Remarks on the inviscid limit problem for the Navier-Stokes equations

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ABSTRACT. For data which are analytic only close to the boundary of the domain, we prove that in the inviscid limit the Navier-Stokes solution converges to the corresponding Euler solution. Compared to earlier results, in this paper we only require boundedness of an integrable analytic norm of the initial data, with respect to the normal variable, thus removing the uniform in viscosity boundedness assumption on the vorticity. As a consequence, we may allow the initial vorticity to be unbounded close to the set $y = 0$, which we take as the boundary of the domain; in particular the vorticity can grow with the rate $1/y^{1-\delta}$ for y close to 0, for any $\delta > 0$.

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1. Introduction

In this paper, we address the inviscid limit problem for the Navier-Stokes equations

$$\partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla p = 0 \tag{1.1}$$

$$\operatorname{div} u = 0 \tag{1.2}$$

with an incompressible initial datum

$$u|_{t=0} = u_0 \,. \tag{1.3}$$

It is a well-known open problem to identify the initial data u_0 for which the solutions of the system (1.1)–(1.3) converge to the solution of the Euler system

$$\partial_t \bar{u} + \bar{u} \cdot \nabla \bar{u} + \nabla \bar{p} = 0 \tag{1.4}$$

$$\operatorname{div} \bar{u} = 0 \tag{1.5}$$

with the initial condition

$$\bar{u}|_{t=0} = u_0 \,. \tag{1.6}$$

While the question is settled in the absence of boundaries ($\mathbb{R}^2$ or $\mathbb{T}^2$) [53, 58, 30, 4, 8, 12, 49], the problem in a domain Ω with boundary remains to a large extend unresolved, due to a mismatch in boundary conditions: The Navier-Stokes solution satisfies the no-slip boundary condition $u|_{\partial\Omega} = 0$, while the Euler one may allow for tangential slip at the boundary since $\bar{u} \cdot n|_{\partial\Omega} = 0$. The convergence $u \to \bar{u}$ in the topology of the energy norm $L^\infty(0,T;L^2(\Omega))$ is equivalent to the condition

$$\nu \int_0^T \int_{\operatorname{dist}(y,\partial\Omega)\le c_0\nu} |\nabla u|^2 \to 0 \qquad \text{as} \qquad \nu \to 0 \,, \tag{1.7}$$

which is commonly referred to as Kato's criterion [31]. Above, $c_0 > 0$ is an arbitrarily chosen positive constant. Note that (1.7) demands non-anomalous energy dissipation in a very thin ($\nu \ll \sqrt{\nu}$) boundary layer around the boundary.

In this paper, we are interested in identifying general conditions on initial data which permit us to conclude the convergence of the Navier-Stokes solution toward the Euler one, as $\nu \to 0$. A classical result of Sammartino and Caflisch [56, 57] states that this indeed holds for initial data which is analytic in Ω. Subsequently, Maekawa [46] proved that the same is true if the initial data u_0 has vorticity $\omega_0 = \operatorname{curl} u_0$ that is compactly supported in the interior of Ω. Both approaches relied on the Prandtl boundary layer expansion, by establishing its validity. Then, Nguyen-Nguyen have found in [54] a proof of the Sammartino-Caflisch analyticity result without the use of Prandtl correctors by using a combination of analytic norms which suitably encode the boundary layer behavior of the solution. Recently, in [38] the authors of the present paper bridged the results of Sammartino-Caflisch and Maekawa by proving that the inviscid limit holds for data which are analytic in a vicinity of the boundary, and Sobolev regular otherwise. In [38], three norms were used; two norms in the analytic near-boundary region (a combination of a uniform weighted norm and an integrable norm), and a Sobolev norm away from the boundary. The analytic norms are defined in a wedge-like analytic sector with respect to the normal variable. Finally, the third named author of the present paper proved the analogous statement in three space dimensions in [61]. We would also like to point the important result of Gerard-Varet, Maekawa, and Masmoudi [19], who proved that the inviscid limit holds for initial data in a proximity of a monotone and convex shear flow, with only Gevrey-2 regularity in the tangential variable.

The main purpose of this paper is to further relax the requirements on the initial data from [38]: We no longer demand the boundedness of the vorticity in an analytic sector with respect to the normal variable, and instead only require its integrability. Recall that in [38] we used two analytic norms in a constant neighborhood of the boundary. The first norm $X(t)$ was a weighted L^∞ type norm, and $Y(t)$ was an L^1 type norm. The first norm allows an $\mathcal{O}(\nu^{-1/2})$ size of the vorticity in the boundary layer, which is then in turn used in Kato's criterion to imply the inviscid limit. However, the use of this norm restricts the initial data to those which are bounded in a neighborhood of the boundary. Here, we eliminate the norm $X(t)$ from the analysis, thus allowing more general initial vorticity. In particular, we only require analytic integrability in a wedge close to the boundary, such as initial vorticity of the type $f(x)/y^{1-\delta}$ for $y \leq c_0$, where c_0 is constant, and Sobolev regularity for $y \geq c_0$. In addition, the approach presented here provides a simple proof of the inviscid limit problem with data analytic close to the boundary. Note also that the integrability requirement is natural and consistent with the paper [10] by Constantin et al, where it is shown that uniform integrability of the Navier-Stokes vorticity is sufficient for a weak-* convergence on compact sets in the interior of the domain (see also [7]). Also, as opposed to [38], we no longer use norms which are ν-dependent (note that the weight w used in the definition of the X-norm in [38] depends on the viscosity ν).

In the rest of the introduction we briefly summarize available results on the topic of inviscid limit; for a more comprehensive review, see [47]. The papers [32, 59, 62] provide alternatives to the Kato criterion by considering the vorticity or using tangential/vertical derivatives of the velocity. In [32], the vanishing viscosity limit is related to accumulation of vorticity on the boundary, showing that L^q norms of the vorticity may not be suitable for the study of convergence when $q > 1$; also, see [48, 3, 7, 9, 34] for other necessary and sufficient conditions on the vorticity or velocity for the validity of the inviscid limit. We point out that there are several classes of initial data with different symmetry assumptions (e.g. plane-parallel flow, pipe flow, etc.) for which one can conclude that the convergence $u \to \bar{u}$ holds as $\nu \to 0$ (cf. [35, 5, 27, 33, 43, 44, 47, 51, 52]). Finally, the vanishing viscosity limit and the Prandtl expansion is known to hold in various settings for the stationary problem over a flat plate [18, 25, 26, 29]. For other works on the inviscid limit, see [6, 49, 11, 14, 15, 16, 36, 42, 60], while for works on the various aspects of the Prandtl boundary layer theory, cf. [55, 22, 41, 17, 21, 20, 39, 1, 37, 23, 50, 28, 40, 24, 13].

The paper is organized as follows. In Section 2, we introduce the necessary notation, while in Section 3 we state the two main results, the first on a uniform existence time with a uniform estimate on solutions of the Navier-Stokes solution, and the second on the inviscid limit. The next section contains the analytic norm estimate and the estimate for the nonlinearity. In Section 5, we provide a Sobolev type estimate. This proof provides several improvements over the approach in [38] by using the analyticity more directly and by providing new bounds on the derivatives of the velocity and the vorticity. The last two sections contain proofs of the two main theorems; in particular, the last section gives a proof of the inviscid limit by using only the L^1-analytic norm of the vorticity.

2. Notation and norms

For a function $f = f(x,y)$, which is 2π-periodic, we denote the Fourier transform in the x variable by f_ξ. In particular,

$$f(x,y) = \sum_{\xi \in \mathbb{Z}} f_\xi(y) e^{ix\xi}.$$

We fix $\mu_0 \in (0, 1/10)$, and assume that $\mu \in (0, \mu_0)$. Then

$$\Omega_\mu = \{z \in \mathbb{C} : 0 \leq \mathbb{R}\mathrm{e}\, z \leq 1, |\mathbb{I}\mathrm{m}\, z| \leq \mu \mathbb{R}\mathrm{e}\, z\} \cup \{z \in \mathbb{C} : 1 \leq \mathbb{R}\mathrm{e}\, z \leq 1+\mu, |\mathbb{I}\mathrm{m}\, z| \leq 1+\mu-\mathbb{R}\mathrm{e}\, z\}$$

denotes the complex domain for functions of the y variable.

For a sufficiently large constant $\gamma > 0$ to be determined below, which depends on μ_0 and the size of the initial datum (but is independent of ν), we require that t satisfies

$$t \in \left(0, \frac{\mu_0}{2\gamma}\right) . \tag{2.1}$$

We assume, without loss of generality, that $\gamma > \mu_0/2$.

