On the ill-posedness of active scalar equations with odd singular kernels

Igor Kukavica, Vlad Vicol, and Fei Wang

ABSTRACT. We consider active scalar equations with constitutive laws that are *odd* and *very singular*, in the sense that the velocity field loses more than one derivative with respect to the active scalar. We provide an example of such a constitutive law for which the equation is ill-posed: Either Sobolev solutions do not exist, from Gevrey-class datum, or the solution map fails to be Lipschitz continuous in the topology of a Sobolev space, with respect to Gevrey class perturbations in the initial datum. Monday 7th March, 2016.

1. Introduction

In this paper we address the well-posedness of the Cauchy problem for active scalar equations

$$\partial_t\theta + u\cdot\nabla\theta = 0 \tag{1.1}$$

$$\nabla\cdot u = 0 \tag{1.2}$$

$$\theta(x,0) = \theta_0(x) \tag{1.3}$$

posed on $\mathbb{T}^2\times[0,\infty) = [-\pi,\pi]^2\times[0,\infty)$ with a certain constitutive law for the incompressible drift

$$u = T\theta,$$

to be specified precisely below (cf. (1.9) below). The datum θ_0 and the solution θ are taken to have zero mean on $\mathbb{T}^2$.

The study of active scalar equations of type (1.1)–(1.2) is motivated by several important fluid models: When $T = \nabla^\perp(-\Delta)^{-1}$, the system becomes the vorticity formulation of the 2D Euler equations; the case $T = \nabla^\perp(-\Delta)^{-1/2}$ corresponds to the surface quasi-geostrophic equation [**1**]; the constitutive law $T = \partial_1\nabla^\perp(-\Delta)^{-1}$ models flow in an incompressible porous medium with Darcy's law [**2**]; while $T = \nabla^\perp B$, where B is a scalar bounded Fourier multiplier, appears in magneto-geostrophic dynamics [**3**]. The well-posedness theory of active scalar equations (either inviscid or with fractional dissipation) has attracted considerable attention in the last two decades. We refer the readers to [**4**, **5**, **6**, **1**, **7**, **8**, **9**, **2**, **10**, **11**, **3**, **12**, **13**, **14**, **15**, **16**, **17**, **18**, **19**, **20**, **21**, **22**, **23**] and the references therein.

Recently, these equations have been considered with velocity fields determined by a *singular constitutive law* (i.e., the map $T\colon\theta\mapsto u$ is unbounded) which is also *odd* (by which we mean the corresponding Fourier multiplier is an odd function of frequency). In particular, in [**24**] the authors considered the velocity field given by

$$u = \mathcal{R}^\perp\Lambda^\beta\theta, \tag{1.4}$$

where $\beta\in(0,1]$, $\Lambda = \sqrt{-\Delta}$, $\mathcal{R}^\perp = \nabla^\perp\Lambda^{-1}$, and showed the local existence and uniqueness of solutions for (1.1)–(1.3) with (1.4), in the Sobolev space H^4. The two key ingredients in their proof were an estimate for the commutator $\|[\partial_i\Lambda^s, g]f\|_{L^2}$ and the identity

$$\int fAf\,g = -\frac{1}{2}\int f[A,g]f \tag{1.5}$$

which holds for smooth f, g, where $A = \partial_i \Lambda^\beta$ *is an odd operator*. This result was later sharpened in [12] where the local existence was established in H^σ with $\sigma > 2 + \beta$, using an improved commutator estimate.

In contrast when the constitutive law is *even and singular* there are some ill-posedness results available for these equations. In [25, 26], the authors established ill-posedness for even singular constitutive laws, i.e., for operators T that have an even and unbounded Fourier multiplier symbol. The main ingredients in these works were a linear ill-posedness result (severe linear instability) in the spirit of [27, 28], and a classical linear implies nonlinear ill-posedness argument [29, 30].

