SNAP COURSE: EIGENVALUES AND EIGENFUNCTIONS: SOLUTIONS

Question 0.1.

Proof. We are looking to solve the equation:

$$-\Delta\varphi(x,y) = \lambda\varphi(x,y)$$

with either Dirichlet or Neumann boundary conditions. The standard technique is to use separation of variables:

$$\varphi(x,y) = f(x)g(y).$$

Plugging this in and rearrainging gives

$$-\frac{f''}{f} = \lambda + \frac{g''}{g}.$$

Since the left-hand side only depends on x and the right-hand side only depends on y, we conclude that both must equal a constant, c. Thus we reduce to solving the pair of ODEs:

$$\begin{cases} f''(x) = -cf(x) \\ g''(y) = (c-\lambda)g(y). \end{cases}$$

These equations are the same, but with different constants. Using the power series method, one can check that the solution to this equation is $A\sin(nx) + B\cos(nx)$ if the constant is negative, $A\sinh(nx) + B\cosh(nx)$ if the constant is positive, and $Ax + B$ if the constant is zero.

We can now apply our boundary conditions. For the Dirichlet boundary conditions, the only admissible solution is $A\sin(nx) + B\cos(nx)$, and so we get:

$$\begin{cases} f(x) = A_1\sin(nx) + B_1\cos(nx) \\ g(y) = A_2\sin(my) + B_2\cos(my) \\ \lambda = n^2 + m^2 \end{cases}$$

Again by the Dirichlet boundary conditions, it is clear that we cannot have any of the cosine terms, and that $n = k\pi/a$, $m = \ell\pi/b$, for some $k, \ell \in \mathbb{N}$. Thus, in this case, the spectra is:

$$\sigma_D([0,a]\times[0,b]) = \left\{\pi^2\left(\frac{k^2}{a^2} + \frac{\ell^2}{b^2}\right) \mid k,\ell \in \mathbb{N}\right\}.$$

For the Neumann boundary conditions, we also see that either f or g could be a constant. Thus, we have three cases:

$$\begin{cases} f(x) = A_1\sin(nx) + B_1\cos(nx) \\ g(y) = A_2\sin(my) + B_2\cos(my) \\ \lambda = n^2 + m^2 \end{cases} \text{ or } \begin{cases} f(x) = A_1\sin(nx) + B_1\cos(nx) \\ g(y) = C \\ \lambda = n^2 \end{cases}$$

$$\text{or} \begin{cases} f(x) = C \\ g(y) = A_2 \sin(my) + B_2 \cos(my) \\ \lambda = m^2 \end{cases}$$

Now we cannot have any of the sine terms, but we have the same restrictions on n and m. Thus, the spectra is given by:

$$\sigma_N([0,a] \times [0,b]) = \left\{ \pi^2 \left(\frac{k^2}{a^2} + \frac{\ell^2}{b^2} \right) \mid k, \ell \in \mathbb{N} \cup \{0\} \right\}.$$

Rmk: In order to see that this is actually the spectrum, we need to check that we have actually found all the eigenfunctions using this method. The standard method is to check that this set of eigenfunctions is dense in $W_0^{1,2}$ – this is a well-known consequence of the Stone-Wierstrauss Theorem. □

Question 0.2.

Proof. We will again use separation of variables, except in polar coordinates this time:

$$\varphi(r, \theta) = f(r)g(\theta).$$

Plugging this in and rearrainging gives

$$r^2 \left(\frac{f'' + (1/r)f'}{f} + \lambda \right) = -\frac{g''}{g} = c.$$

Thus, we reduce to solving the pair of ODEs:

$$\begin{cases} r^2 f''(r) + rf'(r) + (\lambda r^2 - c)f(r) = 0 \\ g''(\theta) = -cg(\theta). \end{cases}$$

The equation for g is the same as before, so again the solution is either $A\sin(n\theta) + B\cos(n\theta)$ if c is positive, $A\sinh(n\theta) + B\cosh(n\theta)$ if c is negative, or $A\theta + B$ if c is zero. By construction, g needs to be periodic, so the only acceptable solutions are $g(\theta) = A\sin(n\theta) + B\cos(n\theta)$, with $n \in \mathbb{N} \cup \{0\}$ and $c = n^2$.