For a complex valued function f defined on Ω_μ, let

$$\|f\|_{\mathcal{L}^1_\mu} = \sup_{0\le\theta<\mu} \|f\|_{L^1(\partial\Omega_\theta)}\,, \tag{2.2}$$

where the integration is taken over the two directed paths along the boundary of the domain Ω_θ. Using (2.2), we define

$$\|f\|_{Y_\mu} = \sum_{\xi\in\mathbb{Z}} \|e^{\epsilon_0(1+\mu-y)|\xi|} f_\xi\|_{\mathcal{L}^1_\mu}\,, \tag{2.3}$$

where $\epsilon_0 > 0$ is a sufficiently small constant. In (2.3) and elsewhere, we overload the notation and write y instead of $\mathbb{R}\mathrm{e}\, y$ when it appears inside the exponential. Fix $\alpha \in (0,1)$. For t as in (2.1), we introduce the analytic norm

$$\|f\|_{Y(t)} = \sup_{0<\mu<\mu_0-\gamma t} \left(\sum_{0\le i+j\le 1} \left\|\partial_x^i (y\partial_y)^j f\right\|_{Y_\mu} + \sum_{i+j=2} (\mu_0-\mu-\gamma t)^\alpha \left\|\partial_x^i (y\partial_y)^j f\right\|_{Y_\mu} \right). \tag{2.4}$$

As for the Sobolev part of the norm, let

$$\|f\|_S^2 = \|yf\|^2_{L^2(y\ge 1/2)} = \sum_{\xi\in\mathbb{Z}} \|y f_\xi\|^2_{L^2(y\ge1/2)} \tag{2.5}$$

and

$$\|f\|_Z = \sum_{0\le i+j\le 3} \left\|\partial_x^i \partial_y^j f\right\|_S = \sum_{0\le i+j\le 3} \left\|y\partial_x^i\partial_y^j f\right\|^2_{L^2(y\ge 1/2)}\,.$$

Lastly, we denote by

$$|\!|\!|\omega|\!|\!|_t = \|\omega\|_{Y(t)} + \|\omega\|_Z \tag{2.6}$$

the cumulative vorticity norm.

For the simplicity of the exposition, as in [38], we provide proofs for bounds on the $Y(t)$ norm using paths in the real plane. It is not difficult to extend proofs to cover the complex paths, as we show in Appendix A.

3. Main results

Denote by $u = u^\nu$ the solution of the Navier-Stokes system

$$\begin{aligned} &\partial_t u - \nu\Delta u + u\cdot\nabla u + \nabla p = 0 \\ &\operatorname{div} u = 0 \end{aligned}$$

with the initial condition

$$u|_{t=0} = u_0$$

on the half-space domain $\mathbb{H} = \mathbb{T}\times\mathbb{R}_+ = \{(x,y)\in\mathbb{T}\times\mathbb{R}\colon y\ge 0\}$, where $\mathbb{T} = [-\pi,\pi]$, with $\mathbb{T}$-periodic boundary conditions in x, and the no-slip boundary condition

$$u|_{y=0} = 0 \tag{3.1}$$

on $\partial\mathbb{H} = \mathbb{T}\times\{y=0\}$. The initial datum u_0 is divergence free and is assumed to obey the boundary condition (3.1). We assume that the viscosity ν belongs to the range (0,1] throughout. The corresponding vorticity

$$\omega = \omega^\nu = \partial_x u_2 - \partial_y u_1 = \nabla^\perp\cdot u\,, \tag{3.2}$$

where $\nabla^\perp = (-\partial_y, \partial_x)$, satisfies

$$\omega_t - \nu\Delta\omega = -u\cdot\nabla\omega\,, \tag{3.3}$$

with the initial data $\omega_0 = \nabla^\perp\cdot u_0$. The velocity u is recovered by the Biot-Savart law $u = \nabla^\perp\Delta^{-1}\omega$, where Δ^{-1} is the inverse of the Laplacian with the homogeneous boundary conditions at $y=0$. The boundary condition then reads

$$\nu(\partial_y + |\partial_x|)\omega|_{y=0} = \partial_y\Delta^{-1}(u\cdot\nabla\omega)|_{y=0} \tag{3.4}$$

(cf. [2, 45, 46]).

The following is the local existence result providing a ν-independent existence time of solutions.

THEOREM 3.1. *Suppose that ω_0 satisfies*

$$\sum_{i+j\le 2} \left\|\partial_x^i (y\partial_y)^j \omega_0\right\|_{Y_{\mu_0}} + \sum_{i+j\le 4} \left\|\partial_x^i (y\partial_y)^j \omega_0\right\|_{S} \le M < \infty\,. \tag{3.5}$$

Then there exist $\gamma, T > 0$, which depend on M and μ_0, such that the solution ω to the system (3.3) is defined on $[0,T]$ and satisfies

$$\|\omega\|_{Y(t)} + \|\omega\|_{Z} \le CM\,,$$

for all $t \in [0,T]$.

Theorem 3.1 is proven in Section 6 below.

The next theorem provides a consequence of the above result on the inviscid problem with initial data as in (3.5).

THEOREM 3.2. *Assume that ω_0 satisfies (3.5). Then, as $\nu \to 0$ the corresponding Navier-Stokes solution $u = u^\nu$ converges in the norm of $L^\infty(0,T,L^2(\Omega))$ to the solution of the Euler equation $\bar{u}$ with the initial data u_0, on the time interval $[0,T]$ provided in Theorem 3.1. The convergence holds at the rate $\mathcal{O}(\nu^{1/4})$.*

The proof of Theorem 3.2 is given in Section 7 below.

The vorticity formulation of the Navier-Stokes system (3.2)–(3.4) may be rewritten upon taking a Fourier transform in the tangential x variable as

$$\begin{aligned} &\partial_t \omega_\xi - \nu \Delta_\xi \omega_\xi = N_\xi \\ &\nu(\partial_y + |\xi|)\omega_\xi = B_\xi, \qquad \xi \in \mathbb{Z}\,, \end{aligned} \tag{3.6}$$

where

$$N_\xi(s,y) = -(u\cdot\nabla\omega)_\xi(s,y)$$

and

$$B_\xi(s) = (\partial_y \Delta^{-1}(u\cdot\nabla\omega))_\xi(s)|_{y=0} = -(\partial_y \Delta_\xi^{-1} N_\xi(s))|_{y=0}\,. \tag{3.7}$$

Above, we denoted $\Delta_\xi = -\xi^2 + \partial_y^2$, with the Dirichlet boundary condition on $\partial\mathbb{H}$. The mild formulation of the system (3.6) reads

$$\begin{aligned} \omega_\xi(t,y) = &\int_0^\infty G_\xi(t,y,z)\omega_{0\xi}(z)\,dz + \int_0^t\int_0^\infty G_\xi(t-s,y,z)N_\xi(s,z)\,dzds \\ &+ \int_0^t G_\xi(t-s,y,0)B_\xi(s)\,ds\,, \end{aligned} \tag{3.8}$$

where $G_\xi(t,y,z)$ denotes the Green's function.

In the next statement (cf. Nguyen-Nguyen [**54**, Proposition 3.3 and Section 3.3]), we recall the upper bounds on the Green's function G_ξ.

LEMMA 3.3. *The Green's function G_ξ for the system (3.6) is given by*

$$G_\xi = \widetilde{H}_\xi + R_\xi\,,$$

where

$$\widetilde{H}_\xi(t,y,z) = \frac{1}{\sqrt{\nu t}}\left(e^{-\frac{(y-z)^2}{4\nu t}} + e^{-\frac{(y+z)^2}{4\nu t}}\right)e^{-\nu\xi^2 t}$$

is the one dimensional heat kernel for the half space with homogeneous Neumann boundary condition. The residual kernel R_ξ is a function of $y+z$, and it satisfies the bounds

$$|\partial_z^k R_\xi(t,y,z)| \lesssim b^{k+1}e^{-\theta_0 b(y+z)} + \frac{1}{(\nu t)^{(k+1)/2}}e^{-\theta_0\frac{(y+z)^2}{\nu t}}e^{-\frac{\nu\xi^2 t}{8}}, \qquad k\in\mathbb{N}_0\,, \tag{3.9}$$

where $\theta_0 > 0$ is a constant independent of ν, but depends on k. The boundary remainder coefficient b in (3.9) is given by

$$b = b(\xi,\nu) = |\xi| + \frac{1}{\sqrt{\nu}}\,.$$

The implicit constant in (3.9) depends only on k and θ_0.

4. Estimates for the analytic norm and the nonlinearity

In this section, we recall the Y-norm estimates from [38]. Denote

$$\|f\|_{S_\mu} = \sum_\xi \|y f_\xi\|_{L^2(y\geq 1+\mu)}\,,$$

which is a weighted L^2 in y, ℓ^1 in ξ norm, and should not be confused with the norm $\|\cdot\|_S$ in (2.5).

LEMMA 4.1 (**An estimate of the analytic norm**). *Let*

$$\mu_1 = \mu + \frac{1}{4}(\mu_0 - \mu - \gamma s)\,.$$

Then the nonlinear term in (3.8) *is bounded as*

$$\begin{aligned}(\mu_0 - \mu - \gamma s) \sum_{i+j=2} &\left\| \partial_x^i (y\partial_y)^j \int_0^\infty G(t-s,y,z) N(s,z)\,dz \right\|_{Y_\mu} \\ &+ \sum_{i+j\leq 1} \left\| \partial_x^i (y\partial_y)^j \int_0^\infty G(t-s,y,z) N(s,z)\,dz \right\|_{Y_{\mu_1}} \\ &\quad\lesssim \sum_{i+j\leq 1} \|\partial_x^i (y\partial_y)^j N(s,\cdot)\|_{Y_{\mu_1}} + \sum_{i+j\leq 1} \|\partial_x^i \partial_y^j N(s,\cdot)\|_{S_{\mu_1}}\,. \end{aligned} \tag{4.1}$$

The Y_μ norm of the trace kernel term in (3.8) *is estimated as*

$$\begin{aligned}&(\mu_0 - \mu - \gamma s) \sum_{i+j=2} \left\|\partial_x^i (y\partial_y)^j G(t-s,y,0) B(s)\right\|_{Y_\mu} + \sum_{i+j\leq 1} \left\|\partial_x^i (y\partial_y)^j G(t-s,y,0) B(s)\right\|_{Y_{\mu_1}} \\ &\lesssim \sum_{i\leq 1} \left(\|\partial_x^i N(s,\cdot)\|_{Y_{\mu_1}} + \|\partial_x^i N(s,\cdot)\|_{S_\mu}\right)\,.\end{aligned}$$

