The inviscid limit for the Navier-Stokes equations with data analytic only near the boundary

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ABSTRACT. We address the inviscid limit for the Navier-Stokes equations in a half space, with initial datum that is analytic only close to the boundary of the domain, and has finite Sobolev regularity in the complement. We prove that for such data the solution of the Navier-Stokes equations converges in the vanishing viscosity limit to the solution of the Euler equation, on a constant time interval. April 10, 2019

1. Introduction

We consider the Cauchy problem for the 2D incompressible Navier-Stokes equations

$$\partial_t u - \nu\Delta u + u\cdot\nabla u + \nabla p = 0 \tag{1.1}$$

$$\operatorname{div} u = 0 \tag{1.2}$$

$$u|_{t=0} = u_0 \tag{1.3}$$

on the half-space domain $\mathbb{H} = \mathbb{T}\times\mathbb{R}_+ = \{(x,y)\in\mathbb{T}\times\mathbb{R}\colon y\geq 0\}$, with $\mathbb{T} = [-\pi,\pi]$-periodic boundary conditions in x, and the *no-slip boundary condition*

$$u|_{y=0} = 0 \tag{1.4}$$

on $\partial\mathbb{H} = \mathbb{T}\times\{y=0\}$. Here $\nu>0$ is the kinematic viscosity. Formally setting $\nu = 0$ in (1.1)–(1.3) we arrive at the 2D incompressible Euler equations, which are supplemented with the *slip boundary condition* given by $u_2|_{y=0} = 0$.

A fundamental problem in mathematical fluid dynamics is to determine whether in the *inviscid limit* $\nu\to 0$ the solutions of the Navier-Stokes equations converge to those of the Euler equations, in the *energy norm* $L^\infty(0,T;L^2(\mathbb{H}))$, on an $\mathcal{O}(1)$ (with respect to ν) time interval. A classical result of Kato [**30**] relates this problem to the anomalous dissipation of kinetic energy: A necessary and sufficient condition for the inviscid limit to hold in the energy norm is that the total dissipation of the energy in a boundary layer of width $\mathcal{O}(\nu)$, vanishes as $\nu\to 0$. To date it remains an open problem whether this condition holds for any smooth initial datum.

In this paper we prove that the inviscid limit holds for initial datum u_0 for which the associated vorticity $\omega_0 = \nabla^\perp\cdot u_0$ is analytic in an $\mathcal{O}(1)$ strip next to the boundary, and is only Sobolev smooth on the complement of this strip. In particular, our main theorem (cf. Theorem 3.2 below) both implies the seminal result of Sammartino-Caflisch [**53**], which assumes analyticity on the entire half-plane, and also the more recent remarkable result of Maekawa [**44**], which assumes that the initial vorticity vanishes identically in an $\mathcal{O}(1)$ strip next to the boundary.

The fundamental source of difficulties in studying the inviscid limit is the mismatch in boundary conditions between the viscous Navier-Stokes flow (no-slip, $u_1|_{y=0} = u_2|_{y=0} = 0$) and the inviscid Euler flow (slip, $u_2|_{y=0} = 0$). Mathematically, this prohibits us from obtaining ν-independent a priori estimates for solutions of (1.1)–(1.4) in the uniform norm. The main obstacle is to quantify the creation of vorticity at $\partial\mathbb{H}$, which is expected to become unbounded as $\nu\to 0$, at least very close to the boundary.

Concerning the validity of the inviscid limit in the energy norm, in the presence of solid boundaries, for smooth initial datum, two types of results are known. First, we have results which make ν-dependent *assumptions on the family of solutions* u of (1.1)–(1.4), and prove that these assumptions imply (a-posteriori, they are equivalent to) the $L_t^\infty L_x^2$ inviscid limit. This program was initiated in the influential paper of Kato [**30**], who showed that the condition

$$\lim_{\nu\to 0} \int_0^T \int_{\{y\lesssim\nu\}} \nu|\nabla u|^2 dxdydt \to 0 \tag{1.5}$$

is equivalent to the validity of the strong inviscid limit in the energy norm. Refinements and extensions based on Kato's original argument of introducing a boundary layer corrector were obtained for instance in [**3, 6, 7, 31, 33, 46, 54, 56**]; see also the recent review [**45**] and references therein. These results are important because they yield explicit properties that the sequence of Navier-Stokes solutions must obey as $\nu \to 0$ in order for them to have a strong $L_t^\infty L_x^2$ Euler limit. On the other hand, verifying these conditions based on the knowledge of the initial datum only, is in general an outstanding open problem. We emphasize that to date, even the question of whether the weak $L_t^2 L_x^2$ inviscid limit holds (against test functions compactly supported in the interior of the domain), remains open. Conditional results have been established recently in terms of interior structure functions [**9, 11**], or in terms of interior vorticity concentration measures [**8**].

The second class of results are those which only make *assumptions on the initial data* u_0, as $\nu \to 0$. In the seminal works [**52, 53**], Sammartino-Caflisch consider initial data u_0 which are analytic in both the x and y variables on the entire half space, and are well-prepared, in the sense that u_0 satisfies the Prandtl ansatz (1.6) below, at time $t = 0$. Sammartino-Caflisch do not just prove the strong inviscid limit in the energy norm, but they in fact establish the validity of the Prandtl expansion

$$u(x,y,t) = \bar{u}(x,y,t) + u^P\left(x, \frac{y}{\sqrt{\nu}}, t\right) + \mathcal{O}(\sqrt{\nu}) \tag{1.6}$$

for the solution u of (1.1)–(1.4). Here $\bar{u}$ denotes the real-analytic solution of the Euler equations, and u^P is the real-analytic solution of the Prandtl boundary layer equations. We refer the reader to [**1, 10, 17, 28, 35, 36, 37, 39, 47, 51, 52**] for the well-posedness theory for the Prandtl equations, to [**14, 26, 18, 38**] for the identification of ill-posed regimes, and to [**19, 20, 21, 22, 23**] for recent works which show the invalidity of the Prandtl expansion at the level of Sobolev regularity. In [**52, 53**] Sammartino-Caflisch carefully analyze the error terms in the expansion (1.6), and show that they remain $\mathcal{O}(\sqrt{\nu})$ for an $\mathcal{O}(1)$ time interval, by appealing to real-analyticity and an abstract Cauchy-Kowalevski theorem. This strategy has been proven successful for treating the case of a channel [**40, 34**] and the exterior of a disk [**5**]. Subsequently, in a remarkable work [**44**], Maekawa proved that the inviscid limit also holds for initial datum whose associated vorticity is Sobolev smooth and is supported at an $\mathcal{O}(1)$ distance away from the boundary of the domain. The main new device in [**44**] is the use of the vorticity boundary condition in the case of the half space [**2, 43**], using which one may actually establish the validity of the expansion (1.6). Using conormal Sobolev spaces, the authors of [**55**] have obtained an energy based proof for the Caflisch-Sammartino result, while in [**12, 13**] it is shown that Maekawa's result can also be proven solely using energy methods, in 2D and 3D respectively. More recently, Nguyen-Nguyen have found in [**50**] a very elegant proof of the Sammartino-Caflisch result, which for the first time completely avoids the usage of Prandtl boundary layer correctors. Instead, Nguyen-Nguyen appeal to the boundary vorticity formulation, precise bounds for the associated Green's function, and an analysis in boundary-layer weighted spaces. In this paper we use a number of estimates from [**50**], chief among which are the ones for the Green's function for the Stokes system (see Lemma 3.4 below). Lastly, we mention that in a recent remarkable result [**16**], Gerard-Varet-Maekawa-Masmoudi establish the stability in a Gevrey topology in x and a Sobolev topology in y, of Euler+Prandtl shear flows (cf. (1.6)), when the Prandtl shear flow is both monotonic and concave. It is worth noting that in all the above cases the Prandtl expansion (1.6) is valid, and thus the Kato-criterion (1.5) holds. However, in general there is a large discrepancy between the question of the vanishing viscosity limit in the energy norm, and the problem of the validity of the Prandtl expansion. It is not clear to which degree these two problems are related.

Finally, we mention that the vanishing viscosity limit is also known to hold in the presence of certain symmetry assumptions on the initial data, which is maintained by the flow; see e.g. [**4, 27, 32, 41, 42, 45, 48, 49**] and references therein. This symmetry implies that the influence of the Prandtl layer to the bulk flow is weak, and thus in these situations the vanishing viscosity limit may be established by verifying Kato's criterion (1.5). Also, the very recent works [**15, 25, 24, 29**] establish the vanishing viscosity limit and the validity of the Prandtl expansion for the stationary Navier-Stokes equation, in certain regimes.

The main goal of this paper is to bridge the apparent gap between the Sammartino-Caflisch [**52, 53**] and the Maekawa [**44**] results, by proving in Theorem 3.2 that the inviscid limit in the energy norm holds for initial datum ω_0 which is analytic in a strip of $\mathcal{O}(1)$ width close to the boundary, and is Sobolev smooth on the complement of this strip. Evidently, this type of data includes the one considered in [**44, 52, 53**]. Informally, one expects analyticity to only be required near the boundary in order to control the catastrophic growth of boundary layer instabilities, and we confirm this intuition. To the best of our knowledge, our result establishes the inviscid limit in the energy norm for the largest class of initial data, in the *absence of structural or symmetry assumptions*. Theorem 3.2 is a direct consequence of Theorem 3.1, which establishes uniform in ν bounds on the vorticity in a mixed analytic-Sobolev norm. In order to prove Theorem 3.1, we use the mild vorticity formulation approach of Nguyen-Nguyen, which avoids the explicit use of Prandtl correctors, and instead relies on pointwise estimates on the Green's function for the associated Stokes equation [**50**, Proposition 3.3]. The main technical difficulty we need to overcome is the treatment of the layer where the analyticity and the Sobolev regions meet. It is known that analytic functions are not localizable, and that the Biot-Savart law is non-local. Thus, one cannot avoid that the analytic and the Sobolev regions communicate. In order to overcome this difficulty we consider an analyticity radius with respect to both the x and y variables, which vanishes in a precisely controlled time-dependent fashion at an $\mathcal{O}(1)$ distance from the boundary. Moreover, since we cannot afford a derivative loss in the Sobolev region, this estimate is carried over using an energy method, with error terms arising to the spill into the analytic region. Compared to [**50**], we employ several simplifications which provide additional information on the solution of the Navier-Stokes equations in the boundary layer. First, we remove the need for the time dependent weight function, thus not allowing time dependent bursts of vorticity in the secondary boundary layer of size $\sqrt{\nu t}$ from [**50**]. Second, since our solutions are no longer analytic away from the constant sized strip, we no longer require them to decay exponentially as $y \to \infty$. Lastly, the approach considered here allows a wider choice of weights functions in the analytic norm (cf. Remark 2.1 below) which may be used to provide a detailed information about the degeneration of the vorticity as $\nu \to 0$ in a suitably defined boundary layer.

This paper is organized as follows. In Section 2 we introduce the analytic norms X and Y and the Sobolev norm Z used in this paper. Section 3 contains the statements and the proofs of our main results, Theorems 3.1 and 3.2. For this purpose, we also recall there the integral representation of the vorticity formulation of the Navier-Stokes equations, and we collect in Lemmas 3.7, 3.8, 3.9, and 3.10 the main analytic and Sobolev estimates needed to establish our main results. These lemmas are then proven in Section 4 for the X-norm, Section 5 for the Y-norm, Section 6 for the nonlinear terms, and Section 7 for the Sobolev norm.

2. Functional setting

2.1. Notation.

- We use $f_\xi(y) \in \mathbb{C}$ to denote the Fourier transform of $f(x,y)$ with respect to the x variable at frequency $\xi \in \mathbb{Z}$, i.e. $f(x,y) = \sum_{\xi\in\mathbb{Z}} f_\xi(y)e^{ix\xi}$. We also use the notation $u_{i,\xi}(y)$ or $(u_i)_\xi(y)$ for the Fourier transform of u_i in x for $i = 1, 2$.
- $\mathbb{R}\mathrm{e}\, z$ and $\mathbb{I}\mathrm{m}\, z$ stand for the real and imaginary parts of the complex number z.
- For $\mu > 0$ we define the complex domain $\Omega_\mu = \{z \in \mathbb{C} : 0 \le \mathbb{R}\mathrm{e}\, z \le 1, |\mathbb{I}\mathrm{m}\, z| \le \mu \mathbb{R}\mathrm{e}\, z\} \cup \{z \in \mathbb{C} : 1 \le \mathbb{R}\mathrm{e}\, z \le 1 + \mu, |\mathbb{I}\mathrm{m}\, z| \le 1 + \mu - \mathbb{R}\mathrm{e}\, z\}$, which is represented in Figure 1. We assume that $\mu < \mu_0$, where $\mu_0 \in (0, 1/10]$ is a fixed constant.

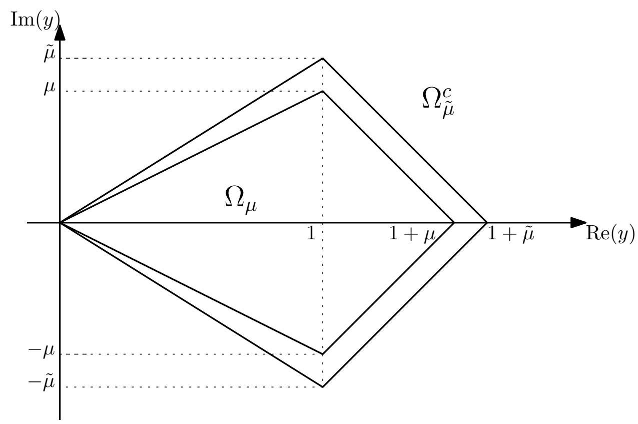

FIGURE 1. Representation of the complex domains Ω_μ and $\Omega_{\widetilde{\mu}}$ for $0<\mu<\widetilde{\mu}$.

- For $y \in \Omega_\mu$ we represent exponential terms of the form $e^{\epsilon_0(1+\mu-\mathbb{R}\mathrm{e}\, y)_+ +|\xi|}$ simply as $e^{\epsilon_0(1+\mu-y)_+ +|\xi|}$. That is, in order to simplify the notation we write y instead of $\mathbb{R}\mathrm{e}\, y$ at the exponential.
- We assume that $\nu \in (0,1]$ and $t \in (0,1]$ throughout.
- The implicit constants in $\lesssim$ depend only on μ_0 and θ_0 (cf. (3.10)), and are thus universal.

2.2. Norms. In this paper, we use norms which capture the features of a solution that is analytic near the boundary and is H^4 smooth at an $\mathcal{O}(1)$ distance away from it. We include two types of analytic norms: the L^∞ based X norm and the L^1 based Y norm, defined in (2.3) and (2.4) respectively. Before these definitions we introduce some notation.

For a sufficiently large constant $\gamma>0$ to be determined in the proof, which depends only on μ_0 and the size of the initial datum via the constant M in (3.1), throughout the paper we let t obey

$$t \in \left(0, \frac{\mu_0}{2\gamma}\right) . \tag{2.1}$$

In order to define the weighted L^∞ based analytic norm X, we introduce the weight function $w\colon [0,1+\mu_0] \to [0,1]$ given by

$$w(y)=\begin{cases}\sqrt{\nu} & ,0<y\le\sqrt{\nu}\\ y & ,\sqrt{\nu}\le y\le 1\\ 1 & ,1\le y\le 1+\mu_0\end{cases} \tag{2.2}$$

and use it to define a weighted analytic norm, with respect to y, as

$$\|f\|_{\mathcal{L}^\infty_{\mu,\nu}} = \sup_{y\in\Omega_\mu} w(\mathbb{R}\mathrm{e}\, y)|f(y)|$$

for a complex function f of the variable $y \in \Omega_\mu$. Throughout the paper, in order to simplify the notation we write $w(y)$ instead of $w(\mathbb{R}\mathrm{e}\, y)$.

Let $\epsilon_0 \in (0,1)$ be a sufficiently small constant to be determined below, which only depends on the parameter θ_0 in (3.10). Moreover, let $\alpha \in \left(0,\frac{1}{2}\right)$ be a fixed constant. Using the $\mathcal{L}^\infty_{\mu,\nu}$ norm, we define

$$\|f\|_{X_\mu} = \sum_{\xi\in\mathbb{Z}} \|e^{\epsilon_0(1+\mu-y)_+ +|\xi|} f_\xi\|_{\mathcal{L}^\infty_{\mu,\nu}} ,$$

and then, with t as in (2.1), we define the analytic X norm as

$$\|f\|_{X(t)} = \sup_{\mu<\mu_0-\gamma t}\left(\sum_{0\le i+j\le 1}\|\partial_x^i(y\partial_y)^j f\|_{X_\mu} + \sum_{i+j=2}(\mu_0-\mu-\gamma t)^{1/2+\alpha}\|\partial_x^i(y\partial_y)^j f\|_{X_\mu}\right) . \tag{2.3}$$

We state in Lemma A.3 a useful analyticity recovery estimate for the X_μ norm.

REMARK 2.1. Throughout the paper, the following properties of the weight are needed:

(a) $w(y) \lesssim w(z)$ for $y \leq z$,
(b) $w(y) \lesssim w(z)$ for $0 < y/2 \leq z \leq 1 + \mu_0$,
(c) $\sqrt{\nu} \lesssim w(y) \lesssim 1$ for $y \in [0, 1 + \mu_0]$,
(d) $y \lesssim w(y)$ for $y \in [0, 1 + \mu_0]$
(e) $w(y)e^{-\frac{y}{C\sqrt{\nu}}} \lesssim \sqrt{\nu}$ for $y \in [0, 1 + \mu_0]$ where $C > 0$ is sufficiently large constant, depending only on θ_0 in (3.10).

It is easy to check that the weight w in (2.2) satisfies (a)–(e). We justify (e) by simply distinguishing the cases $y \leq \sqrt{\nu}$ and $y \geq \sqrt{\nu}$ separately.

Note that, by the above statement, there are other weights for which Theorem 3.1 holds. For instance, we may take

$$w(y) = \min\{\sqrt{\nu}e^{\frac{y}{C\sqrt{\nu}}}, 1\}$$

for a sufficiently large universal constant C. Note that this weight is larger than (2.2), up to a multiplicative constant, and that it satisfies (a)–(e) above.

Next, we define the analytic L^1 based norm. For a complex valued function f defined on Ω_μ, let

$$\|f\|_{\mathcal{L}^1_\mu} = \sup_{0 \leq \theta < \mu} \|f\|_{L^1(\partial\Omega_\theta)} .$$

Using $\mathcal{L}^1_\mu$ we introduce

$$\|f\|_{Y_\mu} = \sum_\xi \|e^{\epsilon_0(1+\mu-y)+|\xi|} f_\xi\|_{\mathcal{L}^1_\mu}$$

and then for t as in (2.1) we define the analytic Y norm by

$$\|f\|_{Y(t)} = \sup_{\mu < \mu_0 - \gamma t} \left(\sum_{0 \leq i+j \leq 1} \left\|\partial_x^i (y\partial_y)^j f\right\|_{Y_\mu} + \sum_{i+j=2} (\mu_0 - \mu - \gamma t)^\alpha \left\|\partial_x^i (y\partial_y)^j f\right\|_{Y_\mu} \right) . \tag{2.4}$$

Note the different time weights when comparing the highest order terms in (2.3) and (2.4). We refer to Lemma A.4 for a useful analyticity recovery estimate for the Y_μ norm.