Recalling that $\lambda > 0$, we then see that the rescaling $s = \sqrt{\lambda} r$ reduces the equation for f to the Bessel equation of order n. We want our eigenfunction to be defined on the whole disk, so we see that $f(r) = J_n(\sqrt{\lambda} r)$, where J_n is the n^{th}-order Bessel function of the first kind. We also remark that, by construction, we need to have $f(0) = 0$ if g is not constant – this is however automatically satisfied, as $J_n(0) = 0$ for all $n \geq 1$.

We can now apply our boundary conditions. For the Dirichlet boundary conditions, we see that we need $J_n(\sqrt{\lambda}) = 0$, and so the specturm is:

$$\sigma_D(\mathbb{D}) = \{\beta_{n,k}^2 \mid k \in \mathbb{N}, n \in \mathbb{N} \cup \{0\}, \text{ and } \beta_{n,k} \text{ is the } k^{\text{th}} \text{ zero of } J_n\}.$$

For the Neumann boundary conditions, we need $J_n'(\sqrt{\lambda}) = 0$, and so the specturm is:

$$\sigma_N(\mathbb{D}) = \{\gamma_{n,k}^2 \mid k \in \mathbb{N}, n \in \mathbb{N} \cup \{0\}, \text{ and } \gamma_{n,k} \text{ is the } k^{\text{th}} \text{ zero of } J_n'\}.$$

Rmk: Again, completness of this set of eigenfunctions is a well-known consequence of the Stone-Wierstrauss Theorem.

□

Question 0.3.

Proof. The translational property is immediate – if $f(x)$ is an eigenfunction on Ω, then $f(x + x_0)$ is an eigenfunction on $\Omega + x_0$ with the same eigenvalue, and visa-versa. The scaling property is similar;, suppose $f(x)$ is an eigenfunction on Ω with eigenvalue λ. Then $f(\alpha^{-1}x)$ is an eigenfunction on $\alpha\Omega$ with eigenvalue $\alpha^{-2}\lambda$, and visa-versa. □

Question 0.4.

Proof. Suppose that A is an $(n \times n)$-orthogonal matrix. Recall then that A^{-1} is also an orthogonal matrix, and that, when viewed as vectors in $\mathbb{R}^n$, we norm of the rows is one. Let $A^{-1} = (a_{ij})_{1\le i,j\le n}$, and suppose that $u(x)$ is an eigenfunction on Ω with eigenvalue λ. Then we claim $u(y) = u(A^{-1}x)$ is an eigenfunction on $A\Omega$. Indeed, by the chain rule:

$$\frac{\partial^2}{\partial y_k^2}[u(A^{-1}x)] = a_{k,1}^2 \frac{\partial^2 u}{\partial x_1^2}(A^{-1}x) + \ldots + a_{k,n}^2 \frac{\partial^2 u}{\partial x_n^2}(A^{-1}x)$$

$$\implies \Delta[u(A^{-1}x)] = \left(\sum_{k=1}^n a_{k,1}^2\right) \frac{\partial^2 u}{\partial x_1^2}(A^{-1}x) + \ldots + \left(\sum_{k=1}^n a_{k,n}^2\right) \frac{\partial^2 u}{\partial x_n^2}(A^{-1}x)$$

$$= [\Delta u](A^{-1}x) = \lambda u(A^{-1}x).$$

In particular, if $u(x)$ is harmonic, then so too is $u(A^{-1}x)$.

□

Question 0.5.