For the initial datum term in (3.8) *we have*

$$\begin{aligned}\sum_{i+j\leq 2} &\left\| \partial_x^i (y\partial_y)^j \int_0^\infty G(t,y,z)\omega_0(z)\,dz \right\|_{Y_\mu} \\ &\lesssim \sum_{i+j\leq 2} \|\partial_x^i (y\partial_y)^j \omega_0\|_{Y_\mu} + \sum_{i+j\leq 2}\sum_\xi \|\xi^i \partial_y^j \omega_{0\xi}\|_{L^1(y\geq 1+\mu)}\,.\end{aligned} \tag{4.2}$$

Next, we provide analytic and Sobolev estimates for the nonlinearity

$$N = u\cdot\nabla\omega.$$

The estimates are based on the Biot-Savart laws

$$u_{1,\xi}(y) = \frac{1}{2}\left(-\int_0^y e^{-|\xi|(y-z)}(1-e^{-2|\xi|z})\omega_\xi(z)\,dz + \int_y^\infty e^{-|\xi|(z-y)}(1+e^{-2|\xi|y})\omega_\xi(z)\,dz\right) \tag{4.3}$$

and

$$u_{2,\xi}(y) = \frac{-\hat{\imath}\xi}{2|\xi|}\left(\int_0^y e^{-|\xi|(y-z)}(1-e^{-2|\xi|z})\omega_\xi(z)\,dz + \int_y^\infty e^{-|\xi|(z-y)}(1-e^{-2|\xi|y})\omega_\xi(z)\,dz\right)\,, \tag{4.4}$$

(cf. [46]), where $\hat{\imath}$ denotes the imaginary unit.

First, we provide a pointwise inequality for the velocity in terms of the vorticity.

LEMMA 4.2. *Let $\mu\in(0,\mu_0-\gamma s)$. Then*

$$\sum_\xi \sup_{y\in\Omega_\mu} |e^{\epsilon_0(1+\mu-y)|\xi|}|\,|(\partial_x^i (y\partial_y)^j u_1)_\xi| \lesssim \|\partial_x^{i+j}\omega\|_{Y_\mu} + \|\partial_x^{i+j}\omega\|_{S_\mu} + j(\|\omega\|_{Y_\mu} + \|y\partial_y\omega\|_{Y_\mu}) \tag{4.5}$$

and

$$\sum_\xi \sup_{y\in\Omega_\mu} |e^{\epsilon_0(1+\mu-y)|\xi|}|\left|\left((y\partial_y)^j\left(\frac{\partial_x^i u_2}{y}\right)\right)_\xi\right| \lesssim \|\partial_x^{i+1}\omega\|_{Y_\mu} + \|\partial_x^{i+1}\omega\|_{S_\mu} \tag{4.6}$$

hold for all $i,j\in\mathbb{N}_0$ such that $0\leq i+j\leq 1$.

PROOF OF LEMMA 4.2. The inequality (4.6) is established in [38, Lemma 6.3]. Likewise, (4.5) is proven in [38, Lemma 6.3], except for the case $(i,j)=(0,1)$. (As in [38], we treat only the case when the variables and integration paths are real valued; cf. also the appendix). Thus, let $i=0$ and $j=1$. Differentiating (4.3), we obtain

$$\begin{aligned} y\partial_y u_{1,\xi} &= \frac{y}{2}\int_0^y e^{-|\xi|(y-z)}(1-e^{-2|\xi|z})|\xi|\omega_\xi(s,z)\,dz \\ &\quad + \frac{y}{2}\int_y^{1+\mu} e^{-|\xi|(z-y)}(1+e^{-2|\xi|y})|\xi|\omega_\xi(s,z)\,dz \\ &\quad + \frac{y}{2}\int_{1+\mu}^{\infty} e^{-|\xi|(z-y)}(1+e^{-2|\xi|y})|\xi|\omega_\xi(s,z)\,dz \\ &\quad - y\int_y^{1+\mu} e^{-|\xi|(z-y)}e^{-2|\xi|y}|\xi|\omega_\xi(s,z)\,dz \\ &\quad - y\int_{1+\mu}^{\infty} e^{-|\xi|(z-y)}e^{-2|\xi|y}|\xi|\omega_\xi(s,z)\,dz - y\omega_\xi(y) \\ &= I_1+I_2+I_3+I_4+I_5+I_6\,. \end{aligned} \tag{4.7}$$

Using

$$e^{\epsilon_0(1+\mu-y)|\xi|}e^{-|y-z||\xi|} \le e^{\epsilon_0(1+\mu-z)|\xi|}e^{\epsilon_0(z-y)|\xi|}e^{-|y-z||\xi|} \le e^{\epsilon_0(1+\mu-z)|\xi|}$$

provided $\epsilon_0 \le 1$, we obtain

$$\begin{aligned} e^{\epsilon_0(1+\mu-y)|\xi|}(|I_1|+|I_2|+|I_4|) &\lesssim \int_0^{1+\mu} e^{\epsilon_0(1+\mu-z)|\xi|}|\xi||\omega_\xi(s,z)|\,dz \\ &\lesssim \||\xi|e^{\epsilon_0(1+\mu-y)|\xi|}\omega\|_{\mathcal{L}^1_\mu}, \end{aligned}$$

which leads to

$$\sum_\xi e^{\epsilon_0(1+\mu-y)|\xi|}(|I_1|+|I_2|+|I_4|) \lesssim \|\partial_x\omega\|_{Y_\mu}\,.$$

Also,

$$e^{\epsilon_0(1+\mu-y)|\xi|}(|I_3|+|I_5|) \lesssim \int_{1+\mu}^{\infty} |\xi||\omega_\xi(s,z)|\,dz \lesssim \|z|\xi|\omega_\xi\|_{L^2(z\ge 1+\mu)}\,,$$

from where

$$\sum_\xi e^{\epsilon_0(1+\mu-y)|\xi|}(|I_3|+|I_5|) \lesssim \|\partial_x\omega\|_{S_\mu}.$$

For the last term $I_6=-y\omega_\xi(y)$ in (4.7), we have

$$\sum_\xi \sup_{y\in\Omega_\mu} e^{\epsilon_0(1+\mu-y)|\xi|}|y||\omega_\xi(y)| \lesssim \|\omega\|_{Y_\mu}+\|y\partial_y\omega\|_{Y_\mu}+\|\partial_x\omega\|_{Y_\mu}\ , \tag{4.8}$$

using the Fundamental Theorem of Calculus. The last term in (4.8) appears when the y derivative falls on the exponential. □

In the following lemma, we state the analytic estimate for the nonlinearity.

LEMMA 4.3. *Let $\mu\in(0,\mu_0-\gamma s)$ be arbitrary. For all s, we have the inequalities*

$$\|N(s,\cdot)\|_{Y_\mu} \lesssim \sum_{i\le 1}\left(\|\partial_x^i\omega\|_{Y_\mu}+\|\partial_x^i\omega\|_{S_\mu}\right)\sum_{i+j=1}\|\partial_x^i(y\partial_y)^j\omega\|_{Y_\mu} \tag{4.9}$$

and

$$\begin{aligned} \sum_{i+j=1}\|\partial_x^i(y\partial_y)^jN(s,\cdot)\|_{Y_\mu} &\lesssim \left(\sum_{0\le j\le 1}\|(y\partial_y)^j\omega\|_{Y_\mu}+\sum_{1\le i\le 2}\left(\|\partial_x^i\omega\|_{Y_\mu}+\|\partial_x^i\omega\|_{S_\mu}\right)\right)\sum_{i+j=1}\|\partial_x^i(y\partial_y)^j\omega\|_{Y_\mu} \\ &\quad + \sum_{i\le 1}\left(\|\partial_x^i\omega\|_{Y_\mu}+\|\partial_x^i\omega\|_{S_\mu}\right)\sum_{i+j=2}\|\partial_x^i(y\partial_y)^j\omega\|_{Y_\mu}\,. \end{aligned} \tag{4.10}$$

PROOF OF LEMMA 4.3. The inequality (4.9) is proven in [38, Lemma 6.4]. For $i+j=1$, by the definition of Y_μ norm and Young's inequality, we have

$$\begin{aligned}\|\partial_x^i (y\partial_y)^j N(s,\cdot)\|_{Y_\mu} \lesssim \|\partial_x \omega\|_{Y_\mu} \sum_\xi \sup_{y\in\Omega_\mu} e^{\epsilon_0(1+\mu-y)|\xi|} |(\partial_x^i (y\partial_y)^j u_1)_\xi| \\ + \|y\partial_y \omega\|_{Y_\mu} \sum_\xi \sup_{y\in\Omega_\mu} e^{\epsilon_0(1+\mu-y)|\xi|} \left| \left((y\partial_y)^j \left(\frac{\partial_x^i u_2}{y} \right) \right)_\xi \right| \\ + \|\partial_x^{i+1} (y\partial_y)^j \omega\|_{Y_\mu} \sum_\xi \sup_{y\in\Omega_\mu} e^{\epsilon_0(1+\mu-y)|\xi|} |(u_1)_\xi| \\ + \|\partial_x^i (y\partial_y)^{j+1} \omega\|_{Y_\mu} \sum_\xi \sup_{y\in\Omega_\mu} e^{\epsilon_0(1+\mu-y)|\xi|} \left| \left(\frac{u_2}{y} \right)_\xi \right| .\end{aligned}$$

The proof of (4.10) is then concluded by an application of Lemma 4.2. ☐

Finally, we state inequalities for the Sobolev norm of the nonlinear term.

