SNAP HW PROBLEMS

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Some of the problems below are exercises, some require more background but hopefully stimulate curiosity. The more difficult ones are given $*$'s.

1. Problems about the wave kernel

1.1. Euclidean space and flat torus.

PROBLEM **1.1.** *Prove the formula for the solution of the 3D Euclidean wave equation from the Darboux-Euler formula.*

$$\begin{aligned} u(x,t) &= t\bar{g}(x,t) + \partial_t(t\bar{f}(x,t)) \\ &= \frac{1}{4\pi t}\int_{|y-x|=t} g(y)dS(y) + \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|y-x|=t} f(y)dS(y)\right). \end{aligned} \tag{1}$$

PROBLEM **1.2.** *Prove the Kirchhoff formula from the above solution formula.*

PROBLEM **1.3.** *Generalize the pullback proof of the Euclidean 3D formula for the fundamental solution to all $\mathbb{R}^{5+1}$..*

PROBLEM **1.4.** *The cosine wave kernel $E(t,x,y)$ always exhibits the finite propagation speed of the wave equation: it is zero for $r(x,y) > t$. Is the same true for the half-wave propagator $U(t,x,y)$ of $e^{it\sqrt{-\Delta}}$? Consider special cases such as Euclidean space or Hyperbolic space. (*) (Or think about the Paley-Wiener theorem).*

PROBLEM **1.5.** *Let $\mathbf{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the Euclidean flat torus. Let $E_{\mathbb{R}^n}(t,x,y)$, resp. $S_{\mathbb{R}^n}(t,x,y)$, be the Euclidean cosine, resp. sine, wave kernel on $\mathbb{R}^n$. Also let $U_{\mathbb{R}^n}(t) = \exp it\sqrt{-\Delta}$ on $\mathbb{R}^n$ and let $U_{\mathbb{R}^n}(t,x,y)$ be its Schwartz kernel. Let $E_{\mathbf{T}^n}$ (etc.) be the cosine (etc.) wave kernel of the flat torus.*

(1) *Show that*

$$E_{\mathbf{T}^n}(t,x,y) = \sum_{k\in\mathbb{Z}^n} E_{\mathbb{R}^n}(t,x,y+k)$$

and justify that the sum converges. Similarly for the sine wave kernel.

(2) *(*) Show that*

$$U_{\mathbf{T}^n}(t,x,y) = \sum_{k\in\mathbb{Z}^n} U_{\mathbb{R}^n}(t,x,y+k)$$

and explain in what senses the sum converges. Recall that

$$U(t,x,y) = C_n\, it((t+i0))^2 - r(x,y)^2)^{-\frac{n+1}{2}}, \tag{2}$$

1.2. **Round sphere $\mathbb{S}^2$.** The eigenvalues of $-\Delta_{\mathbb{S}^2}$ are$k(k+1) = (k+\frac{1}{2})^2 - \frac{1}{4}$. It is common to renormalize the Laplacian to $-\Delta_{\mathbb{S}^2} + \frac{1}{4}$..

PROBLEM **1.6.** *Let $U_{\mathbb{S}^n} = \exp it\sqrt{-\Delta + \frac{1}{4}}$. Show that*

$$U(\pi)f(x) = f(-x).$$

1.3. **Hyperbolic space and hyperbolic quotients.** Let $\mathbb{D}$ be the Poincare (hyperbolic disc) with metric $\frac{|dz|^2}{(1-r^2)^2}$, resp. $\mathbb{H}^2$ be the hyperbolic plane with metric $\frac{ds_0^2}{y^2}$, where $|dz|^2, ds_0^2$ denote the Euclidean metrics. They are normalized to have constant curvature -4. Let $(z, b) \in \mathbb{D} \times B$ where $B = \partial\mathbb{D}$ is the unit circle (it is $\mathbb{R}$ for $\mathbb{H}^2$. Define the Helgason bracket $\langle z, b\rangle$ to be the signed distance of the horocycle $O_{z,b}$ through z and touching B at b. Recall that a horocycle is a Euclidean circle tangent to the boundary. In the upper half plane model, horizontal lines are horocycles through ∞.

PROBLEM **1.7.**

(1) *Show that the hyperbolic Laplacian of $\mathbb{H}^2$ is $y^2\Delta_0$ where Δ_0 is the Euclidean Laplacian.*

(2) *Show that $\varphi_{\lambda,b}(z) := e^{(i\lambda+1)\langle z,b\rangle}$ is an eigenfunction of Δ of eigenvalue $-(\lambda^2+1)$. (Hint: use the upper half plane model and let $b = \infty$. Then use isometries.)*

(3) *Generalize the formula for the fundamental solution of the wave equation to $\mathbb{H}^5$.*

2. PROBLEMS ABOUT SPECTRAL PROJECTIONS KERNELS

PROBLEM **2.1.** *Let $\Pi_k : L^2(\mathbb{S}^2) \to \mathcal{H}_k$ be the orthogonal projection onto the degree k spherical harmonics. Thus, $\Pi_k f = f$ for $f \in \mathcal{H}_k$. Let $\Phi_k^p(x) = \frac{\Pi_k(x,p)}{\sqrt{\Pi_k(p,p}}$. (It is known as a zonal spherical harmonic).*