Proof. Let $\widetilde{\lambda}_n := \inf_{X\in\phi_n(V)} \sup_{u\in X} \rho(u)$. By the first version of the variational principle, we have that $\lambda_n = \sup_{u\in\text{span}\{u_1,\ldots,u_n\}} \rho(u)$, so the inequality $\widetilde{\lambda}_n \le \lambda_n$ is immediate. For the other direction, consider some $X \in \phi_n(V)$. For dimensional reasons, we see that $H_{n-1}(V) \cap X \neq \{0\}$, and so there is a non-zero element $u_0 \in H_{n-1}(V) \cap X$. It thus follows that:

$$\lambda_n = \inf_{u\in H_{n-1}(V)} \rho(u) \le \rho(u_0) \le \sup_{u\in X} \rho(u),$$

and so taking the infimum over all X gives the other inequality $\lambda_n \le \widetilde{\lambda}_n$. □

Question 0.6.

Proof. Consider a rectangle Ω with sides of length a and b, and $b < a$. Then as we have seen, $\mu_1(\Omega) = \pi^2/(4a^2)$. The trick is to have $1 < a < \sqrt{2}$ and b sufficently small so that, after rotating, Ω fits inside the unit square along the diagonal. The lowest Neumann eigenvalue for the unit square is $\pi^2/4 < \pi^2/(4a^2)$, as $a > 1$.

□

Question 0.7.

Proof. Let $\mathbb{D}_-$ and $\mathbb{D}_+$ be the largest disk contained in Ω and the smallest disk containing Ω respectively. By the scaling properties of the Laplacian, we know that $\lambda_1(\mathbb{D}_-) = R_-^{-2}\lambda_1(\mathbb{D})$ and $\lambda_1(\mathbb{D}_+) = R_+^{-2}\lambda_1(\mathbb{D})$, and so by domain monotonicity we have:

$$\frac{\lambda_1(\mathbb{D})}{R_+^2} \le \lambda_1(\Omega) \le \frac{\lambda_1(\mathbb{D})}{R_-^2}.$$

We have also already seen that $\sigma_D(\mathbb{D}) = \{\beta_{n,k}^2\}$, with $k \in \mathbb{N}$, $n \in \mathbb{N} \cup \{0\}$ and $\beta_{n,k}$ the k^{th} zero of the n^{th}-order Bessel function of the first kind – thus, we only need to determine the smallest such $\beta_{n,k}$. It is $\beta_{0,1}$.

□

Question 0.8.

Proof. The upper-bound follows directly from domain monotonicity, which is itself a direct consequence of the second version of the variational principle. The lower-bound is similar – the space of test functions for $\sigma_D(\Omega)$ is $W_0^{1,2}(\Omega)$, while the space of test functions for $\sigma_{DN}(\Omega_1 \cup \Gamma \cup \Omega_2)$ is $\{u \in W^{1,2}(\Omega_1) \mid u|_{\partial\Omega_1 \setminus \Gamma} = 0\} \oplus \{u \in W^{1,2}(\Omega_2) \mid u|_{\partial\Omega_2 \setminus \Gamma} = 0\}$; clearly the former embeds into the later, and so by the min-max theorem, the inequality follows.

Rmk: We are being imprecise here when we write $u|_{\partial\Omega_1 \setminus \Gamma} = 0$ for $u \in W^{1,2}(\Omega_1)$ – indeed u is not obviously even defined on $\partial\Omega_1$. There is a way to make sense of this, but we will not discuss it here.

□

Question 0.9.

Proof. Let us compute $\sigma_{DN}(\Omega_1)$ first:

$$-\Delta \sin((1/2+m)\pi x)\sin(n\pi y) = \pi^2((1/2+m)^2+n^2)\sin((1/2+m)\pi x)\sin(n\pi y)$$

$$\implies \sigma_{DN}(\Omega_1) = \left\{\pi^2\left(\left(\frac{1}{2}+m\right)^2+n^2\right) \mid m \in \mathbb{N}\cup\{0\}, n \in \mathbb{N}\right\}.$$

Now for $\sigma_{DN}(\Omega_2)$, define:

$f_{k,\ell} = \sin((1/2+k)\pi x/\sqrt{2})\sin((1/2+\ell)\pi y/\sqrt{2}) - \sin((1/2+\ell)\pi x/\sqrt{2})\sin((1/2+k)\pi y/\sqrt{2})$.