LEMMA 4.4. *Let $\mu \in (0, \mu_0 - \gamma s)$ be arbitrary. We have*

$$\|N(s,\cdot)\|_{S_\mu} \lesssim \left(\|\omega\|_{Y_\mu} + \|\omega\|_{S_\mu} \right) \sum_{i+j=1} \|\partial_x^i \partial_y^j \omega\|_{S_\mu} \tag{4.11}$$

and

$$\begin{aligned}\sum_{i+j=1} \|\partial_x^i \partial_y^j N(s)\|_{S_\mu} \lesssim \sum_{i+j\le 1} \left(\|\partial_x^i \partial_y^j \omega\|_{Y_\mu} + \|\partial_x^i \partial_y^j \omega\|_{S_\mu} \right) \sum_{i+j\le 1} \|\partial_x^i \partial_y^j \omega\|_{S_\mu} \\ + \left(\|\omega\|_{Y_\mu} + \|\omega\|_{S_\mu} \right) \sum_{i+j=2} \|\partial_x^i \partial_y^j \omega\|_{S_\mu} .\end{aligned} \tag{4.12}$$

PROOF OF LEMMA 4.4. For the proof, cf. [38, Lemma 6.5]. ☐

5. The Sobolev estimate

The main goal of this section is to estimate the Sobolev part of the norm

$$\sum_{i+j\le 3} \left\| \partial_x^i \partial_y^j \omega \right\|_S = \sum_{i+j\le 3} \left\| y \partial_x^i \partial_y^j \omega \right\|_{L^2_{x,y}(y\ge 1/2)} = \sum_{i+j\le 3} \left(\sum_\xi \| y \xi^i \partial_y^j \omega_\xi \|^2_{L^2(y\ge 1/2)} \right)^{1/2} .$$

We first state a lemma which estimates the velocity in terms of the vorticity away from the boundary. Recall the notation $\|\!|\cdot|\!\|_t$ from (2.6).

LEMMA 5.1. *Let t be such that $\gamma t \le \mu_0/2$. Then for all $\delta \in (0, 3/4)$*

$$\sum_{0\le i+j\le 2} \|\partial_x^i \partial_y^j u(t)\|_{L^\infty_{x,y}(y\ge\delta)} \lesssim \|\!|\omega|\!\|_t \tag{5.1}$$

and

$$\sum_{i+j=3} \left\| \partial_x^i \partial_y^j u(t) \right\|_{L^2_{x,y}(y\ge\delta)} \lesssim \|\!|\omega|\!\|_t , \tag{5.2}$$

where the implicit constants depend on δ. Also, we have

$$\sum_{0\le i+j\le 2} \|\partial_x^i \partial_y^j \omega(t)\|_{L^\infty_{x,y}(\delta\le y\le 3/4)} \lesssim \|\!|\omega|\!\|_t \tag{5.3}$$

and

$$\sum_{i+j=3} \left\| \partial_x^i \partial_y^j \omega(t) \right\|_{L^2_{x,y}(y\ge\delta)} \lesssim \|\!|\omega|\!\|_t \tag{5.4}$$

under the condition $\gamma t \le \mu_0/2$.

Note that $t \le \mu_0/2\gamma$ implies
$$\mu_0 - \gamma t \ge \frac{\mu_0}{2}$$
and thus the range $\mu < \mu_0 - \gamma t$ in the definition (2.4) of the analytic norm includes values of μ which are greater than $\mu_0/4$.

In the proof of Lemma 5.1, we need an estimate for high order derivatives of the vorticity in the domains away from the boundary. The proof of Lemma 5.1 is given after the proof of Lemma 5.3 below.

LEMMA 5.2. *Assume that $\mu \ge \mu_0/4$. For every $i, j \in \mathbb{N}_0$ and $\delta > 0$, we have*
$$\|\partial_x^i \partial_y^j \omega\|_{L^q(\delta \le y \le 3/4)} \lesssim \|\omega\|_{Y_\mu}, \qquad 1 \le q \le \infty\,, \tag{5.5}$$
where the implicit constants depend on $i + j$ and δ.

The constants depend on μ_0; however since μ_0 is a fixed constant, we do not point out this dependence.

PROOF OF LEMMA 5.2. By Hölder's inequality, it is sufficient to prove (5.5) for $q = \infty$. Let
$$\phi(y) = y\psi\left(\frac{y}{\delta}\right)\,,$$
where $\psi \in C^\infty$ is a non-decreasing function such that $\psi = 0$ for $0 \le y \le 1/4$ and $\psi = 1$ for $y \ge 3/4$. Then we write
$$\begin{aligned}
&\|\partial_x^i \partial_y^j \omega\|_{L^\infty(\delta \le y \le 3/4)}\\
&\lesssim \left\|\sum_\xi |\xi^i \partial_y^j \omega_\xi|\right\|_{L_y^\infty(\delta \le y \le 3/4)} \lesssim \sum_\xi \|\xi^i \partial_y^j \omega_\xi\|_{L_y^\infty(\delta \le y \le 3/4)}\\
&\lesssim \sum_\xi \left\|y^{j+1}\xi^i \psi\left(\frac{y}{\delta}\right)\partial_y^j \omega_\xi\right\|_{L_y^\infty(\delta \le y \le 3/4)} \lesssim \sum_\xi \left\|y^{j+1}\xi^i \psi\left(\frac{y}{\delta}\right)\partial_y^j \omega_\xi\right\|_{L_y^\infty(0 \le y \le 3/4)}\\
&\lesssim \sum_\xi \left\|y^{j}\xi^i \psi\left(\frac{y}{\delta}\right)\partial_y^j \omega_\xi\right\|_{L_y^1(0 \le y \le 3/4)} + \sum_\xi \left\|y^{j+1}\xi^i \psi\left(\frac{y}{\delta}\right)\partial_y^{j+1} \omega_\xi\right\|_{L_y^1(0 \le y \le 3/4)}\\
&\quad + \frac{1}{\delta}\sum_\xi \left\|y^{j+1}\xi^i \psi'\left(\frac{y}{\delta}\right)\partial_y^j \omega_\xi\right\|_{L_y^1(0 \le y \le 3/4)}\,,
\end{aligned}$$
and thus, by $\psi(y) \lesssim 1$ and $y\psi'(y) \le 1$ for all $y \ge 0$, we have
$$\|\partial_x^i \partial_y^j \omega\|_{L^\infty(\delta \le y \le 3/4)} \lesssim \sum_\xi \|y^j \xi^i \partial_y^j \omega_\xi\|_{L_y^1(0 \le y \le 3/4)} + \sum_\xi \|y^{j+1}\xi^i \partial_y^{j+1}\omega_\xi\|_{L_y^1(0 \le y \le 3/4)}\,.$$
Using the Cauchy estimates on both terms, we get
$$\|\partial_x^i \partial_y^j \omega\|_{L^\infty(\delta \le y \le 3/4)} \lesssim \sum_\xi \|\xi^i \omega_\xi\|_{\mathcal{L}^1_{\mu/2}} \lesssim \sum_\xi \|e^{\epsilon_0(1+\mu-y)|\xi|}\omega_\xi\|_{\mathcal{L}^1_\mu} = \|\omega\|_{Y_\mu}\,,$$
and the lemma is established. □

LEMMA 5.3. *Assume that $\mu \ge \mu_0/4$. For every $i, j \in \mathbb{N}_0$ and $\delta > 0$, we have*
$$\|\partial_x^i \partial_y^j u\|_{L^q(\delta \le y \le 3/4)} \lesssim \|\omega\|_{Y_\mu} + \|\omega\|_{S_\mu}\,, \qquad 1 \le q \le \infty\,, \tag{5.6}$$
where the implicit constant depends on $i + j$ and δ.

PROOF OF LEMMA 5.3. Using induction, as well as the identities $\partial_x u_1 + \partial_y u_2 = 0$ and $\omega = \partial_x u_2 - \partial_y u_1$, we obtain
$$\partial_y^j u_1 = -\partial_y^{j-1}\omega + \partial_x^2 \partial_y^{j-3}\omega - \cdots + (-1)^{j/2}\partial_x^{j-2}\partial_y \omega + (-1)^{j/2}\partial_x^j u_1 \tag{5.7}$$
$$\partial_y^j u_2 = \partial_x \partial_y^{j-2}\omega - \partial_x^3 \partial_y^{j-4}\omega + \cdots - (-1)^{j/2}\partial_x^{j-1}\omega + (-1)^{j/2}\partial_x^j u_2\,, \tag{5.8}$$
valid for even j, and
$$\partial_y^j u_1 = -\partial_y^{j-1}\omega + \partial_x^2 \partial_y^{j-3}\omega - \cdots + (-1)^{(j+1)/2}\partial_x^{j-1}\omega - (-1)^{(j+1)/2}\partial_x^j u_2 \tag{5.9}$$
$$\partial_y^j u_2 = \partial_x \partial_y^{j-2}\omega - \partial_x^3 \partial_y^{j-4}\omega + \cdots - (-1)^{(j+1)/2}\partial_x^{j-2}\partial_y \omega + (-1)^{(j+1)/2}\partial_x^j u_1\,, \tag{5.10}$$

which holds for odd j. In view of Lemma 5.2 and (5.7)–(5.10), it thus suffices to prove (5.6) for $j = 0$. Note also that we only need to consider the case $q = \infty$. By the Biot-Savart law (4.3), we have