On the other hand, we are not aware of *any ill-posedness result for* (1.1)–(1.3) *with an odd constitutive law*. This is due to the special cancellation property (1.5), which technically reduces the order of the constitutive law by one. The main purpose of this paper is to establish an ill-posedness result for (1.1)–(1.3) with an odd constitutive law

$$u = T\theta = \mathcal{R}^\perp \Lambda^\beta M \theta \tag{1.6}$$

where $1 < \beta \leq 2$ and M is a zero-order, scalar, even Fourier multiplier operator with symbol m. That is, $\widehat{M\theta}(k) = m(k)\hat{\theta}(k)$, where the specific Fourier multiplier symbol m we consider is given by

$$m(k) = k_1^2 |k|^{-2} \phi(k) \tag{1.7}$$

for all $k \in \mathbb{Z}^2_*$, where we define

$$\phi(k) = \begin{cases} -1, & k_2 = \pm 1, \\ 1, & \text{otherwise.} \end{cases} \tag{1.8}$$

For simplicity of notation, we denote by $\phi(i\nabla)$ the Fourier multiplier operator with symbol $\phi(k)$. We may then rewrite (1.6)–(1.8) concisely as

$$u = -\mathcal{R}^\perp \mathcal{R}_1^2 \phi(i\nabla) \Lambda^\beta \theta \tag{1.9}$$

where we recall that $\beta \in (1, 2]$ throughout this paper.

Our main result (Theorem 2.2 below) is to prove that the active scalar equation (1.1)–(1.3) with constitutive law (1.9) is ill-posed in any Gevrey space G^s with $s > (4-\beta)/((\beta-1)(3-\beta))$. Here, by ill-posedness we mean that:

- either, given an initial datum θ_0 in G^s, there is no local in time Sobolev solution $\theta \in L^\infty_t H^{\beta/2+1+\delta}_x$, where $\delta > 0$ is arbitrary;
- or, the solution map $\theta_0 \to \theta$ is not Lipschitz continuous in $H^{\beta/2+1+\delta}$, with respect to G^s perturbations in the initial datum (see Definition 2.2 below).

We note that the condition $\beta > 1$ is strictly necessary for our result to hold. Indeed, for $\beta \in (0, 1]$, by the results in [24, 12], we know that for the system (1.1)–(1.3) and (1.9) we have local existence and uniqueness in the Sobolev space H^σ with $\sigma > 2 + \beta$. Additionally, using the identity (1.5) and the aforementioned commutator estimates, one may show that for $\beta \in (0, 1]$ the equations are locally Lipschitz $(H^{\beta/2+1}, H^{\beta+3})$ well-posed, in the sense of Definition 2.2 below. Thus, the requirement $\beta \in (1, 2]$ in this paper is necessary and sufficient.

The proof of our main result is in the spirit of the earlier Lipschitz ill-posedness works [25, 26] with the main difference being that the eigenfunctions are constructed as sums of sines and cosines; this interplay leads to an eigenvalue problem involving a continued fraction with all positive signs (cf. (3.11) below). Such continued fractions require a different treatment than [25, 26] due to non-monotonicity of the corresponding approximation sequence. Finally, we need a way to bound the Sobolev and Gevrey norms ensuring that the L^2 norm of the solutions grow arbitrarily fast in any short period of time.

The paper is organized as follows. In Section 2, we linearize the active scalar equation around a steady solution and state our main linear and nonlinear results. In Section 3, we give the proof of the ill-posedness in Sobolev spaces and Gevrey classes for the linear equation. Finally, in Section 4, we conclude the ill-posedness for the non-linear problem in the corresponding Sobolev and Gevrey spaces using a perturbation argument.

2. The linearized problem and main results

Consider a scalar function of the form $\bar{\theta}(x,t) = \bar{\theta}(x_2)$, where $\bar{\theta}(x_2)$ is a real function in $\mathbb{T}$ with zero mean. According to (1.9) we have that $\bar{u} = T\bar{\theta} = 0$, in view of the differentiation with respect to x_1 inherent in T. Thus, any such $\bar{\theta}$ is a steady state of (1.1). We choose a particular function

$$\bar{\theta}(x_2) = \cos(x_2) \tag{2.1}$$

and consider the linearization of the nonlinear term in (1.1) around it. Denote the corresponding linear operator by

$$\begin{aligned} L\theta &= -\bar{u}\cdot\nabla\theta - u\cdot\nabla\bar{\theta} = \mathcal{R}_1^3\phi(i\nabla)\Lambda^\beta\theta\,\partial_2\bar{\theta} \\ &= -\sin(x_2)\,\mathcal{R}_1^3\phi(i\nabla)\Lambda^\beta\theta. \end{aligned} \tag{2.2}$$

We shall use the method of continued fractions (see e.g. [**25, 27, 26, 28**]) to prove that the operator L has a sequence of eigenvalues whose positive real parts diverge to infinity (cf. Theorem 2.3 below).