(1) *Show that $||\Phi_k^p||_{L^2} = 1$.*

(2) *Show that $||\Phi_k^p||_{L^\infty} = \sqrt{2k+1}$.*

(3) *Show that Φ_k^o has the maximal L^∞ norm among all $f \in \mathcal{H}_k$ with $||f||_{L^2} = 1$.*

3. PROBLEMS ABOUT WEAK* LIMITS AND MICROLOCAL DEFECT MEASURES

The purpose of defect measures and the more precise microlocal defect measures is to give a concrete object that characterizes why a non-compact sequence u_n with $||u_n||_{L^2} = 1$ is non-compact in L^2. Two famous sequences are

(1) non-compactness due to concentration: $u_n(x) = \epsilon_n^{-d/2}\varphi(\frac{x-x_0}{\epsilon_n})$ where $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $\epsilon_n \to 0$;

(2) non-compactness due to oscillation: $u_n(x) = \varphi(x)e^{\frac{2\pi i x\cdot\xi}{h_n}}$. where $h_n \to 0$.

PROBLEM **3.1.** *Show that both sequences are non-compact in L^2. Then compute their defect measures and semi-classical measures.*

Problem **3.2.** *Let γ be the equator of $\mathbb{S}^2$ and let φ_ℓ^γ be the L^2 normalized Gaussian beam along γ. Namely, $\varphi_\ell^\gamma = \frac{\Gamma(\ell+1)}{\Gamma(\ell+\frac{1}{2})}\Re(x_1+ix_2)^\ell$. Consider the matrix elements $\rho_\ell^\gamma(A) := \langle A\varphi_\ell^\gamma, \varphi_\ell^\gamma\rangle$. The microlocal defect measure (quantum limit) of this sequence of eigenfunctions is the probability measure $d\mu^\gamma$ on $S^*(\mathbb{S}^2)$ given by*

$$\int_{S^*S^2} a d\mu^\gamma := \lim_{\ell\to\infty} \rho_\ell^\gamma(Op(a)),$$

where $Op(a)$ is any pseudo-differential operator on S^2 with principal symbol a.

(1) *Show that $d\mu^\gamma = \delta_\gamma$, where $\int a\delta_\gamma = \int_\gamma a ds$. Note that γ is a Hamilton orbit on $S^*\mathbb{S}^2$, so this makes sense. At least, prove it for $a(x,\xi) = f(x)$.*

Remark **3.1.** *Using this observation, one can construct a sequence of eigenfunctions whose weak* limit is $\sum_{j=1}^n \delta_{\gamma_j}$ for any set $\{\gamma_1, \ldots, \gamma_n\}$ of closed geodesics. Can you guess what the sequence is?*

From this fact it follows that the set of weak limits of sequences of eigenfunctions is the set of all geodesic-flow-invariant probability measures on $S^*\mathbb{S}^2$. Hence, there are no constraints on what kind of invariant measures can arise as weak* limits.*

4. Problems about nodal sets

Problem **4.1.** *Let γ be the equator of $\mathbb{S}^2$ and let φ_ℓ^γ be the L^2 normalized Gaussian beam along γ. Namely, $\varphi_\ell^\gamma = \Re\frac{\Gamma(\ell+1)}{\Gamma(\ell+\frac{1}{2})}(x_1+ix_2)^\ell$. Let (θ, φ) be spherical coordinates with θ the rotational coordinate and φ the distance from the north pole.*

(1) *Show that $\varphi_\ell^\gamma(\theta,\varphi) = C_\ell(\cos \ell\theta)(\sin\varphi)^\ell$ for some $C_\ell > 0$.*

(2) *Calculate the order of magnitude of the doubling exponents $\beta(p,r)$ of r^N on $[0,T]$. Then do the same for φ_ℓ^γ in balls $B(p,r)$ centered at any point p and of any radius r. In particular, let p be the north pole.*

Problem **4.2.** *Let $\mathcal{N}_{\Phi_k^p}$ be the nodal set of the zonal spherical harmonic. Show that the 'normalized currents of integration'*

$$\langle \frac{1}{k}[\mathcal{N}_{\Phi_k^p}], f\rangle := \int_{\mathcal{N}_{\Phi_k^p}} f dS$$

over the nodal sets of the (real-valued) Gaussian beam Do the same for φ_k^γ. tends as $k\to\infty$ to the surface average $⨍_{\mathbb{S}^2} f dS$.

Problem **4.3.** (1) *Calculate the length(s) of the nodal line(s) for the zonal spherical harmonic and for the Gaussian beam of $\mathbb{S}^2$.*

(2) *A guessing game (do not prove that your answer is correct): Which spherical harmonic* $f_k \in \mathcal{H}_k$ *has the longest nodal line? Which one has the shortest?*

(Remark: The longest one is known and is not hard to prove using Crofton's formula. It has not been determined rigorously which one has the shortest nodal line, though there is an obvious candidate).

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