$$\|\xi^i u_\xi\|_{L_y^\infty(\delta\le y\le 3/4)} \lesssim \sup_{\delta\le y\le 3/4}\left(\int_0^\infty |\xi|^i e^{-|y-z||\xi|}|\omega_\xi(z)|\,dz\right) \lesssim \|\xi^i\omega_\xi\|_{L_y^1(0\le y\le 1+\mu/2)} + \|\omega_\xi\|_{L_y^1(1,\infty)}\,,$$

where we also used $|\xi|^i e^{-(z-y)|\xi|} \lesssim 1$ for $y \le 3/4$ and $z \ge 1$. Therefore,

$$\begin{aligned}
\|\partial_x^i u\|_{L^\infty(\delta\le y\le 3/4)} &\lesssim \sum_\xi \|\xi^i u_\xi\|_{L_y^\infty(\delta\le y\le 3/4)} + \sum_\xi \|\omega_\xi\|_{L_y^1(1,\infty)}\\
&\lesssim \sum_\xi \|\xi^i\omega_\xi\|_{L_y^1(0\le y\le 1+\mu/2)} + \sum_\xi \|\omega_\xi\|_{L_y^1(1,\infty)}\\
&\lesssim \sum_\xi \|e^{\epsilon_0(1+\mu-y)|\xi|}\omega_\xi\|_{L_y^1(y\le 1+\mu)} + \|\omega\|_{S_\mu}\\
&\lesssim \|\omega\|_{Y_\mu} + \|\omega\|_{S_\mu}\ ,
\end{aligned}$$

and the proof is concluded. □

PROOF OF LEMMA 5.1. We start with the case $i = j = 0$, when we write

$$\|u\|_{L^\infty(y\ge\delta)} \le \|u\|_{L^\infty(y\ge 0)} \lesssim \sum_\xi \|u_\xi\|_{L^\infty(y\ge 0)} \lesssim \sum_\xi \int_0^\infty |\omega_\xi(z)|\,dz \lesssim \|\omega\|_{Y_\mu} + \|\omega\|_{S_\mu}\ ,$$

where we used the Biot-Savart laws (4.3)–(4.4). Next, fix i, j such that $1 \le i + j \le 2$. First, we have

$$\begin{aligned}
\|\partial_x^i\partial_y^j u(t)\|_{L_{x,y}^\infty(y\ge\delta)} &\lesssim \|\partial_x^i\partial_y^j u(t)\|_{L_{x,y}^\infty(\delta\le y\le 3/4)} + \|\partial_x^i\partial_y^j u(t)\|_{L_{x,y}^\infty(y\ge 3/4)}\\
&\lesssim \|\omega\|_{Y_\mu} + \|\omega\|_{S_\mu} + \|\partial_x^i\partial_y^j u(t)\|_{L_{x,y}^\infty(y\ge 3/4)}\,,
\end{aligned}\tag{5.11}$$

where we used (5.6) in the last step. Based on $\Delta u = (-\partial_2\omega, \partial_1\omega)$ and the elliptic estimates, we may bound the last term in in (5.11) by $\|\omega\|_{S_\mu}$. Thus we have established (5.1).

When $i + j = 3$, we write

$$\begin{aligned}
\|\partial_x^i\partial_y^j u(t)\|_{L_{x,y}^2(y\ge\delta)} &\lesssim \|\partial_x^i\partial_y^j u(t)\|_{L_{x,y}^2(\delta\le y\le 3/4)} + \|\partial_x^i\partial_y^j u(t)\|_{L_{x,y}^2(y\ge 3/4)}\\
&\lesssim \|\omega\|_{Y_\mu} + \|\omega\|_{S_\mu} + \|\partial_x^i\partial_y^j u(t)\|_{L_{x,y}^2(y\ge 3/4)}
\end{aligned}$$

and proceed as above.

The inequalities (5.3) and (5.4) are established directly using (5.5). The proof proceeds as above except that we do not need to use the elliptic estimates to pass from u to ω. □

The following lemma provides the main estimate for the Sobolev norm.

LEMMA 5.4. *We have*

$$\sum_{i+j\le 3} \|y\partial_x^i\partial_y^j\omega(t)\|_{L^2(y\ge 1/2)}^2 \lesssim \Bigg(\sum_{i+j\le 3}\|y\partial_x^i\partial_y^j\omega_0\|_{L^2(y\ge 1/4)}^2 + 1 + t\sup_{s\in[0,t]}\|\omega(s)\|_s^3\Bigg)\exp\big(t\sup_{s\in[0,t]}\|\omega(s)\|_s\big)\,,\tag{5.12}$$

for all $0 < t < \mu_0/2\gamma$.

PROOF OF LEMMA 5.4. Denote

$$\phi(y) = y\psi(y)\,,$$

where $\psi \in C^\infty$ is a non-decreasing function such that $\psi = 0$ for $0 \le y \le 1/4$ and $\psi = 1$ for $y \ge 1/2$. Observe that $\|yf\|_{L_{x,y}^2(y\ge 1/2)} \le \|\phi f\|_{L^2(\mathbb{H})}$. The energy equality

$$\begin{aligned}
&\frac12\frac{d}{dt}\|\phi\partial^\alpha\omega\|_{L^2(\mathbb{H})}^2 + \nu\|\phi\nabla\partial^\alpha\omega\|_{L^2(\mathbb{H})}^2\\
&\quad = 2\int_{\mathbb{H}} u_2\phi'\phi|\partial^\alpha\omega|^2 - \sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\int_{\mathbb{H}}\partial^\beta u\cdot\nabla\partial^{\alpha-\beta}\omega\partial^\alpha\omega\phi^2 - 2\nu\int_{\mathbb{H}}\phi'\partial^\alpha\omega\partial_y\partial^\alpha\omega\phi\,,
\end{aligned}$$

which holds for $\alpha \in \mathbb{N}_0^2$, leads to a pointwise in time estimate for the quantity

$$Q = \sum_{i+j\le 3} \|\phi \partial_x^i \partial_y^j \omega\|_{L^2(\mathbb{H})}^2 \,,$$

which reads

$$\begin{aligned}
\frac{1}{2}\frac{dQ}{dt} &\lesssim \Big(\nu + \|u_2\|_{L^\infty(y\ge 1/4)} + \sum_{1\le i+j\le 2} \|\partial_x^i \partial_y^j u\|_{L^\infty_{x,y}(y\ge 1/4)}\Big) Q \\
&\quad + \Big(\sum_{i+j=3} \big\|\partial_x^i\partial_y^j u(t)\big\|_{L^2_{x,y}(y\ge 1/4)}\Big) \|\phi \nabla \omega\|_{L^\infty(\mathbb{H})} Q^{1/2} \\
&\quad + \Big(\nu + \|u_2\|_{L^\infty_{x,y}(1/4\le y\le 1/2)}\Big) \sum_{i+j\le 3} \|\partial_x^i\partial_y^j \omega\|^2_{L^2_{x,y}(1/4\le y\le 1/2)} \\
&\lesssim \Big(\nu + \|\!|\omega|\!\|_t\Big) Q + \|\!|\omega|\!\|_t \,\|\phi\nabla\omega\|_{L^\infty(\mathbb{H})} Q^{1/2} + \Big(\nu + \|\!|\omega|\!\|_t\Big)\|\!|\omega|\!\|_t^2 \,.
\end{aligned}$$

In the second term, we use (5.2) to write

$$\begin{aligned}
\|\phi\nabla\omega\|_{L^\infty(\mathbb{H})} &\lesssim \|\nabla(\phi\omega)\|_{L^\infty(\mathbb{H})} + \|\phi\omega\|_{L^\infty(\mathbb{H})} + \|\omega\|_{L^\infty_{x,y}(1/4\le y\le 1/2)} \\
&\lesssim \sum_{i+j\le 3} \big\|\partial_x^i\partial_y^j(\phi\omega)\big\|_{L^2(\mathbb{H})} + \|\omega\|_{Y_\mu} \lesssim \sum_{i+j\le 3}\big\|\phi\partial_x^i\partial_y^j(\omega)\big\|_{L^2(\mathbb{H})} + \|\omega\|_{Y_\mu} \\
&\lesssim Q^{1/2} + \|\omega\|_{Y_\mu} \;.
\end{aligned}$$

Thus, we obtain

$$\frac{dQ}{dt} \lesssim (1 + \|\!|\omega(t)|\!\|_t) Q + (1 + \|\!|\omega(t)|\!\|_t)\|\!|\omega(t)|\!\|_t^2 \,,$$

from where, using the Grönwall inequality and $t \lesssim 1$

$$Q(t) \lesssim \Big(Q(0) + t\sup_{s\in[0,t]} \|\!|\omega(s)|\!\|_s^2 + t \sup_{s\in[0,t]}\|\!|\omega(s)|\!\|_s^3\Big) \exp(t\sup_{s\in[0,t]}\|\!|\omega(s)|\!\|_s) \,,$$

and (5.12) follows by the properties of the functions ϕ and ψ. □

6. Proof of the main local existence theorem

In this section, we prove Theorem 3.1.

