P_\alpha(K)/4 < \sum n_k 2^{-(k+1)\alpha} = P_\alpha(K)/4.$$

Therefore, for $N(K, 2^{k+1})$

$$N(K, 2^{-(k+2)})2^{-(k+2)t} \geq n_k 2^{-(k+2)t} \geq 2^{-2t}(1 - 2^{(t-\alpha)}).$$

Thus,

$$\lim_{\delta \to 0} N(K,\delta)\delta^t > lim_{k\to\infty} N(K, 2^{-k})2^{-kt} > 0.$$

Thus, $Mdim(K) \geq t$ for all $t \leq dim_P(K)$.

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a

Monotonicity and empty set follow. For $P_\alpha(A \cup B) = P_\alpha(A) + P_\alpha(B)$, we use the δ packings of A,B to cover $A \cup B$ and vice-versa. Let $A_i \subset \bigcup_n E_{in}$ with $\sum P_0^\alpha(E_{in}) \leq P_0^\alpha(A_i) + \frac{\varepsilon}{2^i}$ then

$$\begin{aligned} P^\alpha(A) &\leq \sum_i \sum_n P_0^\alpha(E_{in}) \\ &\leq \sum_i P_0^\alpha(A_i) + \varepsilon. \end{aligned}$$

b

The upper bound $dim_P(K) \leq \overline{Mdim}(K)$: using $dim_P(\bigcup K_i) = sup_i dim(K_i)$ we obtain

$$dim_P(K) \leq dim_P(\bigcup K_i) = sup_i dim_P(K_i)$$

because $P^\alpha(K) \leq P_0(K)$

$$\leq sup_i dim_{\hat{P}}(K_i)$$

from previous exercise

$$= sup_i Mdim(K_i).$$

$$= \overline{Mdim}(K)$$

For lower bound $dim_P(K) \geq \overline{Mdim}(K)$: consider $t < dim_P(K)$, then $P_0^t(K) = 0$ and so $P_0^t(K_i) < \infty$ for some $K \subset \bigcup K_i$. By taking disjoint B_j with $|B_j| = \varepsilon$ then for $N_\varepsilon(K)$ the largest number of such balls we obtain

$$N_\varepsilon(K_i)\varepsilon^t \leq P_\delta^s(K_i) < \infty.$$

and so $\overline{Mdim}(K) \leq t$.

c

Because $|\bar{K_i}| = |K_i|$.

d

Follows from $\overline{Mdim}(K) \geq Hdim(K)$.

e

Since $Pdim(K) = \overline{Mdim}(K)$ we want set K s.t. $\overline{Mdim}(K) > \underline{Mdim}(K)$.

Let $k_n := 10^n$: remove 1/3 if $k_{2n} < k < k_{2n+1}$ and 3/5 on the next step $k_{2n-1} < k < k_{2n}$. So we have upper bound by taking $3^{-k} < \delta < 3^{-k+1}$

$$\begin{aligned} &\underline{\lim}\frac{logN_\delta}{log\delta} \\ &\leq \underline{\lim}\frac{\log 2^k}{\log 3^k} \\ &= log_3 2 \end{aligned}$$

and we have lower bound because any $\delta-$sized cover intersects at least one interval $(3/5)^{-k}$ size, and so at least 2^k are needed to cover K

$$\begin{aligned} &\overline{\lim}\frac{logN_\delta}{log\delta} \\ &\geq \overline{\lim}\frac{\log 2^k}{\log(3/5)^k} \\ &= \log_{3/5} 2 < log_3(2). \end{aligned}$$