We also define a weighted L^2 norm (with respect to both x and y) by

$$\|f\|_S^2 = \|yf\|^2_{L^2(y \geq 1/2)} = \sum_\xi \|yf_\xi\|^2_{L^2(y \geq 1/2)}$$

and a weighted version of the Sobolev H^3 norm as

$$\|f\|_Z = \sum_{0 \leq i+j \leq 3} \left\|\partial_x^i \partial_y^j f\right\|_S .$$

Further below, it is convenient to also use a weighted L^2 in y, ℓ^1 in ξ norm S_μ given by

$$\|f\|_{S_\mu} = \sum_\xi \|yf_\xi\|_{L^2(y \geq 1+\mu)} .$$

Lastly, for fixed $\mu_0, \gamma > 0$, and with t which obeys (2.1), we introduce the notation

$$|\!|\!| \omega |\!|\!|_t = \|\omega\|_{X(t)} + \|\omega\|_{Y(t)} + \|\omega\|_Z$$

for the *cumulative* time-dependent norm used in this paper.

3. Main results

We denote by $\omega = \nabla^\perp \cdot u$ the scalar vorticity associated to the the velocity field u, where $\nabla^\perp = (-\partial_y, \partial_x)$. The following is the main result of the paper.

THEOREM 3.1. *Let $\mu_0 > 0$ and assume that ω_0 is such that*

$$\sum_{i+j\leq 2} \left\|\partial_x^i (y\partial_y)^j \omega_0\right\|_{X_{\mu_0}} + \sum_{i+j\leq 2} \left\|\partial_x^i (y\partial_y)^j \omega_0\right\|_{Y_{\mu_0}} + \sum_{i+j\leq 4} \left\|\partial_x^i (y\partial_y)^j \omega_0\right\|_{S} \leq M < \infty \,. \tag{3.1}$$

Then there exists a $\gamma > 0$ and a time $T > 0$ depending on M and μ_0, such that the solution ω to the system (3.4) *satisfies*

$$\sup_{t\in[0,T]} \|\omega(t)\|_t \leq CM \,. \tag{3.2}$$

The above result immediately implies the following statement.

THEOREM 3.2. *Let ω_0 be as in Theorem 3.1. Denote by u^ν the solution of the Navier-Stokes equation* (1.1)–(1.4) *with viscosity $\nu > 0$, defined on $[0, T]$, where T is as given in Theorem 3.1. Also, denote by $\bar{u}$ the solution of the Euler equations with initial datum ω_0, which is defined globally in time. Then we have*

$$\lim_{\nu\to 0} \sup_{t\in[0,T]} \|u^\nu(t) - \bar{u}(t)\|_{L^2(\mathbb{H})} = 0 \,.$$

The proof of Theorem 3.1 is given in Section 3.4, while the proof of Theorem 3.2 is given in Section 3.5.

REMARK 3.3. Note that the condition on the initial datum in both theorems depends on ν since the first norm in (3.1), the X norm, depends on it. However, it is easy to find sufficient ν-independent conditions which guarantee the bound. For instance, by using $w(y) \leq 1$ we see that a sufficient condition for

$$\sum_{i+j\leq 2} \left\|\partial_x^i (y\partial_y)^j \omega_0\right\|_{X_{\mu_0}} \leq \frac{M}{3}$$

to hold is that

$$\sum_{i+j\leq 2} \sum_{\xi\in\mathbb{Z}} \left(\sup_{y\in\Omega_{\mu_0}} \left| e^{\epsilon_0(1+\mu_0-y)+|\xi|} \partial_x^i (y\partial_y)^j \omega_{0,\xi}(y) \right| \right) \leq \frac{M}{C} \,, \tag{3.3}$$

for a sufficiently large universal constant $C > 0$. A sufficient condition for (3.3) is

$$\sum_{\xi\in\mathbb{Z}} \left(\sup_{y\in\Omega_{\bar{\mu}_0}} \left| e^{\epsilon_0(1+\bar{\mu}_0-y)+|\xi|} \omega_{0,\xi}(y) \right| \right) \leq \frac{M}{C}$$

where $\bar{\mu}_0 > \mu_0$.

3.1. Vorticity formulation. In this paper, we use the vorticity formulation of the Navier-Stokes equations (1.1)–(1.4). Taking the curl of the momentum equation (1.1), i.e. by applying $\nabla^\perp \cdot$ to it, gives

$$\omega_t + u \cdot \nabla\omega - \nu\Delta\omega = 0 \,, \tag{3.4}$$

where u is recovered by the Biot-Savart law $u = \nabla^\perp \Delta^{-1}\omega$. The boundary condition in this setting was introduced in [**2, 43, 44**] and is given by

$$\nu(\partial_y + |\partial_x|)\omega = \partial_y \Delta^{-1}(u \cdot \nabla\omega)|_{y=0} \,. \tag{3.5}$$

The condition (3.5) follows from $\partial_t u_1|_{y=0} = 0$, the Biot-Savart law, and the vorticity equation (3.4).

3.2. Integral representation of the solution to the Navier-Stokes equations. For $\xi \in \mathbb{Z}$, denote by

$$N_\xi(s,y) = -(u \cdot \nabla\omega)_\xi(s,y)$$

the Fourier transform in the x variable of the nonlinear term in the vorticity formulation of the Navier-Stokes system. Also, let

$$B_\xi(s) = (\partial_y \Delta^{-1}(u\cdot\nabla\omega))_\xi(s)|_{y=0} = -(\partial_y \Delta_\xi^{-1} N_\xi(s))|_{y=0}\,,$$

where

$$\Delta_\xi = -\xi^2 + \partial_y^2$$

is considered with a Dirichlet boundary condition at $y=0$. After taking a Fourier transform in the tangential x variable, the system (3.4)–(3.5) may be rewritten as

$$\begin{aligned} \partial_t \omega_\xi - \nu\Delta_\xi \omega_\xi &= N_\xi \\ \nu(\partial_y + |\xi|)\omega_\xi &= B_\xi\,, \end{aligned} \tag{3.6}$$

for $\xi \in \mathbb{Z}$. Denoting the Green's function for this system by $G_\xi(t,y,z)$, we may represent the solution of this system as

$$\omega_\xi(t,y) = \int_0^\infty G_\xi(t,y,z)\omega_{0\xi}(z)\,dz + \int_0^t\int_0^\infty G_\xi(t-s,y,z)N_\xi(s,z)\,dzds + \int_0^t G_\xi(t-s,y,0)B_\xi(s)\,ds, \tag{3.7}$$

where $\omega_{0\xi}(z)$ is the Fourier transform of the initial data. A proof of this formulation can be found in [**50**].

The next lemma gives an estimate of the Green's function G_ξ of the Stokes system. For its proof, we refer to [**50**, Proposition 3.3 and Section 3.3].

LEMMA 3.4 ([**50**]). *The Green's function G_ξ for the system (3.6) is given by*

$$G_\xi = \widetilde{H}_\xi + R_\xi\,, \tag{3.8}$$

where

$$\widetilde{H}_\xi(t,y,z) = \frac{1}{\sqrt{\nu t}}\left(e^{-\frac{(y-z)^2}{4\nu t}} + e^{-\frac{(y+z)^2}{4\nu t}}\right)e^{-\nu\xi^2 t} \tag{3.9}$$

is the one dimensional heat kernel for the half space with homogeneous Neumann boundary condition. The residual kernel R_ξ has the property $(\partial_y - \partial_z)R_\xi(t,y,z) = 0$, meaning that it is a function of $y+z$, and it satisfies the bounds

$$|\partial_z^k R_\xi(t,y,z)| \lesssim b^{k+1}e^{-\theta_0 b(y+z)} + \frac{1}{(\nu t)^{(k+1)/2}}e^{-\theta_0\frac{(y+z)^2}{\nu t}}e^{-\frac{\nu\xi^2 t}{8}}, \qquad k\in\mathbb{N}_0\,, \tag{3.10}$$

where $\theta_0 > 0$ is a constant independent of ν. The boundary remainder coefficient b in (3.10) is given by

$$b = b(\xi,\nu) = |\xi| + \frac{1}{\sqrt{\nu}}\,.$$

The implicit constant in (3.10) depends only on k and θ_0.

REMARK 3.5. Based on (3.10), the residual kernel R_ξ satisfies

$$\begin{aligned} |(y\partial_y)^k R_\xi(t,y,z)| &\lesssim b\left((yb)^k + 1\right)e^{-\theta_0 b(y+z)} + \left(\left(\frac{y}{\sqrt{\nu t}}\right)^k + 1\right)\frac{1}{\sqrt{\nu t}}e^{-\theta_0\frac{(y+z)^2}{\nu t}}e^{-\frac{\nu\xi^2 t}{8}} \\ &\lesssim b e^{-\frac{\theta_0}{2}b(y+z)} + \frac{1}{\sqrt{\nu t}}e^{-\frac{\theta_0}{2}\frac{(y+z)^2}{\nu t}}e^{-\frac{\nu\xi^2 t}{8}} \end{aligned} \tag{3.11}$$

for $k\in\{0,1,2\}$, pointwise in $y,z\geq 0$.

REMARK 3.6. As explained in [**50**], the Duhamel formula (3.7) holds not just for real values of $y, z \in [0,\infty)$ but in general for all complex values $y, z \in \Omega_\mu \cup [1+\mu,\infty)$. In this case, for $y \in \Omega_\mu$, we may find $\theta \in [0,\mu)$ such that $y \in \partial\Omega_\theta$. If $\mathbb{I}\mathrm{m}\, y \geq 0$, the integrals from 0 to ∞ in (3.7) become integrals over the complex contour $\gamma_\theta^+ = (\partial\Omega_\theta \cap \{z\colon \mathbb{I}\mathrm{m}\, z \geq 0\}) \cup [1+\theta,\infty)$. A similar contour may be defined for $\mathbb{I}\mathrm{m}\, y < 0$. Moreover, the Green's function $G_\xi(t,y,z)$ from Lemma 3.4, which appears in (3.7), has a natural extension to the complex domain $\Omega_\mu \cup [1+\mu,\infty)$, by complexifying the heat kernels involved. Since for $y \in \Omega_\mu$ we have $|\mathbb{I}\mathrm{m}\, y| \leq \mu \mathbb{R}\mathrm{e}\, y$, for μ small, we have that $|y|$ is comparable to $\mathbb{R}\mathrm{e}\, y$. Therefore, the upper bounds we have available for the complexified heat kernel $\widetilde{H}_\xi$ and for the residual kernel R_ξ may be written in terms of $\mathbb{R}\mathrm{e}\, y, \mathbb{R}\mathrm{e}\, z \geq 0$. Because of this, as in [**50**], when we prove inequalities for the analytic norms X and Y we provide details only for the case when y and z are real-valued. The complex versions of (3.7) and Lemma 3.4 only lead to notational complications due to integration over complex paths, and due to having to write $\mathbb{R}\mathrm{e}\, y$, $\mathbb{R}\mathrm{e}\, z$ at the exponentials in all upper bounds. We omit these details.

3.3. Main estimates. We denote

$$\mu_1 = \mu + \frac{1}{4}(\mu_0 - \mu - \gamma s) \tag{3.12}$$

$$\mu_2 = \mu + \frac{1}{2}(\mu_0 - \mu - \gamma s) \tag{3.13}$$

and observe that

$$0 < \mu < \mu_1 < \mu_2 < \mu_0 - \gamma s\,.$$

LEMMA 3.7 (**Main X norm estimate**). *With μ_1 and μ_2 as in (3.12) and (3.13), the nonlinear term in (3.7) is bounded in the X_μ norm as*

$$\begin{aligned}
&(\mu_0 - \mu - \gamma s) \sum_{i+j=2} \left\| \partial_x^i (y\partial_y)^j \int_0^\infty G(t-s,y,z) N(s,z)\, dz \right\|_{X_\mu} \\
&\quad + \sum_{i+j\leq 1} \left\| \partial_x^i (y\partial_y)^j \int_0^\infty G(t-s,y,z) N(s,z)\, dz \right\|_{X_{\mu_1}} \\
&\quad \lesssim \sum_{i+j\leq 1} \|\partial_x^i (y\partial_y)^j N(s)\|_{X_{\mu_2}} + \frac{1}{(\mu_0-\mu-\gamma s)^{1/2}} \sum_{i+j\leq 1} \|\partial_x^i \partial_y^j N(s)\|_{S_{\mu_2}}\,.
\end{aligned} \tag{3.14}$$

The X_μ norm of the trace kernel term in (3.7) is estimated as

$$\begin{aligned}
&(\mu_0 - \mu - \gamma s) \sum_{i+j=2} \left\| \partial_x^i (y\partial_y)^j G(t-s,y,0) B(s) \right\|_{X_\mu} \\
&\quad + \sum_{i+j\leq 1} \left\| \partial_x^i (y\partial_y)^j G(t-s,y,0) B(s) \right\|_{X_{\mu_1}} \\
&\quad \lesssim \frac{1}{\sqrt{t-s}} \left(\sum_{i\leq 1} \|\partial_x^i N(s)\|_{Y_{\mu_1}} + \|\partial_x^i N(s)\|_{S_{\mu_1}} \right) + \sum_{i\leq 1} \left\| \partial_x^i N(s) \right\|_{X_{\mu_1}}\,.
\end{aligned} \tag{3.15}$$

Lastly, the initial datum term in (3.7) may be bounded in the X_μ norm as

$$\begin{aligned}
&\sum_{i+j\leq 2} \left\| \partial_x^i (y\partial_y)^j \int_0^\infty G(t,y,z) \omega_0(z)\, dz \right\|_{X_\mu} \\
&\quad \lesssim \sum_{i+j\leq 2} \|\partial_x^i (y\partial_y)^j \omega_0\|_{X_\mu} + \sum_{i+j\leq 2} \sum_\xi \|\xi^i \partial_y^j \omega_{0\xi}\|_{L^\infty(y\geq 1+\mu)}\,.
\end{aligned}$$

The proof of Lemma 3.7 is given at the end of Section 4.

LEMMA 3.8 (**Main Y norm estimate**). *Let μ_1 be as defined in (3.12). Then the nonlinear term in (3.7) is bounded in the Y_μ norm as*

$$
\begin{aligned}
&(\mu_0-\mu-\gamma s)\sum_{i+j=2}\left\|\partial_x^i(y\partial_y)^j\int_0^\infty G(t-s,y,z)N(s,z)\,dz\right\|_{Y_\mu}\\
&\quad+\sum_{i+j\le 1}\left\|\partial_x^i(y\partial_y)^j\int_0^\infty G(t-s,y,z)N(s,z)\,dz\right\|_{Y_{\mu_1}}\\
&\quad\lesssim\sum_{i+j\le 1}\|\partial_x^i(y\partial_y)^jN(s)\|_{Y_{\mu_1}}+\sum_{i+j\le 1}\|\partial_x^i\partial_y^jN(s)\|_{S_{\mu_1}}\,. \qquad (3.16)
\end{aligned}
$$

The Y_μ norm of the trace kernel term in (3.7) is estimated as

$$
\begin{aligned}
&(\mu_0-\mu-\gamma s)\sum_{i+j=2}\left\|\partial_x^i(y\partial_y)^jG(t-s,y,0)B(s)\right\|_{Y_\mu}+\sum_{i+j\le 1}\left\|\partial_x^i(y\partial_y)^jG(t-s,y,0)B(s)\right\|_{Y_{\mu_1}}\\
&\quad\lesssim\sum_{i\le 1}\left(\|\partial_x^iN(s)\|_{Y_{\mu_1}}+\|\partial_x^iN(s)\|_{S_\mu}\right)\,. \qquad (3.17)
\end{aligned}
$$

Lastly, the initial datum term in (3.7) may be bounded as

$$
\begin{aligned}
&\sum_{i+j\le 2}\left\|\partial_x^i(y\partial_y)^j\int_0^\infty G(t,y,z)\omega_0(z)\,dz\right\|_{Y_\mu}\\
&\quad\lesssim\sum_{i+j\le 2}\|\partial_x^i(y\partial_y)^j\omega_0\|_{Y_\mu}+\sum_{i+j\le 2}\sum_\xi\|\xi^i\partial_y^j\omega_{0\xi}\|_{L^1(y\ge 1+\mu)}\,. \qquad (3.18)
\end{aligned}
$$

The proof of Lemma 3.8 is provided at the end of Section 5. The next lemma provides inequalities for the nonlinearity.

LEMMA 3.9 (**The X, Y, and S_μ norm estimates for the nonlinearity**). *For any $\mu\in(0,\mu_0-\gamma s)$ we have the inequalities*

$$
\begin{aligned}
\sum_{i+j\le 1}\|\partial_x^i(y\partial_y)^jN(s)\|_{X_\mu}&\lesssim\sum_{i\le 1}\left(\|\partial_x^i\omega\|_{Y_\mu}+\|\partial_x^i\omega\|_{S_\mu}\right)\sum_{i+j\le 2}\|\partial_x^i(y\partial_y)^j\omega\|_{X_\mu}\\
&\quad+\sum_{i\le 2}\left(\|\partial_x^i\omega\|_{Y_\mu}+\|\partial_x^i\omega\|_{S_\mu}\right)\sum_{i+j\le 1}\|\partial_x^i(y\partial_y)^j\omega\|_{X_\mu}\\
&\quad+\|\omega\|_{X_\mu}\sum_{i+j=1}\|\partial_x^i(y\partial_y)^j\omega\|_{X_\mu} \qquad (3.19)
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{i+j\le 1}\|\partial_x^i(y\partial_y)^jN(s)\|_{Y_\mu}&\lesssim\sum_{i\le 1}\left(\|\partial_x^i\omega\|_{Y_\mu}+\|\partial_x^i\omega\|_{S_\mu}\right)\sum_{i+j\le 2}\|\partial_x^i(y\partial_y)^j\omega\|_{Y_\mu}\\
&\quad+\sum_{i\le 2}\left(\|\partial_x^i\omega\|_{Y_\mu}+\|\partial_x^i\omega\|_{S_\mu}\right)\sum_{i+j\le 1}\|\partial_x^i(y\partial_y)^j\omega\|_{Y_\mu}\\
&\quad+\|\omega\|_{X_\mu}\sum_{i+j=1}\|\partial_x^i(y\partial_y)^j\omega\|_{Y_\mu}\,.
\end{aligned}
$$

For the Sobolev norm we have the estimate

$$
\sum_{i+j\le 1}\left\|\partial_x^i\partial_y^jN(s)\right\|_{S_\mu}\lesssim\|\omega\|_s\sum_{i+j\le 3}\left\|\partial_x^i\partial_y^j\omega\right\|_S\,. \qquad (3.20)
$$

The proof of Lemma 3.9 is given at the end of Section 6. Lastly, the following statement provides the estimate of the Sobolev part of the norm.

LEMMA 3.10. *For any* $0 < t < \frac{\mu_0}{2\gamma}$ *the estimate*

$$\sum_{i+j\le 3} \|y\partial_x^i\partial_y^j\omega(t)\|_{L^2(y\ge 1/2)}^2 \lesssim \Big(1 + t\sup_{s\in[0,t]}\|\!|\omega(s)\|\!|_s^3\Big) e^{Ct(1+\sup_{s\in[0,t]}\|\!|\omega(s)\|\!|_s)} \sum_{i+j\le 3}\|y\partial_x^i\partial_y^j\omega_0\|_{L^2(y\ge 1/4)}^2 \tag{3.21}$$

holds, where $C > 0$ *is a constant independent of* γ.

This statement follows from Lemma 7.2 below.