PROOF OF THEOREM 3.1. Denote

$$M_0 = \sum_{i+j\le 2} \|\partial_x^i (y\partial_y)^j \omega_0\|_{Y_{\mu_0}} + \sum_{i+j\le 2}\sum_\xi \|\partial_x^i\partial_y^j \omega_{0,\xi}\|_{L^1(y\ge 1+\mu_0)} + \sum_{i+j\le 3} \|y\partial_x^i\partial_y^j\omega_0\|_{L^2(y\ge 1/4)} \lesssim M \,.$$

Let $t < \mu_0/2\gamma$ and $\mu < \mu_0 - \gamma t$. By (4.1)–(4.2), we have

$$\begin{aligned}
&\sum_{i+j=2} \big\|\partial_x^i(y\partial_y)^j\omega(t)\big\|_{Y_\mu} \\
&\lesssim M_0 + \int_0^t \frac{1}{(\mu_0-\mu-\gamma s)^{1+\alpha}} \sum_{i+j\le 1}\big(\|\partial_x^i(y\partial_y)^j N(s,\cdot)\|_{Y_{\mu_1}} + \|\partial_x^i\partial_y^j N(s,\cdot)\|_{S_{\mu_1}}\big)\, ds \,,
\end{aligned}$$

and then by (4.11)–(4.12),

$$\sum_{i+j=2}\big\|\partial_x^i(y\partial_y)^j\omega(t)\big\|_{Y_\mu} \lesssim M_0 + \int_0^t \frac{\|\!|\omega(s)|\!\|_s^2}{(\mu_0-\mu-\gamma s)^{1+\alpha}}\, ds \lesssim M_0 + \frac{\sup_{0\le s\le t}\|\!|\omega(s)|\!\|_s^2}{\gamma(\mu_0-\mu-\gamma t)^\alpha} \,. \tag{6.1}$$

Similarly, for the lower order terms, i.e., when $i + j \le 1$, we obtain

$$\sum_{i+j\le 1}\big\|\partial_x^i(y\partial_y)^j\omega(t)\big\|_{Y_\mu} \lesssim M_0 + \int_0^t \frac{\|\!|\omega(s)|\!\|_s^2}{(\mu_0-\mu-\gamma s)^{\alpha}}\, ds \lesssim M_0 + \frac{\sup_{0\le s\le t}\|\!|\omega(s)|\!\|_s^2}{\gamma} \,. \tag{6.2}$$

Using (6.1) and (6.2), it follows that

$$\|\omega(t)\|_{Y(t)} \lesssim M_0 + \frac{\sup_{0\le s\le t} \|\!|\omega(s)|\!\|_s^2}{\gamma}\,. \tag{6.3}$$

On the other hand, the Sobolev estimate (5.12) reads

$$\|\omega(t)\|_Z \lesssim \left(M_0 + 1 + t^{1/2} \sup_{s\in[0,t]} \|\!|\omega(s)|\!\|_s^{3/2}\right) \exp\left(t \sup_{s\in[0,t]} \|\!|\omega(s)|\!\|_s\right)\,,$$

and thus by $t \le \mu_0/(2\gamma)$, we get

$$\|\omega(t)\|_Z \lesssim \left(M_0 + 1 + \frac{\mu_0^{1/2}}{\gamma^{1/2}} \sup_{s\in[0,t]} \|\!|\omega(s)|\!\|_s^{3/2}\right) \exp\left(\frac{\mu_0}{2\gamma} \sup_{s\in[0,t]} \|\!|\omega(s)|\!\|_s\right)\,, \qquad 0 < t \le \frac{\mu_0}{2\gamma}. \tag{6.4}$$

Upon adding (6.3) and (6.4), and recalling the definition of $\|\!|\cdot|\!\|_t$ in (2.6), we arrive at the estimate

$$\|\!|\omega(t)|\!\|_t \lesssim M_0 + \frac{\sup_{s\in[0,t]} \|\!|\omega(s)|\!\|_s^2}{\gamma} + \left(M_0 + 1 + \frac{\mu_0^{1/2}}{\gamma^{1/2}} \sup_{s\in[0,t]} \|\!|\omega(s)|\!\|_s^{3/2}\right) \exp\left(\frac{\mu_0}{2\gamma} \sup_{s\in[0,t]} \|\!|\omega(s)|\!\|_s\right)\,,$$

for all $t \in (0, \mu_0/(2\gamma)]$. Since $\|\!|\omega_0|\!\|_0 \lesssim M_0$, the proof is concluded by choosing $\gamma = C(1+M_0^3)$, where $C > 0$ is a sufficiently large constant, and applying a barrier argument. □

The justification of the a priori estimates is obtained as in [**38**, Remark 3.11].

7. Strong inviscid limit

In this final section, we prove Theorem 3.2. The idea is to combine the bounds provided by Theorem 3.1 with the self-regularization of the Navier-Stokes equation to deduce a bound on the vorticity which is uniform all the way up to the boundary, but degenerates at $t = 0$ (cf. (7.5) below). The Kato criterion then concludes the proof.

PROOF OF THEOREM 3.2. First, note that (3.5) implies

$$\sum_\xi \int_0^\infty (1+|\xi|)|\omega_{0\xi}(z)|\,dz \lesssim M\,.$$

Next, we bound the uniform norm of the vorticity. Using (3.8), we have for $\xi \in \mathbb{Z}$ and $(t,y) \in (0,T]\times(0,\infty)$,

$$\begin{aligned} |\omega_\xi(t,y)| &\lesssim \int_0^\infty |G_\xi(t,y,z)||\omega_{0\xi}(z)|\,dz \\ &\quad + \int_0^t\int_0^\infty |G_\xi(t-s,y,z)||N_\xi(s,z)|\,dzds + \int_0^t |G_\xi(t-s,y,0)||B_\xi(s)|\,ds \\ &= I_1 + I_2 + I_3\,. \end{aligned} \tag{7.1}$$

From Lemma 3.3 recall the pointwise bound

$$|G_\xi(t,y,z)| \lesssim \frac{1}{\sqrt{\nu t}} \exp\left(-\frac{(y-z)^2}{4\nu t}\right) + b\exp\big(-b\theta_0(y+z)\big)\,, \tag{7.2}$$

where $\theta_0 > 0$ is a constant and $b = |\xi| + 1/\sqrt{\nu}$. Therefore, we may estimate the first term in (7.1) as

$$\begin{aligned} I_1 &\lesssim \int_0^\infty \left(\frac{1}{\sqrt{\nu t}} \exp\left(-\frac{(y-z)^2}{4\nu t}\right) + b\exp(-b\theta_0(y+z))\right)|\omega_{0\xi}(z)|\,dz \\ &\lesssim \frac{1}{\sqrt{\nu t}}\int_0^\infty |\omega_{0\xi}(z)|\,dz + \int_0^1 \left(\frac{1}{\sqrt{\nu}} + |\xi|\right)|\omega_{0\xi}(z)|\,dz + \int_1^\infty \frac{1}{z}|\omega_{0\xi}(z)|\,dz \\ &\lesssim \frac{1}{\sqrt{\nu t}}\int_0^\infty |\omega_{0\xi}(z)|\,dz + \int_0^\infty (|\xi|+1)|\omega_{0\xi}(z)|\,dz\,, \end{aligned}$$

where we used

$$be^{-b\theta_0(y+z)} \lesssim b\chi_{[0,1]}(y) + \frac{1}{z}\chi_{[1,\infty)}(y) \tag{7.3}$$

in the second inequality. For the second term in (7.1), we also use the bound (7.2) and write

$$\begin{aligned}
I_2 &\lesssim \int_0^t \int_0^\infty \left(\frac{1}{\sqrt{\nu(t-s)}} \exp\left(-\frac{(y-z)^2}{4\nu(t-s)}\right) + b\exp\bigl(-b\theta_0(y+z)\bigr)\right) |N_\xi(s,z)|\, dzds \\
&\lesssim \frac{1}{\sqrt{\nu}} \int_0^t \int_0^\infty \frac{1}{\sqrt{t-s}} |N_\xi(s,z)|\, dzds + \int_0^t \int_0^1 \left(\frac{1}{\sqrt{\nu}} + |\xi|\right) |N_\xi(s,z)|\; dzds + \int_0^t \int_1^\infty \frac{1}{z}\, |N_\xi(s,z)|\; dzds
\end{aligned}$$

by (7.3). For the third term in (7.1), recall from (3.7) and [**38**, (4.29)] that we have

$$B_\xi(s) = -(\partial_y \Delta_\xi^{-1} N_\xi(s))|_{y=0} = \int_0^\infty e^{-|\xi|z} N_\xi(s,z)\, dz\,,$$

and we may bound

$$\begin{aligned}
I_3 &\lesssim \int_0^t \int_0^\infty \frac{1}{\sqrt{\nu(t-s)}} \exp\left(-\frac{y^2}{4\nu t}\right) e^{-|\xi|z} |N_\xi(s,z)|\, dzds + \int_0^t \int_0^\infty b e^{-b\theta_0 y} e^{-|\xi|z} |N_\xi(s,z)|\, dzds \\
&= I_{31} + I_{32}\,.
\end{aligned} \tag{7.4}$$

The first term in (7.4) is estimated by dividing the integration in the z variable to integrals over $[0,1]$ and $[1,\infty]$ obtaining

$$\begin{aligned}
I_{31} &\lesssim \frac{1}{\sqrt{\nu}} \int_0^t \left(\int_0^\infty e^{-|\xi|z} |N_\xi(s,z)|\, dz\right) \frac{1}{\sqrt{t-s}}\, ds \\
&\lesssim \frac{1}{\sqrt{\nu}} \int_0^t \left(\int_0^1 |N_\xi(s,z)|\, dz\right) \frac{1}{\sqrt{t-s}}\, ds + \frac{1}{\sqrt{\nu}} \int_0^t \left(\int_1^\infty |N_\xi(s,z)|\, dz\right) \frac{1}{\sqrt{t-s}}\, ds\,.
\end{aligned}$$