In order to state our main results, we recall here the definition of the Gevrey-space G^s (cf. [**31**]).

DEFINITION 2.1 (**Gevrey-space**). *A function $\theta \in C^\infty(\mathbb{R}^2)$ belongs to the Gevrey class G^s, where $s \geq 1$, if there exists a positive constant $\tau > 0$, called the Gevrey-class radius, such that the G^s_τ-norm is finite, i.e.,*

$$\|\theta\|^2_{G^s_\tau} = \sum_{k\in\mathbb{Z}^2_*} |\hat{\theta}(k)|^2 |k|^4 e^{2\tau|k|^{1/s}} < \infty. \tag{2.3}$$

With this definition, $G^s_\tau = \{\theta \in C^\infty(\mathbb{T}^2)\colon \|\theta\|_{G^s_\tau} < \infty\}$ is an algebra for $s \geq 1, \tau > 0$, and we have $G^s = \bigcup_{\tau>0} G^s_\tau$.

Next we recall the definition of the Lipschitz well-posedness (cf. [**26, 32**]).

DEFINITION 2.2 (**Locally Lipschitz (X,Y) well-posedness**). *Let $Y \subseteq X \subseteq H^{\beta/2+1+\delta}$ be Sobolev spaces, where $\delta > 0$ is arbitrary. We say that the Cauchy problem for the active scalar equation (1.1)–(1.3) with (1.6) is* locally Lipschitz (X,Y) well-posed, *if there exist continuous functions $T, K\colon [0,\infty)^2 \to (0,\infty)$, so that for every pair of initial data $\theta^{(1)}(0,\cdot), \theta^{(2)}(0,\cdot) \in Y$ there exist unique solutions $\theta^{(1)}, \theta^{(2)} \in L^\infty(0,T;X)$ of the initial value problem associated to (1.1)–(1.3) with (1.6), such that*

$$\|\theta^{(1)}(t,\cdot) - \theta^{(2)}(t,\cdot)\|_X \leq K\|\theta^{(1)}(0,\cdot) - \theta^{(2)}(0,\cdot)\|_Y \tag{2.4}$$

for every $t \in [0,T]$, where $T = T(\|\theta^{(1)}(0,\cdot)\|_Y, \|\theta^{(2)}(0,\cdot)\|_Y)$ and $K = K(\|\theta^{(1)}(0,\cdot)\|_Y, \|\theta^{(2)}(0,\cdot)\|_Y)$.

The role of the space $H^{\beta/2+1+\delta}$ in the above definition is to ensure that the ranges of the linear operator L defined in (2.2), and that of the nonlinear operator in (1.1), defined as

$$N[\theta] = T\theta\cdot\nabla\theta,$$

lie in L^2. Here, recall that $u = T\theta$ is given by (1.9). More precisely, by the Sobolev embedding we have

$$\|L\theta\|_{L^2} \leq C\|\theta\|_{H^\beta} \leq C\|\theta\|_{H^{\beta/2+1+\delta}} \tag{2.5}$$

and

$$\|N[\theta]\|_{L^2} = \|T\theta\cdot\nabla\theta\|_{L^2} \leq C\|\theta\|^2_{H^{\beta/2+1+\delta}} \tag{2.6}$$

for a sufficiently large constant $C > 0$. Note that we can set $\delta = 0$ if $\beta < 2$. The role of the two-spaces X and Y in Definition 2.2 is to allow the solution to lose regularity, as it is expected due to the derivative losses of order β (from T) and 1 (from ∇) present in the nonlinearity.

