Differentiating the Non-Differentiable: Problem Set 2

Exercise 9. For $L \in \mathscr{D}'(\mathbb{R}^n)$ and $g \in C^\infty(\mathbb{R}^n)$, define

$$gL : \mathscr{D}(\mathbb{R}^n) \to \mathbb{R}, \quad \langle gL, \phi \rangle := \langle L, g\phi \rangle .$$

Prove that $gL \in \mathscr{D}'(\mathbb{R}^n)$.

Exercise 10. Find all $L \in \mathscr{D}'(\mathbb{R})$ that satisfy $xL = 0$ in the sense of distributions. (Suggestion: First produce a fixed $\psi \in \mathscr{D}(\mathbb{R})$ with $\psi(x) = 1$ for $x \in [-1, 1]$. For $\phi \in \mathscr{D}(\mathbb{R})$, write $\phi = \psi\phi + (1 - \psi)\phi$. Show that $\langle L, (1 - \psi)\phi \rangle = 0$, and then use an appropriate Taylor expansion (with remainder) on ϕ to compute $\langle L, \psi\phi \rangle$.)

Exercise 11. For $g \in C^0(\mathbb{R}^n)$ and $\boldsymbol{x}_0 \in \mathbb{R}^n$ define $\tau_{\boldsymbol{x}_0} g(\boldsymbol{x}) := g(\boldsymbol{x} - \boldsymbol{x}_0)$.

(a) Prove that for $f \in C^0(\mathbb{R}^n)$, $\phi \in \mathscr{D}(\mathbb{R}^n)$, and $\boldsymbol{x}_0 \in \mathbb{R}^n$ we have

$$\int_{\mathbb{R}^n} (\tau_{\boldsymbol{x}_0} f)(\boldsymbol{x}) \phi(\boldsymbol{x}) dV_n(\boldsymbol{x}) = \int_{\mathbb{R}^n} f(\boldsymbol{x}) (\tau_{-\boldsymbol{x}_0} \phi)(\boldsymbol{x}) dV_n(\boldsymbol{x}).$$

(b) To generalize (a), let $L \in \mathscr{D}'(\mathbb{R}^n)$ and $\boldsymbol{x}_0 \in \mathbb{R}^n$. Prove that the map

$$\tau_{\boldsymbol{x}_0} L : \mathscr{D}(\mathbb{R}^n) \to \mathbb{R}, \quad \langle \tau_{\boldsymbol{x}_0} L, \phi \rangle := \langle L, \tau_{-\boldsymbol{x}_0} \phi \rangle$$

is a distribution on $\mathbb{R}^n$.

Exercise 12. For each $\epsilon > 0$, define $L_\epsilon : \mathscr{D}(\mathbb{R}) \to \mathbb{R}$ by

$$\langle L_\epsilon, \phi \rangle = \int_{|x| \geq \epsilon} \frac{\phi(x)}{x} dx = \int_{-\infty}^{-\epsilon} \frac{\phi(x)}{x} dx + \int_{\epsilon}^{\infty} \frac{\phi(x)}{x} dx.$$

(a) Prove that for each $0 < \epsilon < R$ and for every $\phi \in \mathscr{D}(\mathbb{R}^n)$ with $\operatorname{supp}(\phi) \subseteq [-R, R]$,

$$\langle L_\epsilon, \phi \rangle = \int_{\epsilon \leq |x| \leq R} \frac{\phi(x) - \phi(0)}{x} dx.$$

(b) Produce $C = C(R)$, independent of ϵ, so that for all $\phi \in \mathscr{D}(\mathbb{R})$ with $\operatorname{supp}(\phi) \subseteq [-R, R]$,

$$| \langle L_\epsilon, \phi \rangle | \leq C \|\phi\|_1.$$

(c) For each $\phi \in \mathscr{D}(\mathbb{R})$, prove that $\langle L, \phi \rangle := \lim_{\epsilon \to 0+} \langle L_\epsilon, \phi \rangle$ exists. Prove that $L : \mathscr{D}(\mathbb{R}) \to \mathbb{R}$ is linear and satisfies $| \langle L, \phi \rangle | \leq C \|\phi\|_1$ for all $\phi \in \mathscr{D}(\mathbb{R})$ with $\operatorname{supp}(\phi) \subseteq [-R, R]$, where C is the constant from (b), and conclude that $L \in \mathscr{D}'(\mathbb{R})$.

L is the closest thing we have to a regular distribution corresponding to the (not locally integrable) function $\frac{1}{x}$. The distribution L is called the **principal value** of $\frac{1}{x}$, and is denoted $L = p.v.\left(\frac{1}{x}\right)$.

Exercise 13. For $k \in \mathbb{N}$, let $f_k : \mathbb{R} \to \mathbb{R}$ be $f_k(x) = \cos(kx)$. Prove that $\lim_{k\to\infty} f_k = 0$ in the sense of distributions.

Exercise 14. Show that $\lim_{\epsilon\to 0+} \eta_{\epsilon,\mathbf{0}} = \delta$ in the sense of distributions.

Exercise 15. Here's another motivation for our definition of distributional derivatives, this time via the definition of partial derivatives from multivariable calculus. Recall the operator $\tau_{\boldsymbol{x}}$ from Exercise 11. Fix $i \in \{1, \ldots, n\}$, and for $L \in \mathscr{D}'(\mathbb{R}^n)$ and $h \in \mathbb{R}\backslash\{0\}$ define

$$\Delta_h L = \frac{1}{h}(\tau_{-h\boldsymbol{e}_i} L - L) \in \mathscr{D}'(\mathbb{R}^n).$$

Prove that $\lim_{h\to 0} \Delta_h L = \frac{\partial L}{\partial x_i}$ in the sense of distributions.