3.4. Closing the a priori estimates. In this section, we provide the a priori estimates needed to prove Theorem 3.1.

PROOF OF THEOREM 3.1. Define

$$\widetilde{M} = \sum_{i+j\le 2}\|\partial_x^i(y\partial_y)^j\omega_0\|_{X_{\mu_0}} + \sum_{i+j\le 2}\sum_\xi \|\partial_x^i\partial_y^j\omega_{0,\xi}\|_{L^\infty(y\ge 1+\mu_0)}$$

and

$$\overline{M} = \sum_{i+j\le 2}\|\partial_x^i(y\partial_y)^j\omega_0\|_{Y_{\mu_0}} + \sum_{i+j\le 2}\sum_\xi \|\partial_x^i\partial_y^j\omega_{0,\xi}\|_{L^1(y\ge 1+\mu_0)}\,.$$

Note that by (3.1), (6.23) below, and Lemma A.1, we have

$$\widetilde{M} + \overline{M} \lesssim M\,.$$

Let $t < \frac{\mu_0}{2\gamma}$, $s \in (0,t)$, and $\mu < \mu_0 - \gamma t$. First we estimate the $X(t)$ norm of $\omega(t)$. From the mild formulation (3.7), the estimates (3.14)–(3.15), and the bounds (3.19)–(3.20) for the nonlinear term, we obtain

$$\begin{aligned}
\sum_{i+j=2}\big\|\partial_x^i(y\partial_y)^j\omega(t)\big\|_{X_\mu} &\lesssim \int_0^t \left(\frac{\|\!|\omega(s)\|\!|_s^2}{(\mu_0-\mu-\gamma s)^{3/2+\alpha}} + \frac{1}{\sqrt{t-s}}\frac{\|\!|\omega(s)\|\!|_s^2}{(\mu_0-\mu-\gamma s)^{1+\alpha}}\right) ds + \widetilde{M}\\
&\lesssim \sup_{0\le s\le t}\|\!|\omega(s)\|\!|_s^2\left(\frac{1}{\gamma(\mu_0-\mu-\gamma t)^{1/2+\alpha}} + \frac{1}{\sqrt{\gamma}(\mu_0-\mu-\gamma t)^{1/2+\alpha}}\right) + \widetilde{M}\\
&\lesssim \frac{\sup_{0\le s\le t}\|\!|\omega(s)\|\!|_s^2}{\sqrt{\gamma}(\mu_0-\mu-\gamma t)^{1/2+\alpha}} + \widetilde{M}\,,
\end{aligned} \tag{3.22}$$

where we used Lemma A.2. Similarly, we obtain

$$\begin{aligned}
\sum_{i+j\le 1}\big\|\partial_x^i(y\partial_y)^j\omega(t)\big\|_{X_\mu} &\lesssim \int_0^t \left(\frac{\|\!|\omega(s)\|\!|_s^2}{(\mu_0-\mu-\gamma s)^{1/2+\alpha}} + \frac{1}{\sqrt{t-s}}\frac{\|\!|\omega(s)\|\!|_s^2}{(\mu_0-\mu-\gamma s)^{\alpha}}\right) ds + \widetilde{M}\\
&\lesssim \frac{\sup_{0\le s\le t}\|\!|\omega(s)\|\!|_s^2}{\sqrt{\gamma}} + \widetilde{M}\,,
\end{aligned} \tag{3.23}$$

where we again used Lemma A.2. Combining (3.22) and (3.23), we obtain

$$\|\omega(t)\|_{X(t)} \lesssim \frac{\sup_{0\le s\le t}\|\!|\omega(s)\|\!|_s^2}{\sqrt{\gamma}} + \widetilde{M}\,. \tag{3.24}$$

Next we estimate the $Y(t)$ norm of $\omega(t)$. From the mild formulation (3.7), the estimates (3.16)–(3.18), and the bounds (3.19)–(3.20) for the nonlinear term, we obtain

$$\sum_{i+j=2}\big\|\partial_x^i(y\partial_y)^j\omega(t)\big\|_{Y_\mu} \lesssim \int_0^t \frac{\|\!|\omega(s)\|\!|_s^2}{(\mu_0-\mu-\gamma s)^{1+\alpha}}\,ds + \overline{M} \lesssim \frac{\sup_{0\le s\le t}\|\!|\omega(s)\|\!|_s^2}{\gamma(\mu_0-\mu-\gamma t)^{\alpha}} + \overline{M}\,. \tag{3.25}$$

For the lower order derivatives we obtain

$$\sum_{i+j\leq 1} \left\|\partial_x^i (y\partial_y)^j \omega(t)\right\|_{Y_\mu} \lesssim \int_0^t \frac{\|\omega(s)\|_s^2}{(\mu_0-\mu-\gamma s)^\alpha}\, ds + \overline{M} \lesssim \frac{\sup_{0\leq s\leq t} \|\omega(s)\|_s^2}{\gamma} + \overline{M}\,. \tag{3.26}$$

By combining (3.25)–(3.26), we arrive at

$$\|\omega(t)\|_{Y(t)} \lesssim \frac{\sup_{0\leq s\leq t} \|\omega(s)\|_s^2}{\gamma} + \overline{M}\,. \tag{3.27}$$

To conclude, let

$$\mathring{M} = \sum_{i+j\leq 3} \|\partial_x^i\partial_y^j \omega_0\|_{L^2(y\geq 1/4)} \lesssim M\,.$$

Recall that the Sobolev estimate (3.21) yields

$$\|\omega(t)\|_Z \lesssim \left(1 + \frac{\sup_{s\in[0,t]} \|\omega(s)\|_s^{3/2}}{\sqrt{\gamma}}\right) e^{\frac{C}{\gamma}(1+\sup_{s\in[0,t]}\|\omega(s)\|_s)} \mathring{M}\,, \tag{3.28}$$

and this inequality holds pointwise in time for $t < \frac{\mu_0}{2\gamma}$. The constant C and the implicit constants in $\lesssim$ are independent of γ.

Combining (3.24), (3.27), and (3.28), and taking the supremum in time for $t < \frac{\mu_0}{2\gamma}$, we arrive at

$$\begin{aligned}\sup_{t\in[0,\frac{\mu_0}{2\gamma}]} \|\omega(t)\|_t \leq C(\widetilde{M} + \overline{M}) + \frac{C\sup_{t\in[0,\frac{\mu_0}{2\gamma}]} \|\omega(t)\|_t^2}{\sqrt{\gamma}} \\ + C\mathring{M}\left(1 + \frac{\sup_{t\in[0,\frac{\mu_0}{2\gamma}]} \|\omega(t)\|_t^{3/2}}{\sqrt{\gamma}}\right) e^{\frac{C\mu_0}{\gamma}(1+\sup_{t\in[0,\frac{\mu_0}{2\gamma}]}\|\omega(t)\|_t)}\,,\end{aligned}$$

where $C \geq 1$ is a constant that depends only on μ_0. Using a standard barrier argument, one may show that if γ is chosen sufficiently large, in terms of $\widetilde{M}, \overline{M}, \mathring{M}, \mu_0$, we obtain

$$\sup_{t\in[0,\frac{\mu_0}{2\gamma}]} \|\omega(t)\|_t \leq 2C(\widetilde{M} + \overline{M} + \mathring{M})\,,$$

concluding the proof of the theorem. □

REMARK 3.11. In order to justify the above a priori estimates, for each $\delta \in (0,1]$, we apply them on the approximate system

$$\omega_t + u^\delta \cdot \nabla\omega - \nu\Delta\omega = 0\,, \tag{3.29}$$

where u^δ is a regularization of the velocity in the Biot-Savart law (6.2)–(6.3). The boundary condition (3.5) becomes $\nu(\partial_y + |\partial_x|)\omega = \partial_y\Delta^{-1}(u^\delta\cdot\nabla\omega)|_{y=0}$, and the initial condition is replaced by an analytic approximation. The regularized velocity u^δ is obtained from ω by a heat extension to time δ, using a homogeneous version of the boundary condition (3.6), and then computing the Biot-Savart law for this regularized vorticity. Now, in order to justify our a priori estimates, we approximate the initial datum ω_0 with an entire one ω_0^δ. We may show using the approach in this paper that the system (3.29) with entire initial data has a solution which is entire for all time. Then, we perform all the estimates in the present paper on (3.29), obtaining uniform-in-δ upper bounds for the norm $\|\cdot\|_t$ for all $t \in [0, \frac{\mu_0}{2\gamma})$, thus allowing us to pass those bounds to the limit $\delta \to 0$.

3.5. Inviscid limit. This section is devoted to the proof of Theorem 3.2.

PROOF OF THEOREM 3.2. Let $T > 0$ be as in Theorem 3.1. In view of the Kato criterion [30], we only need to consider

$$\begin{aligned}
&\nu \int_0^T\int_{\mathbb{H}} |\nabla u|^2 \,dxdyds = \nu \int_0^T\int_{\mathbb{H}} |\omega|^2 \,dxdyds \\
&\quad = \nu \int_0^T\int_{\{y\le 1/2\}} |\omega|^2 \,dxdyds + \nu \int_0^T\int_{\{y\ge 1/2\}} |\omega|^2 \,dxdyds \\
&\quad \lesssim \sqrt{\nu} \int_0^T \sum_\xi \|e^{\epsilon_0(1-y)_+ |\xi|} w(y)\omega_\xi(s)\|_{L^\infty(y\le 1/2)} \|e^{\epsilon_0(1-y)_+|\xi|}\omega_\xi(s)\|_{L^1(y\le 1/2)}\,ds + \nu \int_0^T \|\omega(s)\|_S^2 \,ds \\
&\quad \lesssim \sqrt{\nu} \int_0^T \|\omega(s)\|_{X(s)} \|\omega(s)\|_{Y(s)} \,ds + \nu \int_0^T \|\omega(s)\|_S^2 \,ds \\
&\quad \lesssim \sqrt{\nu} CM\,.
\end{aligned}$$

Here we used that $\sqrt{\nu} \lesssim w(y)$ and have appealed to the bound (3.2). By [30] it follows that the inviscid limit holds in the topology of $L^\infty(0,T;L^2(\mathbb{H}))$. $\square$

4. Estimates for the X analytic norm

Throughout this section we fix $t > 0$ and $s \in (0,t)$ and provide the X norm estimate of the three integrals appearing in (3.7). We first consider the kernel

$$H_\xi(t,y,z) = \frac{1}{\sqrt{\nu t}} e^{-\frac{(y-z)^2}{4\nu t}} e^{-\nu\xi^2 t}\,. \tag{4.1}$$

In the following lemma, we estimate the derivatives up to order one of the integral involving the nonlinearity.

LEMMA 4.1. *Assume that μ and $\widetilde{\mu}$ obey the conditions*

$$0 < \mu < \widetilde{\mu} < \mu_0 - \gamma s, \qquad \widetilde{\mu} - \mu \ge \frac{1}{C}(\mu_0 - \mu - \gamma s)\,, \tag{4.2}$$

for some constant $C \ge 1$. Then, for $(i,j) = (0,0),(1,0),(0,1)$, we have

$$\begin{aligned}
&\left\| \partial_x^i (y\partial_y)^j \int_0^\infty H(t-s,y,z) N(s,z)\,dz \right\|_{X_\mu} \\
&\qquad \lesssim \|\partial_x^i (y\partial_y)^j N(s)\|_{X_{\widetilde{\mu}}} + \|N(s)\|_{X_{\widetilde{\mu}}} + \frac{1}{(\mu_0-\mu-\gamma s)^{1/2}} \sum_\xi \|\partial_x^i \partial_y^j N_\xi(s)\|_{L^2(y\ge 1+\widetilde{\mu})}\,.
\end{aligned} \tag{4.3}$$

REMARK 4.2. Inspecting the proof of Lemma 4.1 below, we note that only the following properties of the kernel $H_\xi(y,z,t)$ are used. First, we use that either $\partial_y H_\xi(y,z,t) = \partial_z H_\xi(y,z,t)$ or $\partial_y H_\xi(y,z,t) = -\partial_z H_\xi(y,z,t)$, the property allowing us to transfer y derivatives to z derivatives. For the terms I_1 and I_2 we use

$$\left\| \chi_{\{0\le y\le 1+\mu\}} \chi_{\{0\le z\le 3y/4\}} \frac{w(y)}{w(z)} \big(|H_\xi(t,y,z)| + |y\partial_y H_\xi(t,y,z)| \big) \right\|_{L_y^\infty L_z^1} \lesssim 1\,, \tag{4.4}$$

for the term I_3 we need

$$\left\| e^{\epsilon_0(z-y)_+ |\xi|} H_\xi(t,y,z) \right\|_{L_y^\infty L_z^1} \lesssim 1\,, \tag{4.5}$$

while for the term I_4 we additionally use

$$\left\| \chi_{\{0\le y\le 1+\mu\}} \chi_{\{z\ge 1+\widetilde{\mu}\}} e^{\epsilon_0 (z-y)_+ |\xi|} H_\xi(t,y,z) \right\|_{L^\infty_y L^2_z} \lesssim \frac{1}{\sqrt{\widetilde{\mu}-\mu}}\,. \tag{4.6}$$

Observe that the kernel $\widetilde{H}_\xi(t,y,z) - H_\xi(t,y,z) = \frac{1}{\sqrt{\nu t}} e^{-\frac{(y+z)^2}{4\nu t}} e^{-\nu \xi^2 t}$ also obeys these three properties, and thus Lemma 4.1 holds with $H(t,y,z)$ replaced by $\widetilde{H}(t,y,z)$.

PROOF OF LEMMA 4.1. Let $y \in \Omega_\mu$. For simplicity, we only work with $y \in \mathbb{R}$; an adjustment for the complex case is straight-forward and leads only to notational complications.

We start with the proof of (4.3) in the case $(i,j) = (0,1)$. Let $\psi \colon \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth non-increasing cut-off function such that $\psi(x) = 1$ for $0 \le x \le 1/2$, and $\psi(x) = 0$ for $x \ge 3/4$. We first decompose

$$\begin{aligned}
& y\partial_y \int_0^\infty H_\xi(t-s,y,z) N_\xi(s,z)\, dz \\
& = -y \int_0^\infty \partial_z H_\xi(t-s,y,z) N_\xi(s,z)\, dz \\
& = -y \int_0^\infty \psi\left(\frac{z}{y}\right) \partial_z H_\xi(t-s,y,z) N_\xi(s,z)\, dz \\
& \quad - y \int_0^\infty \left(1-\psi\left(\frac{z}{y}\right)\right) \partial_z H_\xi(t-s,y,z) N_\xi(s,z)\, dz \\
& = -y \int_0^{3y/4} \psi\left(\frac{z}{y}\right) \partial_z H_\xi(t-s,y,z) N_\xi(s,z)\, dz - \int_{y/2}^{3y/4} \psi'\left(\frac{z}{y}\right) H_\xi(t-s,y,z) N_\xi(s,z)\, dz \\
& \quad + y \int_{y/2}^{1+\mu} \left(1-\psi\left(\frac{z}{y}\right)\right) H_\xi(t-s,y,z) \partial_z N_\xi(s,z)\, dz + y \int_{1+\mu}^\infty H_\xi(t-s,y,z) \partial_z N_\xi(s,z)\, dz \\
& = I_1 + I_2 + I_3 + I_4\,.
\end{aligned} \tag{4.7}$$

The first three terms represent contributions from the analytic region and the last term from the Sobolev region.

In order to bound I_1, we compute the derivative of H_ξ as

$$y\partial_z H_\xi = -y\partial_y H_\xi = y \frac{(y-z)}{2\nu(t-s)} \frac{1}{\sqrt{\nu(t-s)}} e^{-\frac{(y-z)^2}{4\nu(t-s)}} e^{-\nu\xi^2(t-s)}\,. \tag{4.8}$$

By

$$|y| \le 4|y-z|, \qquad 0 \le z \le \frac{3y}{4} \tag{4.9}$$

we arrive at

$$|y\partial_z H_\xi| \lesssim \frac{1}{\sqrt{\nu(t-s)}} e^{-\frac{(y-z)^2}{8\nu(t-s)}} e^{-\nu\xi^2(t-s)}, \qquad 0 \le z \le \frac{3y}{4} \tag{4.10}$$

and therefore

$$|I_1| \lesssim \int_0^{3y/4} \frac{1}{\sqrt{\nu(t-s)}} e^{-\frac{(y-z)^2}{8\nu(t-s)}} e^{-\nu\xi^2(t-s)} |N_\xi(s,z)|\, dz\,. \tag{4.11}$$

Next, we claim that the weight function obeys the estimate

$$w(y) \le w(z) e^{\frac{(y-z)^2}{64\nu(t-s)}}, \qquad 0 \le z \le \frac{3y}{4}\,. \tag{4.12}$$

In order to prove (4.12) we use that $t-s \le T \le 1$ and estimate

$$w(y) e^{-\frac{(y-z)^2}{64\nu(t-s)}} \le w(y) e^{-\frac{y^2}{256\nu(t-s)}} \le w(y) e^{-\frac{y^2}{256\nu}} \lesssim w(y) e^{-\frac{y}{16\sqrt{\nu}}} \lesssim \sqrt{\nu}\,, \tag{4.13}$$

where we used (4.9) in the first and Remark 2.1(e) in the last step. Then (4.12) follows from $\sqrt{\nu} \lesssim w(z)$ by Remark 2.1(c). Using (4.11), (4.12), and

$$e^{\epsilon_0(1+\mu-y)_+|\xi|} \leq e^{\epsilon_0(1+\mu-z)_+|\xi|}, \qquad 0 \leq z \leq y\,, \tag{4.14}$$

we obtain

$$\begin{aligned}|e^{\epsilon_0(1+\mu-y)_+|\xi|}w(y)I_1| &\lesssim \int_0^{3y/4} e^{\epsilon_0(1+\mu-z)_+|\xi|}\frac{1}{\sqrt{\nu(t-s)}}e^{-\frac{(y-z)^2}{16\nu(t-s)}}e^{-\nu\xi^2(t-s)}w(z)|N_\xi(s,z)|\,dz\\ &\lesssim \|N_\xi(s)\|_{\mathcal{L}^\infty_{\mu,\nu}}\int_0^\infty \frac{1}{\sqrt{\nu(t-s)}}e^{-\frac{(y-z)^2}{16\nu(t-s)}}\,dz\\ &\lesssim \|N_\xi(s)\|_{\mathcal{L}^\infty_{\mu,\nu}}\,.\end{aligned}$$

Summing in ξ yields the bound

$$\|I_1\|_{X_\mu} \lesssim \|N(s)\|_{X_\mu}\,. \tag{4.15}$$

Next, we consider the term I_2 on the right side of (4.7). Since $\|\psi'\|_{L^\infty} \lesssim 1$, we directly obtain

$$|I_2| \lesssim \int_{y/2}^{3y/4}\frac{1}{\sqrt{\nu(t-s)}}e^{-\frac{(y-z)^2}{4\nu(t-s)}}e^{-\nu\xi^2(t-s)}|N_\xi(s,z)|\,dz\,,$$

which shows that I_2 obeys the same estimate as I_1 (cf. (4.11) above). Since the regions of integration also match, the same proof as for (4.15) gives

$$\|I_2\|_{X_\mu} \leq \|N(s)\|_{X_\mu}\,.$$

The term I_3 in (4.7), which we recall equals

$$I_3 = y\int_{y/2}^{1+\mu}\left(1-\psi\left(\frac{z}{y}\right)\right)H_\xi(t-s,y,z)\partial_z N_\xi(s,z)\,dz\,, \tag{4.16}$$

is treated slightly differently. Since $z \geq y/2$ we may trade a power of y for a power of z, and we also have $w(y) \lesssim w(z)$ for $z \geq y/2$ by Remark 2.1(b), where the implicit constant is independent of ν. Therefore,

$$\begin{aligned}\|I_3\|_{X_\mu} &= \sum_\xi \sup_{y\in\Omega_\mu} w(y)e^{\epsilon_0(1+\mu-y)_+|\xi|}|I_3|\\ &\lesssim \sum_\xi \sup_{y\in\Omega_\mu}\int_{y/2}^{1+\mu} e^{\epsilon_0(1+\mu-y)_+|\xi|}H_\xi(t-s,y,z)w(z)|z\partial_z N_\xi(s,z)|\,dz\,.\end{aligned}$$