The first main result in this paper asserts that singular odd active scalar equations are locally Lipschitz ill-posed in Sobolev spaces.

THEOREM 2.1. *Assume $\beta \in (1,2]$, let $\delta > 0$, and define $r = \beta/2 + 1 + \delta$. Then the system (1.1)–(1.3) with (1.6) is locally Lipschitz (H^r, H^s) ill-posed for any $s > r$.*

In fact, in view of the bound (2.9), one may obtain a stronger ill-posedness result: The solution is not well-posed in a class of Gevrey spaces.

THEOREM 2.2. *Let $\beta \in (1,2]$, $r = \beta/2 + 1 + \delta$ for some $\delta > 0$, and let $s > (4-\beta)/(\beta-1)(3-\beta)$. Then the system (1.1)–(1.3) with (1.6) is locally Lipschitz (H^r, G^s_τ) ill-posed for $\tau > 0$.*

Both theorems above hold with $\delta = 0$ in the case $\beta < 2$. The main ingredient in the proofs of Theorems 2.1 and 2.2 is the following bound on the eigenvalues of the linearized operator L defined in (2.2).

THEOREM 2.3. *The linearized operator L defined in (2.2) has a sequence of entire real-analytic eigenfunctions $\{\theta_k\}_{k\geq 1}$ with corresponding eigenvalues $\{\lambda_k\}_{k\geq 1}$, such that*

$$\lambda_k \geq C_0^{-1} k^{\beta-1} \tag{2.7}$$

for all $k \geq 1$, where $C_0 \geq 10$ is a universal constant. Moreover, we may normalize θ_k such that either one of the following statements holds:

(a) *Given $s \geq 0$*

$$\|\theta_k\|_{H^s} = 1 \quad \textit{and} \quad \|\theta_k\|_{L^2} \geq C_s^{-1} k^{-s(4-\beta)/(3-\beta)} \tag{2.8}$$

for all $k \geq 1$, where $C_s \geq 1$ is a constant that depends only on s;

(b) *given $s \geq 1$ and $\tau > 0$*

$$\|\theta_k\|_{G^s_\tau} = 1 \quad \textit{and} \quad \|\theta_k\|_{L^2} \geq C_{s,\tau}^{-1} \exp\left(-C_{s,\tau} k^{(4-\beta)/(s(3-\beta))}\right) \tag{2.9}$$

for all $k \geq 1$, where $C_{s,\tau} \geq 1$ is a constant that depends only on s, τ.

A standard perturbation argument (see e.g. [**29, 30**]) then implies the ill-posedness (1.1)–(1.3) with the constitutive law (1.9).

3. Proof of linear ill-posedness

PROOF OF THEOREM 2.3. Fix an integer $k \geq 1$. We seek an eigenvalue-eigenfunction pair $(\lambda, \theta) = (\lambda_k, \theta_k)$ for L, with θ oscillating at frequency k with respect to x_1, i.e., we are looking for θ of the form

$$\theta(x_1, x_2) = -\, c_1 \sin(kx_1)\cos(x_2) + \sin(kx_1) \sum_{\text{odd } n\geq 3} c_n \cos(nx_2) + \cos(kx_1) \sum_{\text{even } n\geq 2} c_n \sin(nx_2) \tag{3.1}$$

where the real coefficients $\{c_n\}_{n\geq 1}$ are to be determined. Using

$$\phi(i\nabla)\theta(x_1,x_2) = \sin(kx_1) \sum_{\text{odd } n\geq 1} c_n \cos(nx_2) + \cos(kx_1) \sum_{\text{even } n\geq 2} c_n \sin(nx_2)$$

we get

$$\begin{aligned} L\theta &= k^3 \cos(kx_1) \sum_{\text{odd } n\geq 1} (n^2+k^2)^{(\beta-3)/2} c_n \cos(nx_2)\sin(x_2) \\ &\quad - k^3 \sin(kx_1) \sum_{\text{even } n\geq 2} (n^2+k^2)^{(\beta-3)/2} c_n \sin(nx_2)\sin(x_2) \\ &= \cos(kx_1) \sum_{\text{odd } n\geq 1} k^3 (n^2+k^2)^{(\beta-3)/2} c_n \frac{\sin((n+1)x_2) - \sin((n-1)x_2)}{2} \\ &\quad + \sin(kx_1) \sum_{\text{even } n\geq 2} k^3 (n^2+k^2)^{(\beta-3)/2} c_n \frac{\cos((n+1)x_2) - \cos((n-1)x_2)}{2}. \end{aligned} \tag{3.2}$$