On the other hand, for the second term in (7.4), we write

$$\begin{aligned}
I_{32} &\lesssim \int_0^t \int_0^1 b e^{-b\theta_0 y} e^{-|\xi|z} |N_\xi(s,z)|\, dzds + \int_0^t \int_1^\infty b e^{-b\theta_0 y} e^{-|\xi|z} |N_\xi(s,z)|\, dzds \\
&\lesssim \int_0^t \int_0^1 \left(|\xi| + \frac{1}{\sqrt{\nu}}\right) |N_\xi(s,z)|\, dzds + \frac{1}{\sqrt{\nu}} \int_0^t \int_1^\infty |N_\xi(s,z)|\, dzds + \int_0^t \int_1^\infty \frac{1}{z} |N_\xi(s,z)|\, dzds
\end{aligned}$$

since $|\xi| e^{-|\xi|z} \lesssim 1/z$. Collecting the bounds for I_1, I_2, and I_3 and using

$$\int_1^\infty |N_\xi(s,z)|\, dz \lesssim \|zN_\xi(s,z)\|_{L^2(z\geq 1)}\,,$$

we obtain

$$\begin{aligned}
|\omega_\xi(t,y)| &\lesssim \frac{1}{\sqrt{\nu t}} \int_0^\infty |\omega_{0\xi}(z)|\, dz + \int_0^\infty (|\xi|+1)|\omega_{0\xi}(z)|\, dz + \frac{1}{\sqrt{\nu}} \int_0^t \int_0^1 \frac{1}{\sqrt{t-s}}\, |N_\xi(s,z)|\; dzds \\
&\quad + \int_0^t \int_0^1 \left(\frac{1}{\sqrt{\nu}} + |\xi|\right) |N_\xi(s,z)|\; dzds + \int_0^t \int_1^\infty \frac{1}{z}\, |N_\xi(s,z)|\; dzds \\
&\quad + \frac{1}{\sqrt{\nu}} \int_0^t \frac{1}{\sqrt{t-s}} \|zN_\xi(s,z)\|_{L^2(z\geq 1)}\, ds\,.
\end{aligned}$$

Therefore, since $|\omega(t,x,y)| \lesssim \sum_\xi |\omega_\xi(y)|$ holds pointwise in x and y, upon recalling the definition of the $Y(t)$ norm in (2.4), we get

$$\begin{aligned}
|\omega(t,x,y)| &\lesssim \frac{1}{\sqrt{\nu t}} \sum_\xi \int_0^\infty |\omega_{0\xi}(z)|\, dz + \sum_\xi \int_0^\infty (|\xi|+1)|\omega_{0\xi}(z)|\, dz + \frac{\sqrt{T}}{\sqrt{\nu}} \sup_{s\in[0,T]} \sum_\xi \|N_\xi(s,z)\|_{L^1(z\leq 1)} \\
&\quad + \int_0^t \sum_\xi |\xi| \|N_\xi(s,z)\|_{L^1(z\leq 1)} ds + \frac{\sqrt{T}}{\sqrt{\nu}} \sup_{s\in[0,T]} \sum_\xi \|zN_\xi(s,\cdot)\|_{L^2(z\geq 1)},
\end{aligned}$$

since $T \lesssim 1$. At this point we appeal to Lemma 4.3 and Lemma 4.4 (with $\mu > 0$ arbitrarily small, in particular, we can let $\mu = (\mu_0 - \gamma t)/2$) to bound the nonlinear terms in the above estimate, which results in the estimate

$$|\omega(t,x,y)| \lesssim \frac{1}{\sqrt{\nu t}} \sum_\xi (|\xi|+1) \int_0^\infty |\omega_{0\xi}(z)|\, dz + \int_0^t \frac{1}{(\mu_0 - \gamma s)^\alpha} ds \sup_{s\in[0,T]} \|\omega(s)\|_s^2 + \frac{\sqrt{T}}{\sqrt{\nu}} \sup_{s\in[0,T]} \|\omega(s)\|_s^2,$$

as $\nu \leq 1$ and $T \lesssim 1$. Since by the assumption (3.5) we have $\sum_\xi (1+|\xi|)\int_0^\infty |\omega_{0\xi}(z)|\,dz \lesssim M$, by Theorem 3.1 we obtain

$$|\omega(t,x,y)| \lesssim \frac{1}{\sqrt{\nu t}}\,, \tag{7.5}$$

where the implicit constant depends on M and on μ_0 (recall that $\gamma \gtrsim 1$).

To get the convergence of u^ν to $\bar{u}$, we apply Kato's criterion from [31] by estimating

$$\begin{aligned}
&\nu \int_0^T\!\!\int_{\mathbb{H}} |\nabla u|^2\,dxdyds = \nu \int_0^T\!\!\int_{\mathbb{H}} |\omega|^2\,dxdyds \\
&\quad= \nu \int_0^T\!\!\int_{\{y\leq 1/2\}} |\omega|^2\,dxdyds + \nu \int_0^T\!\!\int_{\{y\geq 1/2\}} |\omega|^2\,dxdyds \\
&\quad\lesssim \nu \int_0^T\!\!\int_{\{y\leq 1/2\}} \frac{1}{\sqrt{\nu s}}|\omega|\,dxdyds + \nu \int_0^T\!\!\int_{\{y\geq 1/2\}} |\omega|^2\,dxdyds \\
&\quad\lesssim \nu \int_0^T \frac{1}{\sqrt{\nu s}} \sum_\xi \|e^{\epsilon_0(1-y)|\xi|}\omega_\xi(s)\|_{L^1(y\leq 1/2)}\,ds + \nu\int_0^T \|\omega(s)\|_S^2\,ds \\
&\quad\lesssim \nu \int_0^T \frac{1}{\sqrt{\nu s}}\,\|\omega(s)\|_{Y(s)}\,ds + \nu \int_0^T \|\omega(s)\|_S^2\,ds \lesssim \sqrt{\nu}\,,
\end{aligned}$$

where we used (7.5) in the first inequality; note that the implicit constants depend on M, μ_0, and T. Using [31], we obtain that the inviscid limit holds in the strong topology of $L^\infty(0,T;L^2(\mathbb{H}))$ with a rate of $\mathcal{O}(\nu^{1/4})$, which is known to be optimal (see e.g. [47, 34]). $\square$

Appendix A. An estimate of the analytic norm using complex paths

In this paper, for the simplicity and clarity of arguments, we have used real paths to establish various inequalities involving norms over complex paths. In order to illustrate how the arguments can be adapted to the general situation, we provide here the proof of (4.1) from Lemma 4.1 using complex paths.

LEMMA A.1. *Let $\mu \in (0,\mu_0-\gamma s)$ and $y \in \Omega_\mu$ be arbitrary. For $(i,j)\in\mathbb{N}_0^2$ with $i+j\leq 1$, we have*

$$\left\|\partial_x^i (y\partial_y)^j \int_{\Gamma_y} H(t-s,y,z)N(s,z)\,dz\right\|_{Y_\mu} \lesssim \|\partial_x^i(y\partial_y)^j N(s)\|_{Y_\mu} + \|N(s)\|_{Y_\mu} + \|\partial_x^i\partial_y^j N(s)\|_S \tag{A.1}$$

where $\Gamma_y = \partial\Omega_\theta \cup \{x\in\mathbb{R} : x \geq 1+\theta\}$ with θ such that $y \in \partial\Omega_\theta$.

Note that the Γ_y is a directed path starting at 0, passing through y and connecting to $+\infty$.

PROOF OF LEMMA A.1. We start with the case $(i,j)=(0,1)$. Let $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a smooth non-increasing cut-off function such that $\psi(x)=1$ for $0\leq x\leq 1/2$, and $\psi(x)=0$ for $x\geq 3/4$. Also, denote

$$\Pi(a,b) = \{z\in\mathbb{C} : a < \mathbb{R}\mathrm{e}\, z < b\}$$

the complex strip corresponding to the real values a and b, where $b>a$. For every $\xi\in\mathbb{Z}$, we have

$$\begin{aligned}
&y\partial_y \int_{\Gamma_y} H_\xi(t-s,y,z)N_\xi(s,z)\,dz \\
&= -y\int_{\Gamma_y\cap\Pi(0,3\mathbb{R}\mathrm{e}\,y/4)} \psi\left(\frac{\mathbb{R}\mathrm{e}\,z}{\mathbb{R}\mathrm{e}\,y}\right)\partial_z H_\xi(t-s,y,z)N_\xi(s,z)\,dz \\
&\quad - \frac{y}{\mathbb{R}\mathrm{e}\,y}\int_{\Gamma_y\cap\Pi(\mathbb{R}\mathrm{e}\,y/2,3\mathbb{R}\mathrm{e}\,y/4)} \psi'\left(\frac{\mathbb{R}\mathrm{e}\,z}{\mathbb{R}\mathrm{e}\,y}\right) H_\xi(t-s,y,z)N_\xi(s,z)\,dz \\
&\quad + y\int_{\Gamma_y\cap\Pi(\mathbb{R}\mathrm{e}\,y/2,1+\mu)} \left(1-\psi\left(\frac{\mathbb{R}\mathrm{e}\,z}{\mathbb{R}\mathrm{e}\,y}\right)\right) H_\xi(t-s,y,z)\partial_z N_\xi(s,z)\,dz \\
&\quad + y\int_{\Gamma_y\cap\Pi(1+\mu,\infty)} H_\xi(t-s,y,z)\partial_z N_\xi(s,z)\,dz \\
&= I_1+I_2+I_3+I_4\,.
\end{aligned} \tag{A.2}$$

