Differentiating the Non-Differentiable: Problem Set 3

Exercise 16. (Product Rule) Prove that if $g \in C^\infty(\mathbb{R}^n)$, $L \in \mathscr{D}'(\mathbb{R}^n)$, and $i \in \{1, \dots, n\}$, then $\frac{\partial}{\partial x_i}(gL) = \frac{\partial g}{\partial x_i} L + g \frac{\partial L}{\partial x_i}$.

Exercise 17. The **Heaviside function** is defined as

$$H : \mathbb{R} \to \mathbb{R}, \quad H(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Prove that $H' = \delta$ in the sense of distributions on $\mathbb{R}$.

Exercise 18. Prove that $\frac{d}{dx} \ln|x| = p.v.\frac{1}{x}$ in the sense of distributions on $\mathbb{R}$.

Exercise 19. Prove the following result using the outline provided.

Theorem 0.1 (Antiderivatives)**.** *Let $L \in \mathscr{D}'(\mathbb{R})$. There exists $L^{(-1)} \in \mathscr{D}'(\mathbb{R})$ such that $L = \frac{dL^{(-1)}}{dx}$. Moreover, if $S \in \mathscr{D}'(\mathbb{R})$ satisfies $\frac{dS}{dx} = L$, then there exists a constant $C \in \mathbb{R}$ with $S = L^{(-1)} + C$ (as distributions). In particular, if $L \in \mathscr{D}'(\mathbb{R})$ with $\frac{dL}{dx} = 0$, then L is given by integration against a constant function.*

Fix $\omega \in \mathscr{D}(\mathbb{R})$ with $\int_{\mathbb{R}} \omega(x)dx = 1$. For $\phi \in \mathscr{D}(\mathbb{R})$ define

$$\Phi(x) = \int_{-\infty}^{x} \left[\phi(t) - \omega(t) \int_{\mathbb{R}} \phi(s)ds\right] dt.$$

(a) Prove that $\Phi \in \mathscr{D}(\mathbb{R})$ whenever $\phi \in \mathscr{D}(\mathbb{R})$.

(b) Define

$$L^{(-1)} : \mathscr{D}(\mathbb{R}) \to \mathbb{R}, \quad \left\langle L^{(-1)}, \phi \right\rangle := -\langle L, \Phi \rangle.$$

Prove that $L^{(-1)} \in \mathscr{D}'(\mathbb{R})$ and $\frac{dL^{(-1)}}{dx} = L$.

(c) Finish the proof.

Exercise 20. Suppose that $L \in \mathscr{D}'(\mathbb{R})$ and that $g \in C^\infty(\mathbb{R})$. If $L' + gL = f \in C^0(\mathbb{R})$ in the sense of distributions, then there exists $u \in C^1(\mathbb{R})$ such that $L = L_u$ and $u' + gu = f$.

Exercise 21. Produce a fundamental solution for Laplace's operator $\Delta = \frac{d^2}{dx^2}$ on $\mathbb{R}$. That is, produce a locally integrable function $N : \mathbb{R} \to \mathbb{R}$ such that $\frac{d^2 N}{dx^2} = N'' = \delta$.

Exercise 22. Regularization gives us yet another way to motivate distributional derivatives. Let $L \in \mathscr{D}'(\mathbb{R}^n)$, and fix $i \in \{1, \ldots, n\}$. Let $f_j \in C^\infty(\mathbb{R}^n)$ with $f_j \to L$ in the sense of distributions. Prove that $\frac{\partial f_j}{\partial x_i} \to \frac{\partial L}{\partial x_i}$ in the sense of distributions as well.

Exercise 23. Let $f \in \mathscr{D}(\mathbb{R}^2)$. Prove that there exists $u \in C^\infty(\mathbb{R}^2)$ satisfying $\Delta u = f$.