We introduce the positive coefficients

$$p_n = p_{n,k} = 2(n^2+k^2)^{(3-\beta)/2} k^{-3} \tag{3.3}$$

omitting the k dependence for ease of notation. Then for all $n \geq 1$ we have

$$p_n \geq 2n^{3-\beta} k^{-3}, \tag{3.4}$$

where by assumption $3 - \beta \geq 1$. With (3.3) we simplify (3.2) as

$$\begin{aligned} L\theta = \cos(kx_1) \sum_{\text{even } n \geq 2} \left(\frac{c_{n-1}}{p_{n-1}} - \frac{c_{n+1}}{p_{n+1}} \right) \sin(nx_2) - \frac{c_2}{p_2} \sin(kx_1) \cos(x_2) \\ + \sin(kx_1) \sum_{\text{even } n \geq 3} \left(\frac{c_{n-1}}{p_{n-1}} - \frac{c_{n+1}}{p_{n+1}} \right) \cos(nx_2). \end{aligned} \tag{3.5}$$

The equation $\lambda\theta = L\theta$ thus becomes

$$\begin{aligned} &- \lambda c_1 \sin(kx_1) \cos(x_2) + \sin(kx_1) \sum_{\text{odd } n \geq 3} \lambda c_n \cos(nx_2) + \cos(kx_1) \sum_{\text{even } n \geq 2} \lambda c_n \sin(nx_2) \\ &= \cos(kx_1) \sum_{\text{even } n \geq 2} \left(\frac{c_{n-1}}{p_{n-1}} - \frac{c_{n+1}}{p_{n+1}} \right) \sin(nx_2) - \frac{c_2}{p_2} \sin(kx_1) \cos(x_2) \\ &\quad + \sin(kx_1) \sum_{\text{even } n \geq 3} \left(\frac{c_{n-1}}{p_{n-1}} - \frac{c_{n+1}}{p_{n+1}} \right) \cos(nx_2). \end{aligned} \tag{3.6}$$

From (3.6) we obtain the recurrence relationship

$$\frac{c_{n-1}}{p_{n-1}} - \frac{c_{n+1}}{p_{n+1}} = \lambda c_n, \qquad n \geq 2, \tag{3.7}$$

$$\frac{c_2}{p_2} = \lambda c_1, \qquad n = 1. \tag{3.8}$$

Denoting

$$\eta_n = \left(\frac{c_n}{p_n} \right) \left(\frac{c_{n-1}}{p_{n-1}} \right)^{-1}, \qquad n \geq 2$$

the recurrence relation (3.7)–(3.8) becomes

$$\frac{1}{\eta_n} - \eta_{n+1} = \lambda p_n, \qquad n \geq 2 \tag{3.9}$$

$$\eta_2 = \lambda p_1. \tag{3.10}$$

A real number λ yields a solution of (3.9)–(3.10) if and only if it solves the continued fraction equation

$$\lambda p_1 = \cfrac{1}{\lambda p_2 + \cfrac{1}{\lambda p_3 + \cdots}}. \tag{3.11}$$

Note that here the coefficients $\{p_n\}_{n \geq 1}$ are given by (3.3), and that λ is the only unknown.

Next we show the existence of a solution $\lambda > 0$ of (3.11). For this purpose, we need to prove first that the continued fraction on the right side of (3.11) *converges* (note that since the associated sequence is not monotone, this is not immediately clear), and then establish that it *defines a continuous function of* λ *on* $(0, \infty)$.