Note that

$$y\partial_z H_\xi = -y\partial_y H_\xi = y\frac{(y-z)}{2\nu(t-s)}\frac{1}{\sqrt{\nu(t-s)}}e^{-\frac{(y-z)^2}{4\nu(t-s)}}e^{-\nu\xi^2(t-s)}$$

and by

$$|y| \le 10|y-z|, \qquad 0 \le \mathbb{R}\mathrm{e}\, z \le \frac{3\mathbb{R}\mathrm{e}\, y}{4}, \quad y, z \in \Omega_\mu$$

we arrive at

$$|y\partial_z H_\xi| \lesssim \frac{1}{\sqrt{\nu(t-s)}}e^{-\frac{(\mathbb{R}\mathrm{e}\, y - \mathbb{R}\mathrm{e}\, z)^2}{8\nu(t-s)}}e^{-\nu\xi^2(t-s)}, \qquad 0 \le \mathbb{R}\mathrm{e}\, z \le \frac{3\mathbb{R}\mathrm{e}\, y}{4}, \quad y, z \in \Omega_\mu.$$

Therefore,

$$|I_1| \lesssim \int_{\Gamma_y \cap \Pi(0, 3\mathbb{R}\mathrm{e}\, y/4)} \frac{1}{\sqrt{\nu(t-s)}}e^{-\frac{(\mathbb{R}\mathrm{e}\, y - \mathbb{R}\mathrm{e}\, z)^2}{8\nu(t-s)}}e^{-\nu\xi^2(t-s)}|N_\xi(s,z)|\,|dz|. \tag{A.3}$$

Defining

$$(z)_+ = \begin{cases} z, & \text{if } \mathbb{R}\mathrm{e}\, z \ge 0 \\ 0, & \text{otherwise}, \end{cases}$$

the inequality (A.3) implies

$$|e^{\epsilon_0(1+\mu-y)_+ + |\xi|}I_1| \lesssim \int_{\Gamma_y \cap \Pi(0, 3\mathbb{R}\mathrm{e}\, y/4)} \frac{1}{\sqrt{\nu(t-s)}}e^{-\frac{(\mathbb{R}\mathrm{e}\, y - \mathbb{R}\mathrm{e}\, z)^2}{8\nu(t-s)}}e^{-\nu\xi^2(t-s)}|e^{\epsilon_0(1+\mu-z)_+ + |\xi|}N_\xi(s,z)|\,|dz|\,,$$

where we also used

$$|e^{\epsilon_0(1+\mu-y)_+ + |\xi|}| \lesssim |e^{\epsilon_0(1+\mu-z)_+ + |\xi|}|\,,$$

for $y, z \in \Omega_\mu$ such that $0 \le \mathbb{R}\mathrm{e}\, z \le 3\mathbb{R}\mathrm{e}\, y/4$ with μ sufficiently small. Integrating in y, changing the order of integration, and using

$$\left\| \frac{1}{\sqrt{\nu(t-s)}}e^{-\frac{(\mathbb{R}\mathrm{e}\, y - \mathbb{R}\mathrm{e}\, z)^2}{8\nu(t-s)}} \right\|_{L^\infty_z L^1_y} \lesssim 1\,, \tag{A.4}$$

we arrive at

$$\begin{aligned} \left\| e^{\epsilon_0(1+\mu-y)_+ + |\xi|}I_1 \right\|_{\mathcal{L}^1_\mu} &\lesssim \sup_{0\le\theta<\mu} \int_{\partial\Omega_\theta} \int_{\Gamma_y \cap \Pi(0, 3\mathbb{R}\mathrm{e}\, y/4)} \frac{1}{\sqrt{\nu(t-s)}}e^{-\frac{(\mathbb{R}\mathrm{e}\, y - \mathbb{R}\mathrm{e}\, z)^2}{16\nu(t-s)}}|e^{\epsilon_0(1+\mu-z)_+ + |\xi|}||N_\xi(s,z)|\,|dz||dy| \\ &\lesssim \left\| e^{\epsilon_0(1+\mu-z)_+ + |\xi|}N_\xi(s) \right\|_{\mathcal{L}^1_\mu}\,. \end{aligned}$$

Summing over ξ yields the bound

$$\|I_1\|_{Y_\mu} \le \|N(s)\|_{Y_\mu}\,. \tag{A.5}$$

The term I_2 in (A.2) is treated analogously by $\|\psi'\|_{L^\infty} \lesssim 1$ and $|y/\mathbb{R}\mathrm{e}\, y| \lesssim 1$ since $\mathbb{I}\mathrm{m}\, y \le \mu\mathbb{R}\mathrm{e}\, y$, leading to the same upper bound as in (A.5). For the term I_3 in (A.2), we use the inequality

$$|e^{\epsilon_0(1+\mu-y)_+ + |\xi|}| \le |e^{\epsilon_0(1+\mu-z)_+ + |\xi|}e^{\epsilon_0(z-y)_+ + |\xi|}| \le |e^{\epsilon_0(1+\mu-z)_+ + |\xi|}|e^{\epsilon_0(\mathbb{R}\mathrm{e}\, y - \mathbb{R}\mathrm{e}\, z)^2/2\nu(t-s)}e^{\epsilon_0\nu\xi^2(t-s)/2} \tag{A.6}$$

and the bound (A.4) to deduce

$$\begin{aligned} \|I_3\|_{Y_\mu} &= \sum_\xi \left\| e^{\epsilon_0(1+\mu-y)_+ + |\xi|}I_3 \right\|_{\mathcal{L}^1_\mu} \\ &\lesssim \sum_\xi \left\| \int_{\Gamma_y \cap \Pi(\mathbb{R}\mathrm{e}\, y/2, 1+\mu)} \frac{1}{\sqrt{\nu(t-s)}}e^{-\frac{(\mathbb{R}\mathrm{e}\, y - \mathbb{R}\mathrm{e}\, z)^2}{8\nu(t-s)}}|e^{\epsilon_0(1+\mu-z)_+ + |\xi|}z\partial_z N_\xi(s,z)|\,|dz| \right\|_{\mathcal{L}^1_\mu} \\ &\lesssim \sum_\xi \left\| e^{\epsilon_0(1+\mu-z)_+ + |\xi|}|z\partial_z N_\xi(s)| \right\|_{\mathcal{L}^1_\mu} = \|z\partial_z N_\xi(s)\|_{Y_\mu} \end{aligned}$$

where we also used that ϵ_0 is small. To bound I_4, we use (A.6), the fact that ϵ_0 is sufficiently small, and the bound $\mathbb{R}\mathrm{e}\, y \leq 1+\mu \leq \mathbb{R}\mathrm{e}\, z$, to estimate

$$\begin{aligned}|e^{\epsilon_0(1+\mu-y)_+|\xi|} I_4| &\lesssim \int_{\Gamma_y\cap\Pi(1+\mu,\infty)} \frac{1}{\sqrt{\nu(t-s)}} |e^{\epsilon_0(z-y)_+|\xi|}| e^{-\frac{(\mathbb{R}\mathrm{e}\, y-\mathbb{R}\mathrm{e}\, z)^2}{4\nu(t-s)}} e^{-\frac{1}{2}\nu\xi^2(t-s)} |\partial_z N_\xi(s,z)|\,|dz| \\ &\lesssim \int_{\Gamma_y\cap\Pi(1+\mu,\infty)} \frac{1}{\sqrt{\nu(t-s)}} e^{-\frac{(\mathbb{R}\mathrm{e}\, y-\mathbb{R}\mathrm{e}\, z)^2}{8\nu(t-s)}} |\partial_z N_\xi(s,z)|\,|dz|\,.\end{aligned}$$

We integrate the above inequality in y, use (A.4), and sum in ξ to get

$$\|I_4\|_{Y_\mu} \lesssim \sum_\xi \|\partial_z N_\xi(s)\|_{L^1(z\geq 1+\mu)} \lesssim \|\partial_z N(s)\|_{S_\mu}$$

concluding the proof of (A.1) when $(i,j)=(0,1)$.

The estimate (A.1) for $(i,j)=(1,0)$ follows from the bound (A.1) for $(i,j)=(0,0)$ with N replaced by $\partial_x N$. To prove (A.1) for $(i,j)=(0,0)$, we split $\int_{\Gamma_y} H(t-s,y,z)N(s,z)\,dz$ as

$$\begin{aligned}&\int_{\Gamma_y} H_\xi(t-s,y,z)N_\xi(s,z)\,dz \\ &\quad= \int_{\Gamma_y\cap\Pi(0,3\mathbb{R}\mathrm{e}\, y/4)} \psi\left(\frac{\mathbb{R}\mathrm{e}\, z}{\mathbb{R}\mathrm{e}\, y}\right) H_\xi(t-s,y,z)N_\xi(s,z)\,dz \\ &\qquad+ \int_{\Gamma_y\cap\Pi(\mathbb{R}\mathrm{e}\, y/2,1+\mu)} \left(1-\psi\left(\frac{\mathbb{R}\mathrm{e}\, z}{\mathbb{R}\mathrm{e}\, y}\right)\right) H_\xi(t-s,y,z)N_\xi(s,z)\,dz + \int_{1+\mu}^\infty H_\xi(t-s,y,z)N_\xi(s,z)\,dz \\ &\quad= J_1+J_2+J_3\,.\end{aligned}$$

When observing the proof for $(i,j)=(0,1)$, we note that using (A.4) we have

$$\|J_1\|_{Y_\mu} + \|J_2\|_{Y_\mu} \lesssim \|N(s)\|_{Y_\mu}\ .$$

On the other hand, the term J_3 is estimated exactly as I_4 above, and we obtain

$$\|J_3\|_{Y_\mu} \lesssim \|N(s)\|_{S_\mu}\ .$$

This concludes the proof of the lemma. □

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