Now, we use

$$e^{\epsilon_0(1+\mu-y)_+|\xi|} \leq e^{\epsilon_0(1+\mu-z)_+|\xi|}e^{\epsilon_0(z-y)_+|\xi|} \leq e^{\epsilon_0(1+\mu-z)_+|\xi|}e^{\epsilon_0(y-z)^2/2\nu(t-s)}e^{\epsilon_0\nu\xi^2(t-s)/2}\,, \tag{4.17}$$

which follows from

$$e^{2a|\xi|} \leq e^{a^2/c}e^{c\xi^2} \tag{4.18}$$

with suitable $a, c > 0$. Choosing ϵ_0 sufficiently small, we obtain

$$\begin{aligned}\|I_3\|_{X_\mu} &\lesssim \sum_\xi \sup_{y\in\Omega_\mu}\int_{y/2}^{1+\mu}\frac{1}{\sqrt{\nu(t-s)}}e^{-\frac{(y-z)^2}{8\nu(t-s)}}e^{-\nu\xi^2(t-s)}e^{\epsilon_0(1+\mu-z)_+|\xi|}w(z)|z\partial_z N_\xi(s,z)|\,dz\\ &\lesssim \|z\partial_z N(s)\|_{X_\mu}\int_0^\infty \frac{1}{\sqrt{\nu(t-s)}}e^{-\frac{(y-z)^2}{8\nu(t-s)}}\,dz\,,\end{aligned}$$

whence

$$\|I_3\|_{X_\mu} \lesssim \|z\partial_z N(s)\|_{X_\mu}\,. \tag{4.19}$$

It remains to estimate the term I_4 in (4.7), which we recall equals

$$I_4 = y\int_{1+\mu}^{\infty} H_\xi(t-s,y,z)\partial_z N_\xi(s,z)\,dz\,.$$

Using Remark 2.1(a) and (c), the bound (4.17), and choosing ϵ_0 sufficiently small, we obtain

$$\begin{aligned}
&e^{\epsilon_0(1+\mu-y)_+ + |\xi|}w(y)|I_4| \\
&\lesssim \int_{1+\mu}^{1+\widetilde{\mu}} \frac{1}{\sqrt{\nu(t-s)}} e^{\epsilon_0(z-y)_+ + |\xi|} e^{-\frac{(y-z)^2}{4\nu(t-s)}} e^{-\frac12\nu\xi^2(t-s)} e^{\epsilon_0(1+\mu-z)_+ + |\xi|} w(z)|z\partial_z N_\xi(s,z)|\,dz \\
&\quad + \int_{1+\widetilde{\mu}}^{\infty} \frac{1}{\sqrt{\nu(t-s)}} e^{\epsilon_0(z-y)_+ + |\xi|} e^{-\frac{(y-z)^2}{4\nu(t-s)}} e^{-\nu\xi^2(t-s)} |\partial_z N_\xi(s,z)|\,dz \\
&\lesssim \|z\partial_z N_\xi(s)\|_{\mathcal{L}^\infty_{\widetilde{\mu},\nu}} \int_{1+\mu}^{1+\widetilde{\mu}} \frac{1}{\sqrt{\nu(t-s)}} e^{-\frac{(y-z)^2}{8\nu(t-s)}}\,dz \\
&\quad + \int_{1+\widetilde{\mu}}^{\infty} \frac{1}{\sqrt{\nu(t-s)}} e^{-\frac{(y-z)^2}{16\nu(t-s)}} e^{-\frac{(\widetilde{\mu}-\mu)^2}{16\nu(t-s)}} |\partial_z N_\xi(s,z)|\,dz \\
&\lesssim \|z\partial_z N_\xi(s)\|_{\mathcal{L}^\infty_{\widetilde{\mu},\nu}} + \frac{e^{-\frac{(\widetilde{\mu}-\mu)^2}{16\nu(t-s)}}}{(\nu(t-s))^{1/4}} \left(\int_{1+\widetilde{\mu}}^{\infty} \frac{1}{\sqrt{\nu(t-s)}} e^{-\frac{(y-z)^2}{8\nu(t-s)}}\,dz\right)^{1/2} \|\partial_z N_\xi(s,z)\|_{L^2(z\geq 1+\widetilde{\mu})} \\
&\lesssim \|z\partial_z N_\xi(s)\|_{\mathcal{L}^\infty_{\widetilde{\mu},\nu}} + \frac{1}{(\widetilde{\mu}-\mu)^{1/2}} \|\partial_z N_\xi(s,z)\|_{L^2(z\geq 1+\widetilde{\mu})}\,.
\end{aligned}$$

Taking a supremum over $y \in \Omega_\mu$, summing over ξ, and recalling the definition of $\widetilde{\mu}$ in (4.2), we deduce

$$\|I_4\|_{X_\mu} \lesssim \|z\partial_z N_\xi(s)\|_{X_{\widetilde{\mu}}} + \frac{1}{\sqrt{\mu_0-\mu-\gamma s}} \sum_\xi \|\partial_z N_\xi(s,z)\|_{L^2(z\geq 1+\widetilde{\mu})}\,.$$

This concludes the proof of (4.3) with $(i,j)=(0,1)$.

Next, we note that the estimate (4.3) for $(i,j)=(1,0)$ follows from the bound (4.3) with $(i,j)=(0,0)$, by applying the bound to $\partial_x N$ instead of N. Therefore, it only remains to establish (4.3) for $(i,j)=(0,0)$. With ψ as above, we decompose the convolution integral into three integrals as

$$\begin{aligned}
&\int_0^\infty H_\xi(t-s,y,z)N_\xi(s,z)\,dz \\
&= \int_0^{3y/4} \psi\left(\frac{z}{y}\right) H_\xi(t-s,y,z)N_\xi(s,z)\,dz + \int_{y/2}^{1+\mu} \left(1-\psi\left(\frac{z}{y}\right)\right) H_\xi(t-s,y,z)N_\xi(s,z)\,dz \\
&\quad + \int_{1+\mu}^{\infty} H_\xi(t-s,y,z)N_\xi(s,z)\,dz \\
&= J_1 + J_2 + J_3\,. \qquad (4.20)
\end{aligned}$$

Upon inspection, J_1 may be bounded in exactly the same way as the term I_1 earlier, which gives the bound

$$\|J_1\|_{X_\mu} \lesssim \|N(s)\|_{X_\mu}\,.$$

On the other hand, J_2 is estimated exactly as the term I_3 above, and we obtain

$$\|J_2\|_{X_\mu} \lesssim \|N(s)\|_{X_\mu}\,.$$

Lastly, J_3 is bounded just as I_4, by splitting the integral on $[1+\mu,\infty)$ into an integral on $[1+\mu, 1+\widetilde{\mu}]$ and one on $[1+\widetilde{\mu},\infty)$. This results in the bound

$$\|J_3\|_{X_\mu} \lesssim \|N_\xi(s)\|_{X_{\widetilde{\mu}}} + \frac{1}{\sqrt{\mu_0-\mu-\gamma s}} \sum_\xi \|N_\xi(s,z)\|_{L^2(z\geq 1+\widetilde{\mu})}\,,$$

concluding the proof of the lemma. $\square$

LEMMA 4.3. *Let $\mu < \widetilde{\mu}$ obey (4.2). For $(i,j) = (0,0), (1,0), (0,1)$, we have*

$$\left\| \partial_x^i (y\partial_y)^j \int_0^\infty R(t-s,y,z) N(s,z)\, dz \right\|_{X_\mu} \lesssim \|\partial_x^i (y\partial_y)^j N(s)\|_{X_{\widetilde{\mu}}} + \|N(s)\|_{X_{\widetilde{\mu}}} + \frac{1}{(\mu_0 - \mu - \gamma s)^{1/2}} \sum_\xi \|\partial_x^i \partial_y^j N_\xi\|_{L^2(y\geq 1+\widetilde{\mu})}\,. \tag{4.21}$$

PROOF OF LEMMA 4.3. In order to establish (4.21), it suffices to verify the assumptions in Remark 4.2 for the kernel $R_\xi(t,y,z)$. We recall that $\partial_y R_\xi(t,y,z) = -\partial_z R_\xi(t,y,z)$, which is necessary to change y derivatives to z derivatives. First fix $y \in [0, 1+\mu]$ and $z \in [0, 3y/4]$. Then, since $w(z) \gtrsim \sqrt{\nu}$, from (3.11) we have

$$\begin{aligned} \frac{w(y)}{w(z)} \big(|R_\xi(t,y,z)| + |y\partial_y R_\xi(t,y,z)|\big) &\lesssim \frac{w(y)}{w(z)} \left(b e^{-\frac{\theta_0}{2} b(y+z)} + \frac{1}{\sqrt{\nu t}} e^{-\frac{\theta_0}{2}\frac{(y+z)^2}{\nu t}} e^{-\frac{\nu \xi^2 t}{8}} \right) \\ &\lesssim \frac{w(y)}{\sqrt{\nu}} \left(b e^{-\frac{\theta_0}{2} b(y+z)} + \frac{1}{\sqrt{\nu t}} e^{-\frac{\theta_0}{2}\frac{y^2+z^2}{\nu t}} \right). \end{aligned} \tag{4.22}$$

Next, we use $b \geq \sqrt{\nu}^{-1}$ and thus by Remark 2.1(e) we have

$$\frac{w(y)}{\sqrt{\nu}} e^{-\frac{\theta_0}{4} by} \lesssim e^{\frac{y}{C\sqrt{\nu}}} e^{-\frac{\theta_0}{4}\frac{y}{\sqrt{\nu}}} \lesssim 1\,, \tag{4.23}$$

provided that C is sufficiently large (in terms of θ_0). Similarly to (4.13), using Remark 2.1(e) and the fact that $t \leq 1$ we have

$$\frac{w(y)}{\sqrt{\nu}} e^{-\frac{\theta_0}{2}\frac{y^2}{\nu t}} \lesssim \frac{w(y)}{\sqrt{\nu}} e^{-\frac{\theta_0}{2}\frac{y}{\sqrt{\nu}}} \lesssim 1\,.$$

Therefore, the right side of (4.22) is bounded pointwise in y by

$$b e^{-\frac{\theta_0}{4} bz} + \frac{1}{\sqrt{\nu t}} e^{-\frac{\theta_0}{2}\frac{z^2}{\nu t}}\,.$$

Using that $\|be^{-\frac{\theta_0 bz}{4}}\|_{L^1_z} \lesssim 1$ and $\|\frac{1}{\sqrt{\nu t}} e^{-\frac{\theta_0 z^2}{2\nu t}}\|_{L^1_z} \lesssim 1$, the condition (4.4) for R follows.

In order to verify the condition (4.5), we use that $b \geq |\xi|$, and provided ϵ_0 is sufficiently small in terms of θ_0, we obtain from (4.18) that

$$\begin{aligned} e^{\epsilon_0(z-y)+|\xi|} |R_\xi(y,z,t)| &\lesssim e^{\epsilon_0(z-y)+|\xi|} \left(b e^{-\theta_0 bz} + \frac{1}{\sqrt{\nu t}} e^{-\frac{\theta_0}{2}\frac{y^2+z^2}{\nu t}} e^{-\frac{\nu\xi^2 t}{8}} \right) \\ &\lesssim b e^{-\frac{1}{2}\theta_0 bz} + \left(e^{\epsilon_0 z|\xi|} e^{-\frac{\theta_0}{4}\frac{z^2}{\nu t}} e^{-\frac{\nu \xi^2 t}{8}} \right) \frac{1}{\sqrt{\nu t}} e^{-\frac{\theta_0}{4}\frac{z^2}{\nu t}} \\ &\lesssim b e^{-\frac{1}{2}\theta_0 bz} + \frac{1}{\sqrt{\nu t}} e^{-\frac{\theta_0}{4}\frac{z^2}{\nu t}}\,, \end{aligned}$$

and (4.5), for this kernel, follows by integrating in z. Finally, we check (4.6). For this, let $y \in [0, 1+\mu]$ and $z \geq 1 + \widetilde{\mu}$. Then

$$\begin{aligned} e^{\epsilon_0(z-y)+|\xi|} |R_\xi(y,z,t)| &\lesssim e^{\epsilon_0(z-y)+|\xi|} \left(b e^{-\theta_0 b(y+z)} + \frac{1}{\sqrt{\nu t}} e^{-\frac{\theta_0}{2}\frac{y^2+z^2}{\nu t}} e^{-\frac{\nu\xi^2 t}{8}} \right) \\ &\lesssim b e^{-\frac{1}{2}\theta_0 bz} + \frac{1}{\sqrt{\nu t}} e^{-\frac{\theta_0}{4}\frac{z^2}{\nu t}} \lesssim b e^{-\frac{1}{2}\theta_0 bz} + \frac{1}{(\nu t)^{1/4}} \left(\frac{z^2}{\nu t}\right)^{1/4} e^{-\frac{\theta_0}{4}\frac{z^2}{\nu t}} \\ &\lesssim b^{1/2} e^{-\frac{1}{4}\theta_0 bz} + \frac{1}{(\nu t)^{1/4}} e^{-\frac{\theta_0}{8}\frac{z^2}{\nu t}}\,, \end{aligned} \tag{4.24}$$

where we used $z \geq 1 + \widetilde{\mu} \geq 1$ in the third inequality. Finally, note that the L^2 norm of the far right hand side of (4.24) over $[1 + \widetilde{\mu}, \infty)$ is less than a constant. □

Next, we consider the trace kernel.

LEMMA 4.4. *Let $\mu \in (0, \mu_0 - \gamma s)$ be arbitrary. For $(i,j) = (0,0), (1,0), (0,1)$, we have the inequality*

$$\left\| \partial_x^i (y\partial_y)^j G(t-s,y,0) \partial_z \Delta^{-1} N(s,z)|_{z=0} \right\|_{X_\mu} \lesssim \frac{1}{\sqrt{t-s}} (\|\partial_x^i N(s)\|_{Y_\mu} + \|\partial_x^i N(s)\|_{S_\mu}) + \left\| \partial_x^i N(s) \right\|_{X_\mu} . \tag{4.25}$$

PROOF OF LEMMA 4.4. For $\xi \in \mathbb{Z}$, the kernel $G_\xi(t-s,y,0)$ is the sum of two trace operators

$$T_1(t-s,y) = \widetilde{H}_\xi(t-s,y,0) = \frac{2}{\sqrt{\nu(t-s)}} e^{-\frac{y^2}{4\nu(t-s)}} e^{-\nu\xi^2(t-s)} \tag{4.26}$$

and

$$T_2(t-s,y) = R_\xi(t-s,y,0) .$$

Recall that

$$|y\partial_y T_1(t-s,y)| = \frac{1}{\sqrt{\nu(t-s)}} e^{-\frac{y^2}{4\nu(t-s)}} e^{-\nu\xi^2(t-s)} \frac{y^2}{2\nu(t-s)} \lesssim \frac{1}{\sqrt{\nu(t-s)}} e^{-\frac{y^2}{8\nu(t-s)}} e^{-\nu\xi^2(t-s)} \tag{4.27}$$

and

$$|T_2(t-s,y)| + |y\partial_y T_2(t-s,y)| \lesssim b e^{-\frac{1}{2}\theta_0 b y} + \frac{1}{\sqrt{\nu(t-s)}} e^{-\frac{\theta_0}{2}\frac{y^2}{\nu(t-s)}} e^{-\frac{\nu\xi^2(t-s)}{8}} . \tag{4.28}$$

We first prove (4.25) in the case $i = j = 0$. Similarly to the equation (6.2) in Lemma 6.1 below, we have the representation formula

$$\begin{aligned} \left(\partial_z \Delta^{-1} N_\xi(s,z)\right)|_{z=0} &= -\int_0^\infty e^{-|\xi|z} N_\xi(s,z)\,dz \\ &= -\int_0^{1+\mu} e^{-|\xi|z} N_\xi(s,z)\,dz - \int_{1+\mu}^\infty e^{-|\xi|z} N_\xi(s,z)\,dz = I_1 + I_2 . \end{aligned} \tag{4.29}$$

First we treat the T_1 contribution. Using (4.13) and choosing ϵ_0 sufficiently small, we have

$$\begin{aligned} &|e^{\epsilon_0(1+\mu-y)_+|\xi|} w(y) T_1(t-s,y) I_1| \\ &\quad\lesssim \frac{1}{\sqrt{t-s}} e^{\epsilon_0(1+\mu-y)_+|\xi|} e^{-\frac{y^2}{8\nu(t-s)}} e^{-\nu\xi^2(t-s)} \int_0^{1+\mu} e^{-|\xi|z} |N_\xi(s,z)|\,dz \\ &\quad\lesssim \frac{1}{\sqrt{t-s}} \int_0^{1+\mu} e^{-|\xi|z} e^{\epsilon_0(z-y)_+|\xi|} e^{\epsilon_0(1+\mu-z)_+|\xi|} |N_\xi(s,z)|\,dz \\ &\quad\lesssim \frac{1}{\sqrt{t-s}} \int_0^{1+\mu} e^{\epsilon_0(1+\mu-z)_+|\xi|} |N_\xi(s,z)|\,dz , \end{aligned}$$

leading to

$$\|T_1(t-s,y) I_1\|_{X_\mu} \lesssim \frac{1}{\sqrt{t-s}} \|N(s)\|_{Y_\mu} \tag{4.30}$$

upon summing in ξ. For the integral I_2 in (4.29), we similarly use (4.13) and obtain the inequality

$$\begin{aligned} |e^{\epsilon_0(1+\mu-y)_+|\xi|} w(y) T_1(t-s,y) I_2| &\lesssim \frac{1}{\sqrt{t-s}} \int_{1+\mu}^\infty |N_\xi(s,z)|\,dz \\ &\lesssim \frac{1}{\sqrt{t-s}} \|z N_\xi(s,z)\|_{L^2(z \geq 1+\mu)} , \end{aligned}$$

implying

$$\|T_1(t-s,y)I_2\|_{X_\mu} \lesssim \frac{1}{\sqrt{t-s}}\|N(s)\|_{S_\mu}\,. \tag{4.31}$$

For the T_2 contribution, appealing to (4.23) and using $b\sqrt{\nu} = 1 + |\xi|\sqrt{\nu} \lesssim 1 + |\xi| w(z)$ for any $z \geq 0$, we have

$$\begin{aligned}
&|e^{\epsilon_0(1+\mu-y)_+|\xi|} w(y) b e^{-\frac{1}{2}\theta_0 b y} I_1| \\
&\lesssim e^{\epsilon_0(1+\mu-y)_+|\xi|} b\sqrt{\nu} \int_0^{1+\mu} e^{-|\xi|z} |N_\xi(s,z)|\, dz \\
&\lesssim \int_0^{1+\mu} e^{-|\xi|z} e^{\epsilon_0(z-y)_+|\xi|} e^{\epsilon_0(1+\mu-z)_+|\xi|} |N_\xi(s,z)|\, dz \\
&\quad + \int_0^{1+\mu} |\xi| e^{-|\xi|z} e^{\epsilon_0(z-y)_+|\xi|} e^{\epsilon_0(1+\mu-z)_+|\xi|} w(z) |N_\xi(s,z)|\, dz \\
&\lesssim \int_0^{1+\mu} e^{\epsilon_0(1+\mu-z)_+|\xi|} |N_\xi(s,z)|\, dz \\
&\quad + \int_0^{1+\mu} |\xi| e^{-\frac{1}{2}|\xi|z} e^{\epsilon_0(1+\mu-z)_+|\xi|} w(z) |N_\xi(s,z)|\, dz\,.
\end{aligned}$$

The first of the above terms is estimated using the Y_μ norm, while the second one is bounded using the X_μ norm. Here we use that $\left\| |\xi| e^{-\frac{1}{2}|\xi|z} \right\|_{L^1_z} \lesssim 1$. The above estimate, combined with the fact that the second term in the upper bound for T_2 (cf. (4.28)) is estimated just as T_1, leads to the bound

$$\|T_2(t-s,y)I_1\|_{X_\mu} \lesssim \frac{1}{\sqrt{t-s}}\|N(s)\|_{Y_\mu} + \|N(s)\|_{X_\mu}\,. \tag{4.32}$$

For the contribution from T_2 to the second integral in (4.29) we use (4.23) and the bound $\sqrt{\nu} b \lesssim 1 + |\xi|$, to obtain

$$\begin{aligned}
|e^{\epsilon_0(1+\mu-y)_+|\xi|} w(y) b e^{-\frac{1}{2}\theta_0 b y} I_2| &\lesssim e^{\epsilon_0(1+\mu-y)_+|\xi|} \sqrt{\nu} b e^{-|\xi|(1+\mu)} \int_{1+\mu}^{\infty} |N_\xi(s,z)|\, dz \\
&\lesssim \|z N_\xi(s)\|_{L^2(z\geq 1+\mu)}\,.
\end{aligned}$$

Again, since the second term in the upper bound for T_2 (cf. (4.28)) is estimated just as T_1 we obtain

$$\|T_2(t-s,y)I_2\|_{X_\mu} \lesssim \frac{1}{\sqrt{t-s}}\|yN(s)\|_{L^2(y\geq 1+\mu)}\,. \tag{4.33}$$

Combining (4.30), (4.31), (4.32), and (4.33) concludes the proof of (4.25) when $(i,j) = (0,0)$. For $(i,j) = (0,1)$, we use the fact that the conormal derivative of the kernel obeys similar estimates as the kernel itself, which holds in view of (4.27) and (4.28). Lastly, for $(i,j) = (1,0)$, the ∂_x derivative simply acts on the N_ξ term in (4.29), and the above proof applies. ☐

Finally, we estimate the first term in the mild representation of the solution (3.7).