For $n \in \{2, 3, \cdots\}$ and $m \in \{0, 1, 2, 3, \cdots\}$, define

$$F_{n,m}(\lambda) = \cfrac{1}{\lambda p_n + \cfrac{1}{\cdots + \frac{1}{\lambda p_{n+m}}}}.$$

It is clear that $F_{n,m}(\lambda) > 0$ for $\lambda \in (0,\infty)$. From the identity

$$F_{n,m}(\lambda) = \frac{1}{\lambda p_n + F_{n+1,m-1}(\lambda)}$$

valid for $n \in \{2,3,4,\cdots\}$ and $m \in \{1,2,3,\cdots\}$, we deduce

$$\begin{aligned} F_{n,m+1}(\lambda) - F_{n,m}(\lambda) &= \frac{1}{\lambda p_n + F_{n+1,m}(\lambda)} - \frac{1}{\lambda p_n + F_{n+1,m-1}(\lambda)} \\ &= \frac{F_{n+1,m-1}(\lambda) - F_{n+1,m}(\lambda)}{(\lambda p_n + F_{n+1,m}(\lambda))(\lambda p_n + F_{n+1,m-1}(\lambda))} \end{aligned}$$

from where we obtain

$$|F_{n,m+1}(\lambda) - F_{n,m}(\lambda)| \leq \frac{1}{(\lambda p_n)^2} |F_{n+1,m}(\lambda) - F_{n+1,m-1}(\lambda)|.$$

Using induction on m and the bound

$$|F_{l,1}(\lambda) - F_{l,0}(\lambda)| = \left| \frac{1}{\lambda p_l + \frac{1}{\lambda p_{l+1}}} - \frac{1}{\lambda p_l} \right| \leq \frac{1}{\lambda^3 p_l^2 p_{l+1}}$$

which is valid for all $l \geq 2$, we obtain

$$\begin{aligned} |F_{n,m+1}(\lambda) - F_{n,m}(\lambda)| &\leq \frac{1}{(\lambda p_n)^2} \frac{1}{(\lambda p_{n+1})^2} \cdots \frac{1}{(\lambda p_{n+m-1})^2} \frac{1}{\lambda^3 p_{n+m}^2 p_{n+m+1}} \\ &= \frac{1}{\lambda^{2m+3} p_{n+m+1} (p_n p_{n+1} \ldots p_{n+m})^2} \\ &\leq \frac{k^{6m+9}}{2^{2m+3} \lambda^{2m+3}} \left(\frac{(n-1)!^2}{(n+m)!(n+m+1)!} \right)^{3-\beta} \end{aligned}$$

where used (3.4) in the last inequality. Hence,

$$|F_{2,m+l}(\lambda) - F_{2,m}(\lambda)| \leq \sum_{j=0}^{l-1} \frac{k^{6m+6j+9}}{2^{2m+2j+3} \lambda^{2m+2j+3}} \left(\frac{1}{(2+m+j)!(3+m+j)!} \right)^{3-\beta}.$$

Since $3-\beta \geq 1$, the sequence $F_{2,m}(\lambda)$ is uniformly Cauchy on every compact subset of $(0,\infty)$. Therefore, as $m \to \infty$ the sequence $F_{2,m}(\lambda)$ converges uniformly on compact subsets to a continuous function

$$F_2(\lambda) = \cfrac{1}{\lambda p_2 + \cfrac{1}{\lambda p_3 + \cdots}}.$$

Since

$$F_2(\lambda) \leq \frac{1}{\lambda p_2} \tag{3.12}$$

we moreover have $\lim_{\lambda\to\infty} F_2(\lambda) = 0$. The function

$$G(\lambda) = F_2(\lambda) - \lambda p_1 \tag{3.13}$$

is continuous on $(0,\infty)$ and satisfies

$$\lim_{\lambda\to\infty} G(\lambda) = -\infty. \tag{3.14}$$

On the other hand, we have the lower bound

$$G(\lambda) = F_2(\lambda) - \lambda p_1 \geq \frac{1}{\lambda p_2 + \frac{1}{\lambda p_3}} - \lambda p_1 = \frac{1 - \lambda^2 p_1 p_2 - \frac{p_1}{p_3}}{\lambda p_2 + \frac{1}{\lambda p_3}}. \tag{3.15}$$