Then one computes:

$$-\Delta f_{k,\ell} = \pi^2\left(\frac{(1/2+k)^2}{2}+\frac{(1/2+\ell)^2}{2}\right)f_{k,\ell}$$

$$\implies \sigma_{DN}(\Omega_2) = \left\{\frac{\pi^2}{2}\left(\left(\frac{1}{2}+k\right)^2+\left(\frac{1}{2}+\ell\right)^2\right) \mid k,\ell \in \mathbb{N}\cup\{0\}, k > \ell\right\}.$$

We must now verify that these are equal. The trick is to note that:

$$\frac{\pi^2}{2}\left(\left(\frac{1}{2}+m+n\right)^2+\left(\frac{1}{2}+m-n\right)^2\right) = \frac{\pi^2}{2}\left(\left(\frac{1}{2}+m+n\right)^2+\left(\frac{1}{2}+n-m-1\right)^2\right) =$$

$$= \frac{\pi^2}{2}\left(\left(\frac{1}{2}+m\right)^2+(1+2m)n+n^2+\left(\frac{1}{2}+m\right)^2-(1+2m)n+n^2\right) =$$

$$= \pi^2\left(\left(\frac{1}{2}+m\right)^2+n^2\right).$$

This gives us an injection from $\sigma_{DN}(\Omega_1)$ into $\sigma_{DN}(\Omega_2)$. One can now check that it is a bijection easily – note that it is necessary to take into account both expressions listed in the first line, depending on the parity of $k-\ell$.

□

Question 0.10.

Proof. Note that, by question 2, each eigenspace, corresponding to $\beta_{n,k}^2$ say, will be two dimensional (unless $n = 0$), spanned by $J_n(\beta_{n,k}r)\sin(n\theta)$ and $J_n(\beta_{n,k}r)\cos(n\theta)$. Each of these will clearly have the same number of nodal domains, so we can stick to analyzing $J_n(\beta_{n,k}r)\sin(n\theta)$. It is well-known that J_n is oscillatory, and so we see that the number of times it will flip signs as r goes from zero to one will be $k-1$. The number of times that $\sin(n\theta)$ oscillates is $2n$, so we can count the total number of nodal domains to be $2nk$ if $n > 0$, and just k if $n = 0$.

To compute the length of the nodal set, note that the nodal domains will form a radial "grid," and so we can calculate the length easily – it will be $2n+2\pi\sum_{i=1}^{k-1}\frac{\beta_{n,i}}{\beta_{n,k}}$.

We are then left to find the five lowest eigenvalues. Checking a table indicates that they are $\beta_{0,1}, \beta_{1,1}, \beta_{2,1}, \beta_{0,2}$, and $\beta_{3,1}$.

□

Question 0.11.

Proof. Suppose that there were two eigenfunctions with only one nodal domain, which we'll label WLOG as ϕ_1 and ϕ_2. We can also suppose WLOG that $\phi_1, \phi_2 \geq 0$. Then by orthogonality:

$$0 = \int_\Omega \phi_1\phi_2.$$

But this is a contradiction, as $\phi_1\phi_2$ is a non-zero, positive function on Ω, and hence has to have positive integral.

□

Question 0.12.

Proof. Suppose we have the eigenfunction $\phi_{k,\ell} = \sin(k\pi/a\,x)\sin(\ell\pi/b\,y)$. It is clear that the nodal domains will form a $(k+1)\times(\ell+1)$ grid of smaller rectangles, and so each will have side lengths $a/(k+1)$ and $b/(\ell+1)$. Thus, their inradius will be $\min\{a/(k+1), b/(\ell+1)\}$.

□

Question 0.13.