LEMMA 4.5. *Let $\mu \in (0, \mu_0 - \gamma s)$ be arbitrary. For $i + j \leq 2$, the initial datum term in (3.7) satisfies*

$$\begin{aligned}
&\sum_{i+j\leq 2} \left\| \partial_x^i (y\partial_y)^j \int_0^\infty G(t,y,z)\omega_0(z)\, dz \right\|_{X_\mu} \\
&\quad \lesssim \sum_{i+j\leq 2} \|\partial_x^i (y\partial_y)^j \omega_0\|_{X_\mu} + \sum_{i+j\leq 2}\sum_{\xi} \|\xi^i \partial_y^j \omega_{0,\xi}\|_{L^\infty(y\geq 1+\mu)}\,.
\end{aligned} \tag{4.34}$$

PROOF OF LEMMA 4.5. Let $i+j \leq 2$. We recall from (3.8), (3.9), and (4.1) that

$$G_\xi(t,y,z) = H_\xi(t,y,z) + H_\xi(t,y,-z) + R_\xi(t,y,z)\,. \tag{4.35}$$

Accordingly, we divide

$$\begin{aligned}
&\left(\partial_x^i (y\partial_y)^j \int_0^\infty G(t,y,z)\omega_0(z)\,dz\right)_\xi \\
&= \int_0^\infty (\hat{\imath}\xi)^i (y\partial_y)^j H_\xi(t,y,z)\omega_{0,\xi}(z)\,dz + \int_0^\infty (\hat{\imath}\xi)^i (y\partial_y)^j H_\xi(t,y,-z)\omega_{0,\xi}(z)\,dz \\
&\quad + \int_0^\infty (\hat{\imath}\xi)^i (y\partial_y)^j R_\xi(t,y,z)\omega_{0,\xi}(z)\,dz \\
&= J_1 + J_2 + J_3\,.
\end{aligned} \tag{4.36}$$

We first treat the term J_1. Using that

$$(y\partial_y)^j = y^j \partial_y^j + \mathbf{1}_{\{j=2\}} y\partial_y$$

and $y \lesssim 1$, we have, similarly to (4.7),

$$\begin{aligned}
|J_1| &\lesssim \int_0^{3y/4} |(y\partial_y)^j H_\xi(t,y,z)||\xi|^i|\omega_{0,\xi}(z)|\,dz \\
&\quad + \int_{y/2}^{3y/4} \left(\left|\psi'\left(\frac{z}{y}\right)\right| + \left|\psi''\left(\frac{z}{y}\right)\right|\right) |H_\xi(t,y,z)||\xi|^i|\omega_{0,\xi}(z)|\,dz \\
&\quad + \int_{y/2}^{3y/4} \left|\psi'\left(\frac{z}{y}\right)\right| |H_\xi(t,y,z)||\partial_z\omega_{0,\xi}(z)|\,dz \\
&\quad + \int_{y/2}^{1+\mu} |H_\xi(t,y,z)||\partial_z^2\omega_{0,\xi}(z)|\,dz + \int_{1+\mu}^\infty |H_\xi(t,y,z)||\partial_z^2\omega_{0,\xi}(z)|\,dz \\
&\quad + \int_{y/2}^{1+\mu} |H_\xi(t,y,z)||\partial_z\omega_{0,\xi}(z)|\,dz + \int_{1+\mu}^\infty |H_\xi(t,y,z)||\partial_z\omega_{0,\xi}(z)|\,dz \\
&\quad + \mathbf{1}_{\{j\leq 1\}} \int_{y/2}^{1+\mu} |H_\xi(t,y,z)||\xi|^i|\partial_z^j\omega_{0,\xi}(z)|\,dz \\
&\quad + \mathbf{1}_{\{j\leq 1\}} \int_{1+\mu}^\infty |H_\xi(t,y,z)||\xi|^i|\partial_z^j\omega_{0,\xi}(z)|\,dz \\
&= J_{11} + J_{12} + J_{13} + J_{14} + J_{15} + J_{16} + J_{17} + J_{18} + J_{19}\,.
\end{aligned} \tag{4.37}$$

The terms J_{11}, J_{12}, and J_{13} are bounded in the same way as the term I_1 in (4.7) (see (4.11)–(4.15)), leading to the first term in (4.34). The terms J_{14}, J_{16}, and J_{18} are estimated in the same way as the term I_3 in (4.7) (see (4.16)–(4.19)), and are bounded by the first term in (4.34). It is only the Sobolev contributions J_{15}, J_{17}, and J_{19} which need to be treated differently than the term I_4 in (4.7), because here we do not wish to increase the value of μ to $\widetilde{\mu}$. These Sobolev terms are treated in the same way. For instance, for J_{15} we use (4.17) and obtain

$$\begin{aligned}
e^{\epsilon_0(1+\mu-y)_+ + |\xi|} w(y)|J_{15}| &\lesssim \int_{1+\mu}^\infty \frac{1}{\sqrt{\nu t}} e^{\epsilon_0(z-y)_+ + |\xi|} e^{-\frac{(y-z)^2}{4\nu t}} e^{-\frac{1}{2}\nu\xi^2 t} |\partial_z^2\omega_{0,\xi}(z)|\,dz \\
&\lesssim \int_{1+\mu}^\infty \frac{1}{\sqrt{\nu t}} e^{-\frac{(y-z)^2}{8\nu t}} |\partial_z^2\omega_{0,\xi}(z)|\,dz \lesssim \left\|\partial_z^2\omega_{0,\xi}(z)\right\|_{L^\infty(z\geq 1+\mu)}\,.
\end{aligned} \tag{4.38}$$

Thus, the terms J_{15}, J_{17}, and J_{19} contribute to the second term on the right side of (4.34).

The second kernel in (4.35) is treated the same as the first. Likewise, the third kernel in (4.35) is a function of $y+z$ and we may write the analog of the inequality (4.38) and the proof concludes similarly. $\square$

We conclude this section with the proof of Lemma 3.7.

PROOF OF LEMMA 3.7. Given $\mu \in (0, \mu_0 - \gamma t)$ and $s \in (0, t)$, recall that μ_1 and μ_2 are given by (3.12) and (3.13). By the analyticity recovery for the X norm, cf. Lemma A.3 below, the bound for the first term on the left side of (3.14) is a direct consequence of the bound for the second term. Indeed, we have

$$\sum_{i+j=2} \left\| \partial_x^i (y\partial_y)^j \int_0^\infty G(t-s,y,z) N(s,z)\, dz \right\|_{X_\mu}$$
$$\lesssim \frac{1}{\mu_0 - \mu - \gamma s} \sum_{i+j=1} \left\| \partial_x^i (y\partial_y)^j \int_0^\infty G(t-s,y,z) N(s,z)\, dz \right\|_{X_{\mu_1}} .$$

The bound for the second term on the left side of (3.14) follows from Lemma 4.1, Remark 4.2, and Lemma 4.3, applied with $\mu = \mu_1$ and $\widetilde{\mu} = \mu_2$.

Concerning the trace kernel, the estimate for the first term on the left side of inequality (3.15) is a consequence of the bound for the second term in (3.15), the analyticity recovery for the X norm in Lemma A.3, and the increase in analyticity domain from μ to μ_1. The bound for the second term on the left side of (3.15) is a consequence of Lemma 4.4 with $\mu = \mu_1$.

Lastly, the initial datum term is bounded as in Lemma 4.5, which concludes the proof of the lemma. ☐

5. Estimates for the Y analytic norm

LEMMA 5.1. *Let $\mu \in (0, \mu_0 - \gamma s)$ be arbitrary. For $(i,j) = (0,0), (1,0), (0,1)$, we have*

$$\left\| \partial_x^i (y\partial_y)^j \int_0^\infty H(t-s,y,z) N(s,z)\, dz \right\|_{Y_\mu} \lesssim \|\partial_x^i (y\partial_y)^j N(s)\|_{Y_\mu} + \|N(s)\|_{Y_\mu} + \|\partial_x^i \partial_y^j N(s)\|_{S_\mu} . \tag{5.1}$$

REMARK 5.2. Similarly to Remark 4.2, we emphasize that in the proof of Lemma 5.1 we only use several properties of the heat kernel $H_\xi(t,y,z)$. Examining the proof below, one may verify that these properties are: The kernel should be either a function of $y+z$ or $y-z$, and it should obey the estimates

$$\left\| \chi_{\{0\le y\le 1+\mu\}} \chi_{\{0\le z\le 3y/4\}} \big(|H_\xi(t,y,z)| + |y\partial_y H_\xi(t,y,z)| \big) \right\|_{L_y^1 L_z^\infty} \lesssim 1 \tag{5.2}$$

$$\left\| e^{\epsilon_0 (z-y)_+ |\xi|} H_\xi(t,y,z) \right\|_{L_y^1 L_z^\infty} \lesssim 1 . \tag{5.3}$$

It is direct to check that the kernel $\widetilde{H}_\xi(t,y,z) - H_\xi(t,y,z) = H_\xi(t,y,-z) = \frac{1}{\sqrt{\nu t}} e^{-\frac{(y+z)^2}{4\nu t}} e^{-\nu \xi^2 t}$ obeys these two properties. Therefore, the bounds stated in Lemma 5.1 hold with $H(t,y,z)$ replaced by the full kernel $\widetilde{H}(t,y,z)$.

PROOF OF LEMMA 5.1. Let $y \in \Omega_\mu$. For simplicity, we only work with $y \in \mathbb{R}$; an adjustment for the complex case is straight-forward and leads only to notational complications.

We start with the proof of (5.1) in the case $(i,j) = (0,1)$. Let ψ be the cut-off function from the proof of Lemma 4.1. The first conormal derivative is given as in (4.7) by

$$\begin{aligned}
&y\partial_y \int_0^\infty H_\xi(t-s,y,z) N_\xi(s,z)\, dzs \\
&= -y \int_0^{3y/4} \psi\left(\frac{z}{y}\right) \partial_z H_\xi(t-s,y,z) N_\xi(s,z)\, dz - \int_{y/2}^{3y/4} \psi'\left(\frac{z}{y}\right) H_\xi(t-s,y,z) N_\xi(s,z)\, dz \\
&\quad + y \int_{y/2}^{1+\mu} \left(1 - \psi\left(\frac{z}{y}\right)\right) H_\xi(t-s,y,z) \partial_z N_\xi(s,z)\, dz + y \int_{1+\mu}^\infty H_\xi(t-s,y,z) \partial_z N_\xi(s,z)\, dz \\
&= I_1 + I_2 + I_3 + I_4 .
\end{aligned} \tag{5.4}$$

Using the bounds (4.8), (4.9), (4.10), and (4.14), we obtain

$$e^{\epsilon_0(1+\mu-y)_+|\xi|}|I_1| \lesssim \int_0^{3y/4} \frac{1}{\sqrt{\nu(t-s)}} e^{-\frac{(y-z)^2}{8\nu(t-s)}} e^{-\nu\xi^2(t-s)} e^{\epsilon_0(1+\mu-z)_+|\xi|}|N_\xi(s,z)|\, dz\,. \tag{5.5}$$

Integrating in y, changing the order of integration, and using

$$\left\| \frac{1}{\sqrt{\nu(t-s)}} e^{-\frac{(y-z)^2}{8\nu(t-s)}} \right\|_{L_z^\infty L_y^1} \lesssim 1\,, \tag{5.6}$$

we arrive at

$$\begin{aligned}\left\| e^{\epsilon_0(1+\mu-y)_+|\xi|} I_1 \right\|_{\mathcal{L}^1_\mu} &\lesssim \int_0^{1+\mu}\int_0^{3y/4} \frac{1}{\sqrt{\nu(t-s)}} e^{-\frac{(y-z)^2}{16\nu(t-s)}} e^{\epsilon_0(1+\mu-z)_+|\xi|}|N_\xi(s,z)|\, dzdy \\ &\lesssim \left\| e^{\epsilon_0(1+\mu-z)_+|\xi|} N_\xi(s) \right\|_{\mathcal{L}^1_\mu}\,.\end{aligned}$$

Summing over ξ yields the bound

$$\|I_1\|_{Y_\mu} \le \|N(s)\|_{Y_\mu}\,. \tag{5.7}$$

The term I_2 in (5.4) is treated in the same way, by using $\|\psi'\|_{L^\infty} \lesssim 1$, leading to the same upper bound as in (5.7). For the term I_3 in (5.4), we use (4.17), the fact that ε_0 is small, and the bound (5.6), in order to conclude

$$\begin{aligned}\|I_3\|_{Y_\mu} &= \sum_\xi \left\| e^{\epsilon_0(1+\mu-y)_+|\xi|} I_3 \right\|_{\mathcal{L}^1_\mu} \\ &\lesssim \sum_\xi \left\| \int_{y/2}^{1+\mu} \frac{1}{\sqrt{\nu(t-s)}} e^{-\frac{(y-z)^2}{8\nu(t-s)}} e^{\epsilon_0(1+\mu-z)_+|\xi|} |z\partial_z N_\xi(s,z)|\, dz \right\|_{\mathcal{L}^1_\mu} \\ &\lesssim \sum_\xi \left\| e^{\epsilon_0(1+\mu-z)_+|\xi|} |z\partial_z N_\xi(s)| \right\|_{\mathcal{L}^1_\mu} = \|z\partial_z N_\xi(s)\|_{Y_\mu}\,.\end{aligned} \tag{5.8}$$

In order to estimate the term I_4 in (5.4) we appeal to (4.17), use that ϵ_0 is chosen sufficiently small, and the bound $y \le 1+\mu \le z$, to obtain

$$\begin{aligned} e^{\epsilon_0(1+\mu-y)_+|\xi|}|I_4| &\lesssim \int_{1+\mu}^\infty \frac{1}{\sqrt{\nu(t-s)}} e^{\epsilon_0(z-y)_+|\xi|} e^{-\frac{(y-z)^2}{4\nu(t-s)}} e^{-\frac12\nu\xi^2(t-s)} |\partial_z N_\xi(s,z)|\, dz \\ &\lesssim \int_{1+\mu}^\infty \frac{e^{-\frac{(y-z)^2}{8\nu(t-s)}}}{\sqrt{\nu(t-s)}} |\partial_z N_\xi(s,z)|\, dz\,.\end{aligned}$$

Upon integrating in y, using (5.6), and summing in ξ, the above estimate yields

$$\|I_4\|_{Y_\mu} \lesssim \sum_\xi \|\partial_z N_\xi(s)\|_{L^1(z\ge 1+\mu)} \lesssim \|\partial_z N(s)\|_{S_\mu}\,.$$

This concludes the proof of (5.1) with $(i,j)=(0,1)$.

The estimate (5.1) for $(i,j)=(1,0)$ follows from the bound (5.1) with $(i,j)=(0,0)$, by applying the estimate to $\partial_x N$ instead of N. In order to prove (5.1) for $(i,j)=(0,0)$, we decompose, as in (4.20),

$$\begin{aligned}
&\int_0^\infty H_\xi(t-s,y,z)N_\xi(s,z)\,dz\\
&\quad= \int_0^{3y/4}\psi\left(\frac{z}{y}\right)H_\xi(t-s,y,z)N_\xi(s,z)\,dz + \int_{y/2}^{1+\mu}\left(1-\psi\left(\frac{z}{y}\right)\right)H_\xi(t-s,y,z)N_\xi(s,z)\,dz\\
&\qquad+\int_{1+\mu}^\infty H_\xi(t-s,y,z)N_\xi(s,z)\,dz\\
&\quad= J_1+J_2+J_3\,.
\end{aligned}$$

Upon inspection of the proof for $(i,j)=(0,1)$, we see that using (5.6) we obtain

$$\|J_1\|_{Y_\mu}+\|J_2\|_{Y_\mu}\lesssim \|N(s)\|_{Y_\mu}\,.$$

On the other hand, the term J_3 is estimated exactly as the term I_4 above, and we obtain

$$\|J_3\|_{Y_\mu}\lesssim \|N(s)\|_{S_\mu}\,.$$

This concludes the proof of the lemma. □

Next, we state the inequalities involving the remainder kernel R_ξ.

LEMMA 5.3. *Let $\mu\in(0,\mu_0-\gamma s)$ be arbitrary. For $(i,j)\in\{(0,0),(1,0),(0,1)\}$ we have the estimate*

$$\left\|\partial_x^i(y\partial_y)^j\int_0^\infty R_\xi(t-s,y,z)N_\xi(s,z)\,dz\right\|_{Y_\mu}\lesssim \|\partial_x^i(y\partial_y)^jN(s)\|_{Y_\mu}+\|N(s)\|_{Y_\mu}+\|\partial_x^i\partial_y^jN(s)\|_{S_\mu}\,. \tag{5.9}$$

PROOF OF LEMMA 5.3. In order to establish (5.9), we only need to verify that the kernel $R_\xi(t,y,z)$ obeys the conditions stated in Remark 5.2. According to Remark 3.5, in order to obtain (5.2)–(5.3), we only need to prove that

$$\left\|\chi_{\{0\le y\le 1+\mu\}}\chi_{\{0\le z\le 3y/4\}}\left(be^{-\frac12\theta_0 b(y+z)}\right)\right\|_{L^1_yL^\infty_z}\lesssim 1 \tag{5.10}$$

$$\left\|e^{\epsilon_0(z-y)+|\xi|}be^{-\theta_0 b(y+z)}\right\|_{L^1_yL^\infty_z}\lesssim 1\,. \tag{5.11}$$

Indeed, the second term in the upper bound (3.11) for the residual kernel is treated in exactly the same way as $H_\xi(t,y,-z)$, but replacing $\frac14$ with $\frac{\theta_0}{2}$, and this term was addressed in Remark 5.2.

