Differentiating the Non-Differentiable: Problem Set 1

Exercise 1. Prove that if $f : \mathbb{R}^n \to \mathbb{R}$ is locally integrable and if $\phi : \mathbb{R}^n \to \mathbb{R}$ is continuous, then $f\phi : \mathbb{R}^n \to \mathbb{R}$ is locally integrable.

Exercise 2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous at all but perhaps finitely many points. Prove that f is locally integrable if and only if $\displaystyle\int_{\overline{B_R(\mathbf{0})}} |f| dV_n < \infty$ for every $R > 0$.

Exercise 3. Define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(\boldsymbol{x}) = \begin{cases} 0 & \text{if } \boldsymbol{x} = \mathbf{0}, \\ \frac{1}{2}\ln(x_1^2 + \cdots + x_n^2) & \text{if } \boldsymbol{x} \neq \mathbf{0}. \end{cases}$$

Prove that f is locally integrable on $\mathbb{R}^n$ when $n = 1, 2$. (If you are familiar with spherical coordinates in $\mathbb{R}^n$ for $n \geq 3$, then you should prove the result for these values of n as well.)

Exercise 4. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{1}{x}$ $(x \neq 0)$ and $f(0) = 0$. Prove that f is not locally integrable on $\mathbb{R}$.

Exercise 5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally integrable and suppose that f is continuous at $\boldsymbol{a} \in \mathbb{R}^n$. Prove that

$$\lim_{\epsilon \to 0+} \langle L_f, \eta_{\epsilon,\boldsymbol{a}} \rangle = \lim_{\epsilon \to 0+} \int_{\mathbb{R}^n} f(\boldsymbol{x}) \eta_{\epsilon,\boldsymbol{a}}(\boldsymbol{x}) dV_n(\boldsymbol{x}) = f(\boldsymbol{a}).$$

(Suggestion: It may be easier to show that $\displaystyle\lim_{\epsilon \to 0+} |\langle L_f, \eta_{\epsilon,\boldsymbol{a}} \rangle - f(\boldsymbol{a})| = 0$.)

Exercise 6. Let $\phi \in \mathscr{D}(\mathbb{R}^n)$ and fix $i \in \{1, \ldots, n\}$. For $h \in [-1, 1]$ with $h \neq 0$ define $\phi_{h,i}(\boldsymbol{x}) = \frac{\phi(\boldsymbol{x}+h\boldsymbol{e}_i)-\phi(\boldsymbol{x})}{h}$. Prove that $\phi_{h,i} \to \frac{\partial \phi}{\partial x_i}$ in $\mathscr{D}(\mathbb{R}^n)$ as $h \to 0$.

Exercise 7. For $\psi \in C^\infty(\mathbb{R}^n)$ and $\phi \in \mathscr{D}(\mathbb{R}^n)$ we define the **convolution** of ψ and ϕ, $\psi * \phi$, by

$$\psi * \phi : \mathbb{R}^n \to \mathbb{R}, \quad (\psi * \phi)(\boldsymbol{x}) = \int_{\mathbb{R}^n} \psi(\boldsymbol{x} - \boldsymbol{y})\phi(\boldsymbol{y})dV_n(\boldsymbol{y}).$$

(a) Prove that $(\psi * \phi)(\boldsymbol{x}) = (\phi * \psi)(\boldsymbol{x})$.

(b) Prove that $\operatorname{supp}(\psi * \phi) \subseteq \operatorname{supp}(\psi) + \operatorname{supp}(\phi)$.

(c) Prove that $\psi * \phi \in C^\infty(\mathbb{R}^n)$ with $\partial^\alpha(\psi * \phi) = (\partial^\alpha \psi) * \phi = \psi * (\partial^\alpha \phi)$ for every multi-index α, and therefore $\psi * \phi \in C^\infty(\mathbb{R}^n)$. As a consequence, show that $\psi * \phi \in \mathscr{D}(\mathbb{R}^n)$ in the special case where $\operatorname{supp}(\psi)$ is compact.

(d) For $h > 0$, consider the Riemann sum

$$S_h(\boldsymbol{x}) = \sum_{\boldsymbol{m} \in \mathbb{Z}^n} \psi(\boldsymbol{x} - h\boldsymbol{m})\phi(h\boldsymbol{m})h^n.$$

(i) Prove that for each $h > 0$ all but finitely many terms in the sum S_h are identially zero (and therefore the sum is finite).

(ii) Now assume that $\psi \in \mathscr{D}(\mathbb{R}^n)$. Prove that $S_h \in \mathscr{D}(\mathbb{R}^n)$ for all $h > 0$ and that $S_h \to \psi * \phi$ in $\mathscr{D}(\mathbb{R}^n)$ as $h \to 0+$.
(Suggestion: For $\boldsymbol{x} \in \mathbb{R}^n$, first write

$$(\psi * \phi)(\boldsymbol{x}) = \sum_{\boldsymbol{z} \in \mathbb{Z}^n} \int_{h\boldsymbol{m} + [0,h]^n} \psi(\boldsymbol{x} - \boldsymbol{y})\phi(\boldsymbol{y})dV_n(\boldsymbol{y}),$$

and then, for each multi-index α, estimate $|\partial^\alpha S_h(\boldsymbol{x}) - \partial^\alpha(\psi * \phi)(\boldsymbol{x})|$.)

Exercise 8. Prove that if $\phi \in \mathscr{D}(\mathbb{R}^n)$, then $\eta_{\epsilon,\boldsymbol{0}} * \phi \to \phi$ in $\mathscr{D}(\mathbb{R}^n)$ as $\epsilon \to 0+$.