Proof. Let us change notation slightly from the problem and define:

$$P_k^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^{k+m}}{dx^{k+m}}(x^2-1)^k.$$

Then it is clear that $P_{-k}^k(\cos(\phi))) = \sin^k(\phi)$, and so the Gaussian beams are given as:

$$Y_{-k}^k(\theta,\phi) = c_{k,m}\sin^k(\phi)e^{-ik\theta}.$$

We then need to calculate the L^p-norm of Y_{-k}^k and show that there is a constant $C > 0$ so that:

$$\left\|Y_{-k}^k\right\|_p \leq C\lambda^{\delta(p)},$$

for all λ, where λ^2 is an eigenvalue and $\delta(p) = \frac{1}{4} - \frac{1}{2p}$. We know that the eigenvalue of Y_m^k is $k(k+1)$, and observe that for large k we have:

$$k^2 \leq k(k+1) \leq 2k^2,$$

so it is enough to show:

$$\left\|Y_{-k}^k\right\|_p \leq Ck^{\frac{1}{4}-\frac{1}{2p}}.$$

Our first step will be to bound the order of growth of $c_{k,-k}$, which is given by $\left\|\sin^k(\phi)e^{-ik\theta}\right\|_2^{-1}$:

$$\left\|\sin^k(\phi)e^{-ik\theta}\right\|_2^2 = \int_{S^2} \left|\sin^{2k}(\phi)\right| \left|e^{-i2k\theta}\right| dA = 2\pi \int_0^\pi \sin^{2k+1}(\phi)d\phi$$

To evaluate this integral, use integration by parts to see:

$$\int_0^\pi \sin^{2k+1}(\phi)d\phi = 2k\int_0^\pi \cos^2(\phi)\sin^{2k-1}(\phi)d\phi$$
$$\implies \int_0^\pi \sin^{2k+1}(\phi)d\phi = \frac{2k}{2k+1}\int_0^\pi \sin^{2k-1}(\phi)d\phi = \frac{(2k)\cdot\ldots\cdot(2)}{(2k+1)\cdot\ldots\cdot(3)}$$

Now pull the 2 out from each of the terms in the top and bring them down to get:

$$\frac{1}{2\pi}c_{k,-k}^{-2} = \frac{k!}{(k+\frac{1}{2})\cdot(k-\frac{1}{2})\cdot\ldots\cdot(1+\frac{1}{2})}.$$

In order to bound this quantity, we recall Stirling's Formula:

$$\sqrt{2\pi}e^{-n}n^{n+1/2} \le n! \le e^{-n+1}n^{n+1/2},$$

which then gives:

$$\frac{C^{-1}}{\sqrt{k}} \le C^{-1}\frac{e^{-k}k^{k+1/2}}{e^{-k-1/2}(k+1/2)^{k+1}} \le c_{k,-k}^{-2} \le C\frac{e^{-k}k^{k+1/2}}{e^{-k-1/2}(k+1/2)^{k+1}} \le \frac{C}{\sqrt{k}}$$
$$\implies C^{-1}k^{1/4} \le c_{k,-k} \le Ck^{1/4}.$$

We can now perform a similar analysis to bound the L^p-norm:

$$\left\|Y_{-k}^k\right\|_p^p = 2\pi c_{k,-k}^p \int_0^\pi \sin^{pk+1}(\phi)d\phi \le Ck^{p/4}\int_0^\pi \sin^{pk+1}(\phi)d\phi,$$
$$\int_0^\pi \sin^{pk+1}(\phi)d\phi = \frac{(pk)\cdot\ldots\cdot(s+1)}{(pk+1)\cdot\ldots\cdot(s+2)}\int_0^\pi \sin^s(\phi)d\phi$$

where $0 \le s < 2$, and so can be bounded away by a constant. Stirling's Formula then gives:

$$\left\|Y_{-k}^k\right\|_p \le Ck^{1/4}k^{-1/2p} = Ck^{\delta(p)},$$

as desired. One could also have worked with Y_k^k instead – it is the same as Y_{-k}^k up to a sign and a factor of e^{2ki}.

Rmk: One can also show that for $6 \le p \le \infty$, the upper-bound is saturated by the zonal spherical harmonics, i.e. those with $m = 0$. $\square$