In order to establish (5.10), let $y\in[0,1+\mu]$ and $z\in[0,3y/4]$. Then

$$\|be^{-\frac12\theta_0 b(y+z)}\|_{L^1_y}\lesssim \|be^{-\frac12\theta_0 by}\|_{L^1_y}\lesssim 1\,,$$

and (5.10) follows. For (5.11), let $\epsilon_0\le\theta_0$, and observe that $e^{\epsilon_0(z-y)+|\xi|}be^{-\theta_0 b(z+y)}\lesssim be^{-\theta_0 by}$. The inequality (5.11) then follows upon integration in y. □

Next, we consider the Y norm estimate for the trace kernel contribution to (3.7).

LEMMA 5.4. *Let $\mu\in(0,\mu_0-\gamma s)$ be arbitrary. For $0\le i+j\le 1$, we have the inequality*

$$\left\|\partial_x^i(y\partial_y)^jG(t-s,y,0)\partial_z\Delta^{-1}N_\xi(s,z)|_{z=0}\right\|_{Y_\mu}\lesssim \|\partial_x^iN(s)\|_{Y_\mu}+\|\partial_x^iN\|_{S_\mu}\,. \tag{5.12}$$

PROOF OF LEMMA 5.4. First, we note that the case $(i,j)=(1,0)$ follows from the bound (5.12) with $(i,j)=(0,0)$, because the ∂_x derivative commutes with the operator $G(t-s,y,0)\partial_z\Delta^{-1}|_{z=0}$ (see also the formula (5.14) below). Second, we emphasize that the case $(i,j)=(0,1)$ is treated in the same way as the case $(i,j)=(0,0)$, because the conormal derivative $y\partial_y$ of $G(t-s,y,0)$ obeys the same bounds as $G(t-s,y,0)$ itself (see the bounds (4.27)–(4.28) above). Therefore, we only need to consider the case $(i,j)=(0,0)$.

As opposed to the proof of Lemma 4.4, we do not split the kernel $G_\xi(t-s,y,0)$ into two parts. The only property of the kernel which is used in this estimate is

$$\|G_\xi(t-s,y,0)\|_{L^1_y} \lesssim 1\,, \tag{5.13}$$

which follows directly from (4.26) and (4.28).

Using (4.29), we have

$$\partial_z \Delta^{-1} N_\xi(s,z)|_{z=0} = -\int_0^\infty e^{-|\xi|z} N_\xi(s,z)\,dz \tag{5.14}$$

and thus, since ϵ_0 may be taken sufficiently small, we obtain

$$\begin{aligned}
&\left| e^{\epsilon_0(1+\mu-y)+|\xi|} G_\xi(t-s,y,0)\partial_z\Delta^{-1}N_\xi(s,z)|_{z=0}\right| \\
&\quad\lesssim G_\xi(t-s,y,0)\int_0^\infty e^{-|\xi|z} e^{\epsilon_0(z-y)+|\xi|} e^{\epsilon_0(1+\mu-z)+|\xi|}|N_\xi(s,z)|\,dz \\
&\quad\lesssim G_\xi(t-s,y,0)\int_0^{1+\mu} e^{\epsilon_0(1+\mu-z)+|\xi|}|N_\xi(s,z)|\,dz + G_\xi(t-s,y,0)\int_{1+\mu}^\infty |N_\xi(s,z)|\,dz\,.
\end{aligned}$$

Using (5.13) and summing over ξ, we arrive at

$$\left\|G_\xi(t-s,y,0)\partial_z\Delta^{-1}N_\xi(s,z)|_{z=0}\right\|_{Y_\mu} \lesssim \|N(s)\|_{Y_\mu} + \|N(s)\|_{S_\mu}\,,$$

which concludes the proof of the lemma. □

Next, we provide an inequality corresponding to the initial datum.

LEMMA 5.5. *Let $\mu \in (0,\mu_0-\gamma t)$. For $i+j\le 2$, the initial datum term in (3.7) satisfies*

$$\begin{aligned}
&\sum_{i+j\le 2}\left\|\partial_x^i (y\partial_y)^j \int_0^\infty G(t,y,z)\omega_0(z)\,dz\right\|_{Y_\mu} \\
&\qquad\lesssim \sum_{i+j\le 2}\|\partial_x^i(y\partial_y)^j\omega_0\|_{Y_\mu} + \sum_{i+j\le 2}\sum_\xi \|\xi^i\partial_y^j\omega_{0,\xi}\|_{L^1(y\ge 1+\mu)}\,.
\end{aligned} \tag{5.15}$$

PROOF OF LEMMA 5.5. Let $i+j\le 2$. Then we have the decomposition of the kernel (4.35). We start with the first kernel in (4.35) and consider the inequality (4.37), where J_1 is as in (4.36).

Now, the terms J_{11}, J_{12}, and J_{13} are bounded the same as the term I_1 in (5.4) (see (5.5)–(5.7)), giving the first term in (5.15). The terms J_{14}, J_{16}, and J_{18} are estimated in the same way as the term I_3 in (5.4), cf. (5.8), and are bounded by the first term in (4.34). It remains to consider the Sobolev contributions J_{15}, J_{17}, and J_{19}.

For J_{15} we use (4.17) and write

$$\begin{aligned}
e^{\epsilon_0(1+\mu-y)+|\xi|}|J_{15}| &\lesssim \int_{1+\mu}^\infty \frac{1}{\sqrt{\nu(t-s)}} e^{\epsilon_0(z-y)+|\xi|} e^{-\frac{(y-z)^2}{4\nu(t-s)}} e^{-\frac12\nu\xi^2(t-s)}|\partial_z^2\omega_{0,\xi}(z)|\,dz \\
&\lesssim \int_{1+\mu}^\infty \frac{1}{\sqrt{\nu(t-s)}} e^{-\frac{(y-z)^2}{8\nu(t-s)}}|\partial_z^2\omega_{0,\xi}(z)|\,dz\,.
\end{aligned}$$

Upon integrating in y, using Fubini, the estimate (5.6), and summing in ξ, we obtain

$$\|J_{15}\|_{Y_\mu} \lesssim \sum_\xi \left\|\partial_z^2\omega_{0,\xi}\right\|_{L^1(z\ge 1+\mu)}\,.$$

With a similar treatment of J_{17} and J_{19} we obtain the second term on the right side of (5.15).

Since the other kernels in (4.35) are treated completely analogously, the proof is concluded. □

PROOF OF LEMMA 3.8. By increasing the analyticity domain from μ to μ_1, which is defined in (3.12), and using the analyticity recovery for the Y norm in Lemma A.4, we obtain

$$\sum_{i+j=2} \left\| \partial_x^i (y\partial_y)^j \int_0^\infty G(t-s,y,z)N(s,z)\,dz \right\|_{Y_\mu} \lesssim \frac{1}{\mu_0-\mu-\gamma s} \sum_{i+j=1} \left\| \partial_x^i (y\partial_y)^j \int_0^\infty G(t-s,y,z)N(s,z)\,dz \right\|_{Y_{\mu_1}} .$$

Therefore, the bound for the first term on the left of (3.16) is a direct consequence of the estimate for the second term in (3.16). The bound for the second term on the left side of (3.16) follows from Lemma 5.1, Remark 5.2, and Lemma 5.3, with μ replaced by $\mu_1 \in (0, \mu_0 - \gamma s)$.

Similarly, using analytic recovery for the Y norm and increasing the analytic domain from μ to μ_1, we see that the bound for the first term on the left side of (3.17) is a direct consequence of the estimate for the second term. For this later term, the estimate is established in Lemma 5.4, with μ replaced by μ_1. Lastly, the bound (3.18) is proven in Lemma 5.5, concluding the proof of the lemma. □

6. Estimates for the nonlinearity

In this section we provide estimates for the nonlinear term

$$N_\xi = (u\cdot\nabla\omega)_\xi = (u_1\partial_x\omega)_\xi + \left(\frac{u_2}{y} y\partial_y\omega\right)_\xi \tag{6.1}$$

and its $\partial_x^i(y\partial_y)^j$ derivatives, with $i+j \le 1$, in the X_μ, Y_μ, and S_μ norms. We first recall a representation formula of the velocity field in terms of the vorticity.

LEMMA 6.1 (Lemma 2.4 in [**44**]). *The velocity for the system* (3.4)–(3.5) *is given by*

$$u_{1,\xi}(y) = \frac{1}{2}\left(-\int_0^y e^{-|\xi|(y-z)}(1-e^{-2|\xi|z})\omega_\xi(z)\,dz + \int_y^\infty e^{-|\xi|(z-y)}(1+e^{-2|\xi|y})\omega_\xi(z)\,dz \right) \tag{6.2}$$

and

$$u_{2,\xi}(y) = \frac{-\hat{\imath}\xi}{2|\xi|}\left(\int_0^y e^{-|\xi|(y-z)}(1-e^{-2|\xi|z})\omega_\xi(z)\,dz + \int_y^\infty e^{-|\xi|(z-y)}(1-e^{-2|\xi|y})\omega_\xi(z)\,dz \right), \tag{6.3}$$

where $\hat{\imath}$ is the imaginary unit.

As in Remark 3.6 above, the Biot-Savart law of Lemma 6.1 also holds for y in the complex domain $\Omega_\mu \cup [1+\mu,\infty)$. If $y \in \partial\Omega_\theta$ for some $\theta \in [0,\mu)$, and say $\mathbb{I}\mathrm{m}\, y \ge 0$, then the integration from 0 to y in (6.2)–(6.3) is an integration over the complex line $\partial\Omega_\theta \cap \{z\colon \mathbb{I}\mathrm{m}\, z \ge 0, \mathbb{R}\mathrm{e}\, z \le \mathbb{R}\mathrm{e}\, y\}$, while the integration from y to ∞ is an integration over $(\partial\Omega_\theta \cap \{z\colon \mathbb{I}\mathrm{m}\, z \ge 0, \mathbb{R}\mathrm{e}\, y \le \mathbb{R}\mathrm{e}\, z \le 1+\theta\}) \cup [1+\theta,\infty)$.

Moreover, we emphasize here that while (6.3) immediately implies the boundary condition $u_{2,\xi}(0) = 0$, from (6.2) it just follows that $u_{1,\xi}(0) = \int_0^\infty e^{-|\xi|z}\omega_\xi(z)dz$. To see that this integral vanishes, one has to use that it vanishes at time $t=0$, and that its time derivative is given using the vorticity boundary condition (3.6) as $\partial_t u_{1,\xi}(0) = (-\partial_y \Delta_\xi^{-1}(u\cdot\nabla\omega)_\xi)|_{y=0} - \int_0^\infty e^{-|\xi|z}(u\cdot\nabla\omega)_\xi(z)dz = 0$. In the last equality we have used explicitly that the kernel of the operator $(-\partial_y\Delta_\xi^{-1})|_{y=0}$ is given by $e^{-|\xi|z}$. Thus, (3.6) ensures that $u_{1,\xi}(0) = 0$ is maintained by the evolution.

The main estimate concerning the X_μ norm is the following.

LEMMA 6.2. *Let $\mu \in (0, \mu_0-\gamma s)$ be arbitrary. We have the inequalities*

$$\|N(s)\|_{X_\mu} \lesssim \sum_{i\le 1} \left(\|\partial_x^i\omega\|_{Y_\mu} + \|\partial_x^i\omega\|_{S_\mu} \right) \sum_{i+j=1} \|\partial_x^i(y\partial_y)^j\omega\|_{X_\mu} \tag{6.4}$$

and

$$\sum_{i+j=1}\|\partial_x^i(y\partial_y)^jN(s)\|_{X_\mu}\lesssim\left(\|\omega\|_{X_\mu}+\sum_{1\le i\le 2}\left(\|\partial_x^i\omega\|_{Y_\mu}+\|\partial_x^i\omega\|_{S_\mu}\right)\right)\sum_{i+j=1}\|\partial_x^i(y\partial_y)^j\omega\|_{X_\mu}$$
$$+\sum_{i\le 1}\left(\|\partial_x^i\omega\|_{Y_\mu}+\|\partial_x^i\omega\|_{S_\mu}\right)\sum_{i+j=2}\|\partial_x^i(y\partial_y)^j\omega\|_{X_\mu}\,. \tag{6.5}$$

Before the proof of Lemma 6.2, we analyze the first order derivatives of the nonlinear term. By the Leibniz rule, for $i+j=1$, we have

$$\partial_x^i(y\partial_y)^jN_\xi=(\partial_x^i(y\partial_y)^ju_1\partial_x\omega)_\xi+\left((y\partial_y)^j\left(\frac{\partial_x^iu_2}{y}\right)y\partial_y\omega\right)_\xi$$
$$+(u_1\partial_x^{i+1}(y\partial_y)^j\omega)_\xi+\left(\frac{u_2}{y}\partial_x^i(y\partial_y)^{j+1}\omega\right)_\xi\,.$$

Using the triangle inequality we have $e^{\epsilon_0(1+\mu-y)_+|\xi|}\le e^{\epsilon_0(1+\mu-y)_+|\eta|}e^{\epsilon_0(1+\mu-y)_+|\xi-\eta|}$, and thus, by the definition of the X_μ norm and Young's inequality in ξ and η, it follows that

$$\|\partial_x^i(y\partial_y)^jN(s)\|_{X_\mu}\lesssim\|\partial_x\omega\|_{X_\mu}\sum_\xi\sup_{y\in\Omega_\mu}e^{\epsilon_0(1+\mu-y)_+|\xi|}|(\partial_x^i(y\partial_y)^ju_1)_\xi|$$
$$+\|y\partial_y\omega\|_{X_\mu}\sum_\xi\sup_{y\in\Omega_\mu}e^{\epsilon_0(1+\mu-y)_+|\xi|}\left|\left((y\partial_y)^j\left(\frac{\partial_x^iu_2}{y}\right)\right)_\xi\right|$$
$$+\|\partial_x^{i+1}(y\partial_y)^j\omega\|_{X_\mu}\sum_\xi\sup_{y\in\Omega_\mu}e^{\epsilon_0(1+\mu-y)_+|\xi|}|(u_1)_\xi|$$
$$+\|\partial_x^i(y\partial_y)^{j+1}\omega\|_{X_\mu}\sum_\xi\sup_{y\in\Omega_\mu}e^{\epsilon_0(1+\mu-y)_+|\xi|}\left|\left(\frac{u_2}{y}\right)_\xi\right|\,. \tag{6.6}$$

Thus, in order to prove (6.5), we only need to estimate the above norms of the velocity terms. These inequalities are collected in the next lemma.

LEMMA 6.3. *Let $\mu\in(0,\mu_0-\gamma s)$ be arbitrary and let $0\le i+j\le 1$. For the velocity u_1 and its derivatives, we have*

$$\sum_\xi\sup_{y\in\Omega_\mu}e^{\epsilon_0(1+\mu-y)_+|\xi|}|(\partial_x^i(y\partial_y)^ju_1)_\xi|\lesssim\|\partial_x^{i+j}\omega\|_{Y_\mu}+\|\partial_x^{i+j}\omega\|_{S_\mu}+j\,\|\omega\|_{X_\mu}\,, \tag{6.7}$$

while for the second velocity component u_2 the bound

$$\sum_\xi\sup_{y\in\Omega_\mu}e^{\epsilon_0(1+\mu-y)_+|\xi|}\left|\left((y\partial_y)^j\left(\frac{\partial_x^iu_2}{y}\right)\right)_\xi\right|\lesssim\|\partial_x^{i+1}\omega\|_{Y_\mu}+\|\partial_x^{i+1}\omega\|_{S_\mu} \tag{6.8}$$

holds.

PROOF OF LEMMA 6.3. First we prove (6.7), starting with the case $(i,j)=(0,0)$. We decompose the integral (6.2) for u_1 as

$$u_{1,\xi}(y)=\frac{1}{2}\Bigg(-\int_0^y e^{-|\xi|(y-z)}(1-e^{-2|\xi|z})\omega_\xi(s,z)\,dz$$
$$+\left(\int_y^{1+\mu}+\int_{1+\mu}^\infty\right)e^{-|\xi|(z-y)}(1+e^{-2|\xi|y})\omega_\xi(s,z)\,dz\Bigg)$$
$$=I_1+I_2+I_3\,.$$

Note that we have

$$e^{\epsilon_0(1+\mu-y)+|\xi|}e^{-|y-z||\xi|} \leq e^{\epsilon_0(1+\mu-z)+|\xi|}e^{\epsilon_0(z-y)+|\xi|}e^{-|y-z||\xi|} \leq e^{\epsilon_0(1+\mu-z)+|\xi|} \tag{6.9}$$

provided $\epsilon_0 \leq 1$. Hence, we obtain

$$e^{\epsilon_0(1+\mu-y)+|\xi|}(|I_1|+|I_2|) \lesssim \int_0^{1+\mu} e^{\epsilon_0(1+\mu-z)+|\xi|}|\omega_\xi(s,z)|\,dz \lesssim \|e^{\epsilon_0(1+\mu-y)+|\xi|}\omega\|_{\mathcal{L}^1_\mu}\,. \tag{6.10}$$

For the term I_3, using (6.9) we have

$$e^{\epsilon_0(1+\mu-y)+|\xi|}|I_3| \lesssim \int_{1+\mu}^\infty |\omega_\xi(s,z)|\,dz \lesssim \|z\omega_\xi\|_{L^2(z\geq 1+\mu)}\,. \tag{6.11}$$

Summing the bounds (6.10) and (6.11) in ξ, we conclude the proof of (6.7) when $i+j=0$.

The case $(i,j)=(1,0)$ amounts to multiplying by $\hat{\imath}\xi$, and thus the assertion follows by the same proof as for $(i,j)=(0,0)$. Consider now the case $(i,j)=(0,1)$. Taking the conormal derivative of (6.2) gives

$$\begin{aligned} y\partial_y u_{1,\xi} =& \frac{y}{2}\Bigg(\int_0^y e^{-|\xi|(y-z)}(1-e^{-2|\xi|z})|\xi|\omega_\xi(s,z)\,dz \\ &+\int_y^\infty e^{-|\xi|(z-y)}(1+e^{-2|\xi|y})|\xi|\omega_\xi(s,z)\,dz \\ &-2\int_y^\infty e^{-|\xi|(z-y)}e^{-2|\xi|y}|\xi|\omega_\xi(s,z)\,dz\Bigg) - y\omega_\xi(y)\,. \end{aligned} \tag{6.12}$$

The first three terms in (6.12) are treated as in the case $i+j=0$. The presence of the additional factor $|\xi|$ causes ω to be replaced by $\partial_x\omega$ in the upper bounds. For the last term in (6.12), we have

$$\sum_\xi \sup_{y\in\Omega_\mu} e^{\epsilon_0(1+\mu-y)+|\xi|}y|\omega_\xi(y)| \lesssim \sum_\xi \sup_{y\in\Omega_\mu} e^{\epsilon_0(1+\mu-y)+|\xi|}w(y)|\omega_\xi(y)| \lesssim \|\omega\|_{X_\mu}\,,$$

where we have used Remark 2.1(d). This concludes the proof of (6.7) for $(i,j)=(0,1)$.

Next, we prove (6.8), beginning with the case $(i,j)=(0,0)$. Using (6.3) we decompose

$$\begin{aligned} \frac{u_{2,\xi}}{y} =& -\frac{\xi}{|\xi|}\frac{\hat{\imath}}{2y}\Bigg(\int_0^y e^{-|\xi|(y-z)}(1-e^{-2|\xi|z})\omega_\xi(s,z)\,dz \\ &+\left(\int_y^{1+\mu}+\int_{1+\mu}^\infty\right)e^{-|\xi|(z-y)}(1-e^{-2|\xi|y})\omega_\xi(s,z)\,dz\Bigg) \\ =& J_1+J_2+J_3\,. \end{aligned}$$

Using the bound

$$\left|\frac{1-e^{-2|\xi|z}}{y}\right| \lesssim |\xi|, \qquad z\leq y\,,$$

we arrive at

$$\left|\frac{u_{2,\xi}}{y}\right| \lesssim \int_0^y e^{-|\xi|(y-z)}|\xi||\omega_\xi(s,z)|\,dz + \left(\int_y^{1+\mu}+\int_{1+\mu}^\infty\right)e^{-|\xi|(z-y)}|\xi||\omega_\xi(s,z)|\,dz\,. \tag{6.13}$$

Using (6.9) and the same bounds as in (6.10)–(6.11), we obtain the inequality (6.8) for $i+j=0$. The case $(i,j)=(1,0)$ follows from the same argument, by adding an extra x derivative.

It remains to consider the case $(i,j)=(0,1)$. From the incompressibility we have

$$y\partial_y\left(\frac{u_{2,\xi}}{y}\right) = \partial_y u_{2,\xi} - \frac{u_{2,\xi}}{y} = -\hat{\imath}\xi u_{1,\xi} - \frac{u_{2,\xi}}{y}\,. \tag{6.14}$$

The bound for the second term on the right of (6.14) was established in (6.13), whereas the bound for the first term follows by setting $(i,j)=(1,0)$ in (6.7). $\square$

Having established Lemma 6.3, we return to the proofs of (6.4) and (6.5).

PROOF OF LEMMA 6.2. In order to prove (6.4), we use (6.1) and similarly to (6.6) we obtain

$$\|N(s)\|_{X_\mu} \lesssim \|\partial_x\omega\|_{X_\mu} \sum_\xi \sup_{y\in\Omega_\mu} e^{\epsilon_0(1+\mu-y)_+|\xi|}|(u_1)_\xi| + \|y\partial_y\omega\|_{X_\mu} \sum_\xi \sup_{y\in\Omega_\mu} e^{\epsilon_0(1+\mu-y)_+|\xi|}\left|\left(\frac{u_2}{y}\right)_\xi\right| . \tag{6.15}$$

Using Lemma 6.3 with $i+j=0$ we get

$$\|N(s)\|_{X_\mu} \lesssim \big(\|\omega\|_{Y_\mu}+\|\omega\|_{S_\mu}\big)\|\partial_x\omega\|_{X_\mu} + \big(\|\partial_x\omega\|_{Y_\mu}+\|\partial_x\omega\|_{S_\mu}\big)\|y\partial_y\omega\|_{X_\mu} ,$$

and (6.4) is established.

For (6.5), we use the bounds of Lemma 6.3 in (6.6) to obtain

$$\begin{aligned}\sum_{i+j=1}\|\partial_x^i(y\partial_y)^jN(s)\|_{X_\mu} &\lesssim \|\partial_x\omega\|_{X_\mu}\big(\|\partial_x\omega\|_{Y_\mu}+\|\partial_x\omega\|_{S_\mu}+\|\omega\|_{X_\mu}\big)\\ &\quad + \|y\partial_y\omega\|_{X_\mu}\Big(\sum_{i\le1}\|\partial_x^{i+1}\omega\|_{Y_\mu}+\|\partial_x^{i+1}\omega\|_{S_\mu}\Big)\\ &\quad + \Big(\sum_{i+j=1}\|\partial_x^{i+1}(y\partial_y)^j\omega\|_{X_\mu}\Big)\big(\|\omega\|_{Y_\mu}+\|\omega\|_{S_\mu}\big)\\ &\quad + \Big(\sum_{i+j=1}\|\partial_x^{i}(y\partial_y)^{j+1}\omega\|_{X_\mu}\Big)\big(\|\partial_x\omega\|_{Y_\mu}+\|\partial_x\omega\|_{S_\mu}\big) ,\end{aligned}$$

and (6.5) is proven. □

Next, we estimate the term $\partial_x^i(y\partial_y)^jN(s)$ for $0\le i+j\le 1$ in the Y norm.

LEMMA 6.4. *Let $\mu\in(0,\mu_0-\gamma s)$ be arbitrary. For the nonlinear term, we have the inequalities*

$$\|N(s)\|_{Y_\mu} \lesssim \sum_{i\le1}\big(\|\partial_x^i\omega\|_{Y_\mu}+\|\partial_x^i\omega\|_{S_\mu}\big)\sum_{i+j=1}\|\partial_x^i(y\partial_y)^j\omega\|_{Y_\mu} \tag{6.16}$$

and

$$\sum_{i+j=1}\|\partial_x^i(y\partial_y)^jN(s)\|_{Y_\mu} \lesssim \Big(\|\omega\|_{X_\mu}+\sum_{1\le i\le2}\big(\|\partial_x^i\omega\|_{Y_\mu}+\|\partial_x^i\omega\|_{S_\mu}\big)\Big)\sum_{i+j=1}\|\partial_x^i(y\partial_y)^j\omega\|_{Y_\mu} + \sum_{i\le1}\big(\|\partial_x^i\omega\|_{Y_\mu}+\|\partial_x^i\omega\|_{S_\mu}\big)\sum_{i+j=2}\|\partial_x^i(y\partial_y)^j\omega\|_{Y_\mu} . \tag{6.17}$$

PROOF OF LEMMA 6.4. By writing the nonlinear term as in (6.1), and using the definition of the Y_μ norm, we obtain, similarly to (6.15),

$$\|N(s)\|_{Y_\mu} \lesssim \|\partial_x\omega\|_{Y_\mu}\sum_\xi\sup_{y\in\Omega_\mu}e^{\epsilon_0(1+\mu-y)_+|\xi|}|(u_1)_\xi| + \|y\partial_y\omega\|_{Y_\mu}\sum_\xi\sup_{y\in\Omega_\mu}e^{\epsilon_0(1+\mu-y)_+|\xi|}\left|\left(\frac{u_2}{y}\right)_\xi\right| .$$

Using the bounds in Lemma 6.3 with $i+j=0$, we arrive at (6.16).

For $i+j=1$, by the definition of Y_μ norm and Young's inequality, we have as in (6.6)

$$\begin{aligned}\|\partial_x^i (y\partial_y)^j N(s)\|_{Y_\mu} \lesssim \|\partial_x \omega\|_{Y_\mu} \sum_\xi \sup_{y\in\Omega_\mu} e^{\epsilon_0(1+\mu-y)_+|\xi|} |(\partial_x^i (y\partial_y)^j u_1)_\xi| \\ + \|y\partial_y\omega\|_{Y_\mu} \sum_\xi \sup_{y\in\Omega_\mu} e^{\epsilon_0(1+\mu-y)_+|\xi|} \left| \left((y\partial_y)^j \left(\frac{\partial_x^i u_2}{y}\right)\right)_\xi \right| \\ + \|\partial_x^{i+1}(y\partial_y)^j\omega\|_{Y_\mu} \sum_\xi \sup_{y\in\Omega_\mu} e^{\epsilon_0(1+\mu-y)_+|\xi|} |(u_1)_\xi| \\ + \|\partial_x^i (y\partial_y)^{j+1}\omega\|_{Y_\mu} \sum_\xi \sup_{y\in\Omega_\mu} e^{\epsilon_0(1+\mu-y)_+|\xi|} \left|\left(\frac{u_2}{y}\right)_\xi\right| .\end{aligned}$$

The proof of (6.17) is then concluded by an application of Lemma 6.3. □

To conclude this section we consider the Sobolev norm estimates for the nonlinear term.

LEMMA 6.5. *Let $\mu \in (0, \mu_0 - \gamma s)$ be arbitrary. We have*

$$\|N(s)\|_{S_\mu} \lesssim \Big(\|\omega\|_{Y_\mu} + \|\omega\|_{S_\mu}\Big) \sum_{i+j=1} \|\partial_x^i \partial_y^j \omega\|_{S_\mu} \tag{6.18}$$

and

$$\begin{aligned}\sum_{i+j=1} \|\partial_x^i\partial_y^j N(s)\|_{S_\mu} \lesssim \sum_{i+j\le 1} \left(\|\partial_x^i\partial_y^j\omega\|_{Y_\mu} + \|\partial_x^i\partial_y^j\omega\|_{S_\mu}\right) \sum_{i+j\le 1} \|\partial_x^i\partial_y^j\omega\|_{S_\mu} \\ + \left(\|\omega\|_{Y_\mu} + \|\omega\|_{S_\mu}\right) \sum_{i+j=2} \|\partial_x^i\partial_y^j\omega\|_{S_\mu} .\end{aligned} \tag{6.19}$$

PROOF OF LEMMA 6.5. In order to prove (6.18) we write

$$y(u\cdot\nabla\omega) = u_1 y\partial_x\omega + u_2 y\partial_y \omega$$

and thus from Hölder's inequality in y and Young's inequality in ξ we deduce

$$\sum_\xi \left(\|u_{1,\xi}\|_{L^\infty(y\ge 1+\mu)} + \|u_{2,\xi}\|_{L^\infty(y\ge 1+\mu)}\right) \lesssim \sum_\xi \int_0^\infty |\omega_\xi(z)|\, dz \lesssim \|\omega\|_{Y_\mu} + \|\omega\|_{S_\mu} . \tag{6.20}$$

For (6.19), when $i+j=1$, by the Leibniz rule we have

$$y\partial_x^i\partial_y^j(u\cdot\nabla\omega) = \partial_x^i\partial_y^j u_1 y\partial_x\omega + u_1 y\partial_x^{i+1}\partial_y^j\omega + \partial_x^i\partial_y^j u_2 y\partial_y\omega + u_2 y \partial_x^i\partial_y^{j+1}\omega$$

and therefore from Hölder's inequality in y and Young's inequality in ξ we deduce

$$\begin{aligned}\left\|\partial_x^i\partial_y^j N(s)\right\|_{S_\mu} \lesssim \|\partial_x\omega\|_{S_\mu} \sum_\xi \left\||\xi|^i\partial_y^j u_{1,\xi}\right\|_{L^\infty(y\ge 1+\mu)} + \|\partial_y\omega\|_{S_\mu} \sum_\xi \left\||\xi|^i \partial_y^j u_{2,\xi}\right\|_{L^\infty(y\ge1+\mu)} \\ + \left\|\partial_x^{i+1}\partial_y^j\omega\right\|_{S_\mu} \sum_\xi \|u_{1,\xi}\|_{L^\infty(y\ge 1+\mu)} + \left\|\partial_x^i\partial_y^{j+1}\omega\right\|_{S_\mu} \sum_\xi \|u_{2,\xi}\|_{L^\infty(y\ge1+\mu)} .\end{aligned}$$

For the last two terms in the above inequality we appeal to (6.20). For the first two terms, when $(i,j)=(1,0)$ the L^∞ bound on the velocity field is again given by (6.20) with an additional derivative in x, i.e.,

$$\sum_\xi \|(\partial_x u)_{1,\xi}\|_{L^\infty(y\ge1+\mu)} + \|(\partial_x u)_{2,\xi}\|_{L^\infty(y\ge 1+\mu)} \lesssim \|\partial_x\omega\|_{Y_\mu} + \|\partial_x\omega\|_{S_\mu} . \tag{6.21}$$

On the other hand, for $(i,j)=(0,1)$, we use incompressibility and the definition of ω to write

$$\partial_y u_1 = -\omega + \partial_x u_2 \qquad \text{and} \qquad \partial_y u_2 = -\partial_x u_1 . \tag{6.22}$$

For the L^∞ bound on $\partial_x u$ we again appeal to (6.21) whereas for the L^∞ norm of ω we use the fundamental theorem of calculus and Hölder's inequality to estimate

$$\sum_\xi \|\omega_\xi\|_{L^\infty(y\geq 1+\mu)} \lesssim \sum_\xi \|y\partial_y\omega_\xi\|_{L^2(y\geq 1+\mu)} = \|\partial_y\omega\|_{S_\mu} . \tag{6.23}$$

The bound (6.19) now follows by combining all the estimates. □

7. The Sobolev norm estimate

In this section, we provide an estimate on the Sobolev part of the norm

$$\sum_{i+j\leq 3} \left\|\partial_x^i\partial_y^j\omega\right\|_S = \sum_{i+j\leq 3} \left\|y\partial_x^i\partial_y^j\omega\right\|_{L^2_{x,y}(y\geq 1/2)} = \sum_{i+j\leq 3}\left(\sum_\xi \|y\xi^i\partial_y^j\omega_\xi\|^2_{L^2(y\geq 1/2)}\right)^{1/2} . \tag{7.1}$$

For a given norm $\|\cdot\|$ it is convenient to introduce the notation

$$\|D^k u\| = \sum_{i+j=k}\|\partial_x^i\partial_y^j u\| .$$

We first state a lemma which estimates u in terms of ω.

LEMMA 7.1. *Let t be such that $\gamma t\leq \mu_0/2$. Then we have*

$$\sum_{0\leq k\leq 2}\|D^k u(t)\|_{L^\infty_{x,y}(y\geq 1/4)} \lesssim \sum_{i+j\leq 2}\sum_\xi \||\xi|^i\partial_y^j u_\xi(t)\|_{L^\infty(y\geq 1/4)} \lesssim |\!|\!|\omega|\!|\!|_t \tag{7.2}$$

and

$$\left\|D^3 u(t)\right\|_{L^2_{x,y}(y\geq 1/4)} \lesssim |\!|\!|\omega|\!|\!|_t . \tag{7.3}$$

PROOF OF LEMMA 7.1. The first inequality in (7.2), in which the L^∞ norm in x is replaced by an ℓ^1 norm in the ξ variable is merely the Hausdorff-Young inequality. It thus remains to establish the second inequality in (7.2). The case $j=0$ follows by the same argument as (6.20). Indeed, we only replace the norm $L^\infty(y\geq 1+\mu)$ with the norm $L^\infty(y\geq 1/4)$, which has no bearing on the estimates, to obtain

$$\sum_\xi \|(\partial_x^i u)_{1,\xi}\|_{L^\infty(y\geq 1/4)} + \|(\partial_x^i u)_{2,\xi}\|_{L^\infty(y\geq 1/4)} \lesssim \left\|\partial_x^i\omega\right\|_{Y_{\mu/2}} + \left\|\partial_x^i\omega\right\|_{S_{\mu/2}} \tag{7.4}$$

for any $i\leq 2$ and $\mu>0$. In particular, we may take

$$\mu = \frac{\mu_0-\gamma t}{10} . \tag{7.5}$$

Note that since $\gamma t\leq \mu_0/2$ we have $\mu\geq \mu_0/20$. To replace the $S_{\mu/2}$ norm, which is ℓ^1 in ξ, with the S norm, which is ℓ^2 in ξ, we pay an additional price of $1+|\xi|$ (cf. Lemma A.1 below). Additionally, in (7.4) we further appeal to the analyticity recovery for the Y norm, cf. Lemma A.4 below, and obtain

$$\sum_{i\leq 2}\sum_\xi \|(\partial_x^i u)_{1,\xi}\|_{L^\infty(y\geq 1/4)} + \|(\partial_x^i u)_{2,\xi}\|_{L^\infty(y\geq 1/4)} \lesssim \|\omega\|_{Y_\mu} + \sum_{i\leq 3}\left\|\partial_x^i\omega\right\|_S \lesssim |\!|\!|\omega|\!|\!|_t .$$

This concludes the proof of (7.2) when $j=0$ and $i\leq 2$. For the case $j=1$, we use (6.22) to convert the ∂_y derivative into a ∂_x derivative, at a cost of an additional term involving ω. Similarly to (6.23), appealing to Lemma A.1, using that $w(y)\gtrsim 1$ for $y\in[1/4,1/2]$, and with μ as in (7.5) we get

$$\begin{aligned}\sum_\xi \||\xi|^i\omega_\xi\|_{L^\infty(y\geq 1/4)} &\lesssim \sum_\xi \|w(y)e^{\epsilon_0(1+\mu-y)_+|\xi|}|\xi|^i\omega_\xi\|_{L^\infty(1/4\leq y\leq 1/2)} + \sum_\xi \|y\partial_y|\xi|^i\omega_\xi\|_{L^2(y\geq 1/2)} \\ &\lesssim \left\|\partial_x^i\omega\right\|_{X_\mu} + \left\|\partial_x^i\partial_y\omega\right\|_S + \left\|\partial_x^{i+1}\partial_y\omega\right\|_S \lesssim |\!|\!|\omega|\!|\!|_t\end{aligned} \tag{7.6}$$

for $i \leq 1$. The above estimate gives (7.2) for $j = 1$. It only remains to consider the case $(i,j) = (0,2)$. For this purpose, note that

$$\partial_y^2 u_1 = -\partial_y \omega - \partial_x^2 u_1 \qquad \text{and} \qquad \partial_y^2 u_2 = \partial_x \omega - \partial_x^2 u_2 \,, \tag{7.7}$$

which follows from (6.22) and incompressibility. The terms with two x derivatives were already estimated in (7.4), whereas $\partial_x \omega$ was already bounded in (7.6). Lastly, for the term $\partial_y \omega$, we have, similarly to (7.6),

$$\begin{aligned}\sum_\xi \|\partial_y \omega_\xi\|_{L^\infty(y\geq 1/4)} &\lesssim \sum_\xi \|e^{\epsilon_0(1+\mu-y)_+|\xi|} w(y) y \partial_y \omega_\xi\|_{L^\infty(1/4\leq y\leq 1/2)} + \sum_\xi \|y\partial_y^2 \omega_\xi\|_{L^2(y\geq 1/2)} \\ &\lesssim \|y\partial_y \omega\|_{X_\mu} + \left\|\partial_y^2 \omega\right\|_S + \left\|\partial_x \partial_y^2 \omega\right\|_S \lesssim \|\omega\|_t \,,\end{aligned}$$

which gives (7.2).

In order to conclude the proof of the lemma, we need to establish (7.3). For this purpose, fix (i,j) such that $i + j = 3$. To avoid redundancy, we only consider the cases $(i,j) = (3,0)$ and $(i,j) = (0,3)$. First, using Lemma 6.1 and Young's inequality, we have

$$\begin{aligned}\|\partial_x^3 u\|_{L^2(y\geq 1/4)}^2 &\lesssim \sum_\xi \||\xi|^3 u_\xi\|_{L^2(y\geq 1/4)}^2 \\ &\lesssim \sum_\xi \left\| \int_0^\infty e^{-|y-z||\xi|} |\xi|^3 |\omega_\xi|(z)\, dz \right\|_{L^2(y\geq 1/4)}^2 \\ &\lesssim \sum_\xi \||\xi|^{5/2} |\omega_\xi|\|_{L^1(z\leq 1/2)}^2 + \sum_\xi \||\xi|^2 |\omega_\xi|\|_{L^2(z\geq 1/2)}^2 \\ &\lesssim \sum_\xi \|e^{\epsilon_0(1+\mu-z)_+|\xi|} |\omega_\xi|\|_{L^1(z\leq 1/2)}^2 |\xi|^{5/2} e^{-\frac{\epsilon_0}{2}|\xi|} + \sum_\xi \|z|\xi|^2 |\omega_\xi|\|_{L^2(z\geq 1/2)}^2 \\ &\lesssim \|\omega\|_{Y_\mu}^2 + \left\|\partial_x^2 \omega\right\|_S^2 \lesssim \|\omega\|_t^2 \,,\end{aligned}$$

with μ as in (7.5). In the last inequality above we used $\|\cdot\|_{\ell^2} \leq \|\cdot\|_{\ell^1}$. Thus, we have proven (7.3) for $(i,j) = (3,0)$. For the other extremal case, we apply the y derivative to (7.7) and obtain

$$\partial_y^3 u_1 = \partial_y^2 \omega + \partial_x^2 \omega - \partial_x^3 u_2 \qquad \text{and} \qquad \partial_y^3 u_2 = -\partial_x \partial_y \omega + \partial_x^3 u_1 \,.$$

The velocity terms $\partial_x^3 u_1$ and $\partial_x^3 u_2$ were already bounded above. Clearly, we have $\left\|D^2 \omega\right\|_{L^2(y\geq 1/2)} \lesssim \left\|D^2 \omega\right\|_S \lesssim \|\omega\|_t$. On the other hand, similarly to (7.6), we have

$$\left\|D^2 \omega\right\|_{L^2(1/4\leq y\leq 1/2)} \lesssim \left\|D^2 \omega\right\|_{L^\infty(1/4\leq y\leq 1/2)} \lesssim \left\|D^2 \omega\right\|_{X_\mu} \lesssim \|\omega\|_t.$$

In the last inequality we used that μ in (7.5) is bounded from below by $\mu_0/20$. This concludes the proof of the lemma. □

The remainder of this section is devoted to an a priori estimate for the norm $\sum_{i+j\leq 3} \|\partial_x^i \partial_y^j \omega\|_S$. For this purpose, denote

$$\phi(y) = y\bar{\psi}(y) \,, \tag{7.8}$$

where $\bar{\psi} \in C^\infty$ is a non-decreasing function such that $\bar{\psi} = 0$ for $0 \leq y \leq 1/4$ and $\bar{\psi} = 1$ for $y \geq 1/2$. In order to estimate the norm in (7.1), note that

$$\|yf\|_{L^2_{x,y}(y\geq 1/2)} \leq \|\phi f\|_{L^2(\mathbb{H})} \,,$$

so that it suffices to estimate this larger norm.

LEMMA 7.2. *For any* $0 < t < \frac{\mu_0}{2\gamma}$ *the estimate*

$$\sum_{i+j\le 3} \|\phi \partial_x^i \partial_y^j \omega(t)\|_{L^2(\mathbb{H})}^2 \lesssim \Big(1 + t \sup_{s\in[0,t]} \|\omega(s)\|_s^3\Big) e^{Ct(1+\sup_{s\in[0,t]}\|\omega(s)\|_s)} \sum_{i+j\le 3} \|\phi\partial_x^i\partial_y^j \omega_0\|_{L^2(\mathbb{H})}^2$$

holds, where $C > 0$ *is a constant independent of* γ.

PROOF OF LEMMA 7.2. Let $\alpha \in \mathbb{N}_0^2$ be a multi-index with $|\alpha| \le 3$. We apply ∂^α to the vorticity form of the Navier-Stokes equation and test this equation with $\phi^2\partial^\alpha\omega$ to obtain the energy estimate

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\phi\partial^\alpha\omega\|_{L^2(\mathbb{H})}^2 + \nu\|\phi\nabla\partial^\alpha\omega\|_{L^2(\mathbb{H})}^2 \\ &\quad = 2\int_{\mathbb{H}} u_2\phi'\phi|\partial^\alpha\omega|^2 - \sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\int_{\mathbb{H}}\partial^\beta u\cdot\nabla\partial^{\alpha-\beta}\omega\partial^\alpha\omega\phi^2 - 2\nu\int_{\mathbb{H}}\phi'\partial^\alpha\omega\partial_y\partial^\alpha\omega\phi\,. \end{aligned} \tag{7.9}$$

Using the pointwise estimate

$$|\phi'(y)| \lesssim \phi(y) + \chi_{\{1/4\le y\le 1/2\}}$$

on the first and the third term, summing over $|\alpha| \le 3$, and absorbing a part of the third term in (7.9) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\sum_{i+j\le 3}\|\phi\partial_x^i\partial_y^j\omega\|_{L^2(\mathbb{H})}^2 &\lesssim \Big(\nu + \|u_2\|_{L^\infty(y\ge 1/4)} + \sum_{1\le k\le 2}\|D^k u\|_{L^\infty_{x,y}(y\ge 1/4)}\Big)\sum_{i+j\le 3}\|\phi\partial_x^i\partial_y^j\omega\|_{L^2(\mathbb{H})}^2 \\ &\quad + \big\|D^3 u\big\|_{L^2_{x,y}(y\ge 1/4)}\|\phi\nabla\omega\|_{L^\infty(\mathbb{H})}\sum_{i+j\le 3}\|\phi\partial_x^i\partial_y^j\omega\|_{L^2(\mathbb{H})} \\ &\quad + \Big(\nu + \|u_2\|_{L^\infty(\mathbb{H})}\Big)\sum_{i+j\le 3}\|\partial_x^i\partial_y^j\omega\|_{L^2_{x,y}(1/4\le y\le 1/2)}^2\,. \end{aligned} \tag{7.10}$$

Next, note that we have the analytic estimate

$$\begin{aligned}\sum_{i+j\le 3}\|\partial_x^i\partial_y^j\omega\|_{L^2_{x,y}(1/4\le y\le 1/2)}^2 &\lesssim \sum_{i+j\le 3}\sum_{\xi}\big\|\xi^i\partial_y^j\omega_\xi\big\|_{L^2(1/4\le y\le 1/2)}^2 \\ &\lesssim \sum_{i+j\le 3}\sum_{\xi}\Big\|e^{\epsilon_0(1+\mu/2-y)_+|\xi|}\xi^i(y\partial_y)^j\omega_\xi\Big\|_{\mathcal{L}^\infty_{\mu/2,\nu}}^2 \\ &\lesssim \sum_{i+j\le 3}\big\|\partial_x^i(y\partial_y)^j\omega\big\|_{X_{\mu/2}}^2 \end{aligned}$$

uniformly in $\mu > 0$. In particular, we may take $\mu = \frac{\mu_0-\gamma t}{10}$. Here we used essentially that the weight $w(y)$ is comparable to 1 (independently of ν) in the region $\{1/4 \le y \le 1/2\}$. Since $\mu > 0$, we may further use the analyticity recovery Lemma A.3, and estimate

$$\sum_{i+j\le 3}\|\partial_x^i\partial_y^j\omega\|_{L^2_{x,y}(1/4\le y\le 1/2)}^2 \lesssim \|\omega\|_{X_\mu}^2\,. \tag{7.11}$$

Note that we used that $\frac{1}{\mu} = \frac{10}{\mu_0-\gamma t} \le \frac{20}{\mu_0} \lesssim 1$.

For the second term on the right side of (7.10), we appeal to (7.3) and to the estimate

$$\begin{aligned}\|\phi\nabla\omega\|_{L^\infty(\mathbb{H})} &\lesssim \|\nabla(\phi\omega)\|_{L^\infty(\mathbb{H})} + \|\phi\omega\|_{L^\infty(\mathbb{H})} + \|\omega\|_{L^\infty_{x,y}(1/4\le y\le 1/2)} \\ &\lesssim \sum_{i+j\le 3}\big\|\partial_x^i\partial_y^j(\phi\omega)\big\|_{L^2(\mathbb{H})} + \|\omega\|_{X_\mu}\,. \end{aligned}$$

Here we have used the Sobolev embedding $H^2(\mathbb{H}) \subset L^\infty(\mathbb{H})$, the previously established bound (7.11), the Leibniz rule, and the definition of ϕ in (7.8). The resulting inequality is

$$\left\|D^3 u\right\|_{L^2_{x,y}(y\geq 1/4)} \|\phi\nabla\omega\|_{L^\infty(\mathbb{H})} \sum_{i+j\leq 3} \|\phi\partial_x^i\partial_y^j\omega\|_{L^2(\mathbb{H})} \lesssim \|\!|\omega|\!\|_t \sum_{i+j\leq 3} \|\phi\partial_x^i\partial_y^j\omega\|^2_{L^2(\mathbb{H})} + \|\!|\omega|\!\|_t \|\omega\|^2_{X_\mu} .$$

Combining (7.10)–(7.11), and Lemma 7.1 we deduce

$$\frac{1}{2}\frac{d}{dt}\sum_{i+j\leq 3} \|\phi\partial_x^i\partial_y^j\omega\|^2_{L^2(\mathbb{H})} \lesssim (1+\|\!|\omega(t)|\!\|_t) \sum_{i+j\leq 3} \|\phi\partial_x^i\partial_y^j\omega(t)\|^2_{L^2(\mathbb{H})} + (1+\|\!|\omega(t)|\!\|_t)\|\omega(t)\|^2_{X_\mu} .$$

Upon applying the Grönwall inequality, the proof of the lemma is concluded. □

Appendix A. Proofs of some technical lemmas

Here we list some technical lemmas. The next lemma converts an ℓ^1 norm in ξ to an ℓ^2 norm, which is necessary when converting S_μ norms to an S and hence a Z norm.

LEMMA A.1. *Let $\mu \in (0,1)$. We have*

$$\sum_{i+j\leq 2} \|\partial_x^i (y\partial_y)^j\omega\|_{S_\mu} \lesssim \sum_{i+j\leq 2} \left\|\partial_x^i (y\partial_y)^j\omega\right\|_S + \left\|\partial_x^{i+1}(y\partial_y)^j\omega\right\|_S .$$

PROOF OF LEMMA A.1. We have

$$\sum_\xi |v_\xi| \lesssim \left(\sum_\xi (1+|\xi|^2)|v_\xi|^2\right)^{1/2} \tag{A.1}$$

for every v for which the right side is finite. The inequality (A.1) holds since $\sum_\xi (1+|\xi|^2)^{-1} < \infty$. □

LEMMA A.2. *Assume that the parameters $\mu, \mu_0, \gamma, t > 0$ obey $\mu < \mu_0 - \gamma t$. Then, for $\alpha \in (0, \frac{1}{2})$ we have*

$$\int_0^t \frac{1}{\sqrt{t-s}} \frac{1}{(\mu_0-\mu-\gamma s)^{1+\alpha}}\, ds \leq \frac{C}{\sqrt{\gamma}(\mu_0-\mu-\gamma t)^{1/2+\alpha}} \tag{A.2}$$

and

$$\int_0^t \frac{1}{\sqrt{t-s}} \frac{1}{(\mu_0-\mu-\gamma s)^{\alpha}}\, ds \leq \frac{C}{\sqrt{\gamma}} , \tag{A.3}$$

where $C > 0$ is a constant depending on μ_0 and $1/2-\alpha$.

PROOF OF LEMMA A.2. Changing variables $s' = \gamma s$, $t' = \gamma t$, and letting $\mu_0 - \mu = \mu' > 0$, the left side of (A.2) is rewritten and bounded as

$$\begin{aligned} \int_0^{t'} \frac{\sqrt{\gamma}}{\sqrt{t'-s'}} \frac{1}{(\mu'-s')^{1+\alpha}} \frac{ds'}{\gamma} &\leq \frac{1}{\sqrt{\gamma}(\mu'-t')^\alpha} \int_0^{t'} \frac{ds'}{\sqrt{t'-s'}(\mu'-s')} \\ &= \frac{2\arctan\left(\sqrt{\frac{t'}{\mu'-t'}}\right)}{\sqrt{\gamma}(\mu'-t')^{1/2+\alpha}} \lesssim \frac{1}{\sqrt{\gamma}(\mu'-t')^{1/2+\alpha}} = \frac{1}{\sqrt{\gamma}(\mu_0-\mu-\gamma t)^{1/2+\alpha}} . \end{aligned}$$

In order to prove (A.3), we proceed similarly and use $\mu' > t'$ to deduce

$$\int_0^{t'} \frac{\sqrt{\gamma}}{\sqrt{t'-s'}} \frac{1}{(\mu'-s')^\alpha} \frac{ds'}{\gamma} \leq \frac{1}{\sqrt{\gamma}} \int_0^{t'} \frac{ds'}{(t'-s')^{1/2+\alpha}} \lesssim \frac{1}{\sqrt{\gamma}} ,$$

where the implicit constant may depend on μ_0 and $1/2-\alpha$. □

LEMMA A.3 (Analyticity recovery for the X norm). *For $\widetilde{\mu} > \mu \geq 0$, we have*

$$\sum_{i+j=1} \|\partial_x^i (y\partial_y)^j f\|_{X_\mu} \lesssim \frac{1}{\widetilde{\mu} - \mu} \|f\|_{X_{\widetilde{\mu}}}\,.$$

PROOF OF LEMMA A.3. First, let $(i,j) = (1,0)$. According to the definition of the X_μ norm, and using that the bound $(\widetilde{\mu}-\mu)|\xi| e^{\epsilon_0|\xi|((1+\mu-y)_+ - (1+\widetilde{\mu}-y)_+)} \lesssim 1$ holds on Ω_μ, we have

$$\begin{aligned}\|\partial_x f\|_{X_\mu} &= \sum_\xi \|\xi e^{\epsilon_0(1+\mu-y)_+|\xi|} f_\xi\|_{\mathcal{L}^\infty_{\mu,\nu}} \lesssim \frac{1}{\widetilde{\mu}-\mu} \sum_\xi \|e^{\epsilon_0(1+\widetilde{\mu}-y)_+|\xi|} f_\xi\|_{\mathcal{L}^\infty_{\mu,\nu}} \\ &\lesssim \frac{1}{\widetilde{\mu}-\mu} \sum_\xi \|e^{\epsilon_0(1+\widetilde{\mu}-y)_+|\xi|} f_\xi\|_{\mathcal{L}^\infty_{\widetilde{\mu},\nu}} = \frac{1}{\widetilde{\mu}-\mu} \|f\|_{X_{\widetilde{\mu}}}\,.\end{aligned}$$

Next, consider $(i,j) = (0,1)$. By the Cauchy integral theorem, we have

$$\partial_y f_\xi(y) = \int_{C(y,R_y)} \frac{f_\xi(z)}{(y-z)^2}\, dz\,, \tag{A.4}$$

where $C(y,R_y)$ is the circle centered at y with radius R_y. Hence, we have

$$|\partial_y f_\xi(y)| \lesssim \frac{1}{R_y} \sup_{z\in C(y,R_y)} |f_\xi(z)|\,.$$

By taking $R_y = C^{-1}(\widetilde{\mu}-\mu)\mathbb{R}\mathrm{e}\, y$, for a sufficiently large universal constant $C > 0$, we obtain

$$\begin{aligned}\|y\partial_y f\|_{X_\mu} &= \sum_\xi \|e^{\epsilon_0(1+\mu-y)_+|\xi|} y\partial_y f_\xi\|_{\mathcal{L}^\infty_{\mu,\nu}} \lesssim \frac{1}{\widetilde{\mu}-\mu} \sum_\xi \|e^{\epsilon_0(1+\mu-y)_+|\xi|} f_\xi\|_{\mathcal{L}^\infty_{\widetilde{\mu},\nu}} \\ &\lesssim \frac{1}{\widetilde{\mu}-\mu} \sum_\xi \|e^{\epsilon_0(1+\widetilde{\mu}-y)_+|\xi|} f_\xi\|_{\mathcal{L}^\infty_{\widetilde{\mu},\nu}} = \frac{1}{\widetilde{\mu}-\mu} \|f\|_{X_{\widetilde{\mu}}}\,,\end{aligned}$$

concluding the proof. □

LEMMA A.4 (Analyticity recovery for the Y norm). *Let $\mu_0 \geq \widetilde{\mu} > \mu \geq 0$. Then we have*

$$\sum_{i+j=1} \|\partial_x^i (y\partial_y)^j f\|_{Y_\mu} \lesssim \frac{1}{\widetilde{\mu}-\mu} \|f\|_{Y_{\widetilde{\mu}}}\,. \tag{A.5}$$

PROOF OF LEMMA A.4. By the same argument which yielded the X norm estimate in Lemma A.3, we obtain

$$\begin{aligned}\|\partial_x f\|_{Y_\mu} &= \sum_\xi \|\xi e^{\epsilon_0(1+\mu-y)_+|\xi|} f_\xi\|_{\mathcal{L}^1_\mu} \lesssim \frac{1}{\widetilde{\mu}-\mu} \sum_\xi \|e^{\epsilon_0(1+\widetilde{\mu}-y)_+|\xi|} f_\xi\|_{\mathcal{L}^1_\mu} \\ &\lesssim \frac{1}{\widetilde{\mu}-\mu} \sum_\xi \|e^{\epsilon_0(1+\widetilde{\mu}-y)_+|\xi|} f_\xi\|_{\mathcal{L}^1_{\widetilde{\mu}}} = \frac{1}{\widetilde{\mu}-\mu} \|f\|_{Y_{\widetilde{\mu}}}\,.\end{aligned}$$

In order to prove the estimate (A.5) for $(i,j) = (0,1)$, we use (A.4) to bound

$$\|y\partial_y f_\xi\|_{L^1(\partial\Omega_\theta)} = \int_{\partial\Omega_\theta} \left| \int_{C(y,R_y)} \frac{y f_\xi(z)}{(y-z)^2}\, dz \right| dy \lesssim \int_{\partial\Omega_\theta} \int_{C(y,R_y)} \frac{|y f_\xi(z)|}{R_y^2}\, dz dy$$

for any $0 \leq \theta < \mu$. By taking $R_y = C^{-1}(\widetilde{\mu} - \mu)\mathbb{R}\mathrm{e}\, y$ for a sufficiently large universal constant $C > 0$, using that $|y|$ is comparable to $\mathbb{R}\mathrm{e}\, y$ in this region, and applying Fubini's theorem, we obtain

$$\begin{aligned}
\|y\partial_y f_\xi\|_{L^1(\partial\Omega_\theta)} &\lesssim \frac{1}{\widetilde{\mu}-\mu} \int_{\partial\Omega_\theta} \int_{C(y,R_y)} \frac{|f_\xi(z)|}{R_y}\, dz dy \\
&\lesssim \frac{1}{\widetilde{\mu}-\mu} \int_{\partial\Omega_\theta} \int_0^{2\pi} |f_\xi(y + R_y e^{i\phi})|\, d\phi dy \\
&\lesssim \frac{1}{\widetilde{\mu}-\mu} \sup_{\bar{\theta} \in (\theta - \frac{2(\widetilde{\mu}-\mu)}{C}, \theta + \frac{2(\widetilde{\mu}-\mu)}{C})} \|f_\xi\|_{L^1(\partial\Omega_{\bar{\theta}})} \\
&\lesssim \frac{1}{\widetilde{\mu}-\mu} \|f_\xi\|_{\mathcal{L}^1_{\widetilde{\mu}}}\,,
\end{aligned}$$

which proves the claim. Since $e^{\epsilon_0(1+\mu-y)+|\xi|} \leq e^{\epsilon_0(1+\widetilde{\mu}-y)+|\xi|}$, for every $y \in \Omega_\mu$, the lemma follows. $\square$

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