Since $p_3 > p_1$, it follows that there exists $\lambda_0 > 0$, such that

$$G(\lambda_0) > 0. \tag{3.16}$$

From (3.14), (3.16), and the intermediate value theorem, we conclude that there exists $\lambda = \lambda_k \in (\lambda_0, \infty)$ such that

$$G(\lambda_k) = 0 \tag{3.17}$$

providing a solution of (3.11). Note that (3.15) and (3.17) imply

$$0 \geq 1 - \lambda_k^2 p_1 p_2 - \frac{p_1}{p_3} \tag{3.18}$$

providing a lower bound

$$\lambda_k \geq \frac{\sqrt{1 - p_1/p_3}}{\sqrt{p_1 p_2}}.$$

Recalling (3.3), we obtain

$$\lambda_k \geq \frac{k^3 \sqrt{1 - (1 - 8/(9 + k^2))^{(3-\beta)/2}}}{2(4 + k^2)^{(3-\beta)/2}}$$

for all $k \geq 1$. An explicit computation shows that

$$\lim_{k \to \infty} \frac{1}{k^{\beta-1}} \left(\frac{k^3 \sqrt{1 - (1 - 8/(9 + k^2))^{(3-\beta)/2}}}{2(4 + k^2)^{(3-\beta)/2}} \right) = \sqrt{3 - \beta} \geq 1$$

for any $\beta \in (1, 2]$. Using a careful estimate of the object in parentheses above, we further obtain

$$\lambda_k \geq C^{-1} k^{\beta-1} \tag{3.19}$$

for any $k \geq 1$ and any $\beta \in (1, 2]$, where $C \geq 1$ is a constant (independent of β in this range). In particular, $\lambda_k \to \infty$ as $k \to \infty$ at least as fast as the power law $k^{\beta-1}$.

Having found a root $\lambda = \lambda_k$ of (3.11), we next need to estimate the size of the coefficients c_n defining θ in (3.1), and verify that they decay sufficiently fast so that θ is real-analytic. The function

$$F_n(\lambda) = \cfrac{1}{\lambda p_n + \cfrac{1}{\lambda p_{n+1} + \cdots}},$$

obeys the recurrence relationship

$$F_n(\lambda) = \frac{1}{\lambda p_n + F_{n+1}(\lambda)},$$

or equivalently.

$$F_{n+1}(\lambda) = -\lambda p_n + \frac{1}{F_n(\lambda)}. \tag{3.20}$$

Since also

$$F_2(\lambda_k) = \lambda_k p_1, \tag{3.21}$$

by the definition of λ_k, we get, comparing the recurrence relations (3.9)–(3.10) for η_n and (3.20)–(3.21) for $F_n(\lambda_k)$, that

$$F_n(\lambda_k) = \eta_n.$$

Using $c_n = (p_n/p_{n-1})c_{n-1}\eta_n$ for $n \geq 2$ and

$$F_n(\lambda_k) \leq \frac{1}{\lambda_k p_n}, \qquad n \geq 2$$

we get

$$\begin{aligned} c_n &= \eta_n \cdots \eta_2 \frac{p_n\, c_1}{p_1} = F_n(\lambda_k) \cdots F_2(\lambda_k) \frac{p_n\, c_1}{p_1} \\ &\leq \frac{1}{\lambda_k p_n} \frac{1}{\lambda_k p_{n-1}} \cdots \frac{1}{\lambda_k p_2} \frac{p_n\, c_1}{p_1} = \frac{c_1}{\lambda_k^{n-1} p_{n-1} \ldots p_1} \end{aligned}$$

showing rapid convergence of c_n to 0. Without loss of generality we choose

$$c_1 = 1.$$

Then, by (3.4) and (3.19) we have

$$c_n \leq C^n \left(\frac{1}{k^{\beta-1}}\right)^{n-1} \frac{(k^3)^{n-1}}{((n-1)!)^{3-\beta}} \leq n^2 C^n \left(k^{4-\beta}\right)^2 (n!)^{\beta-3} \tag{3.22}$$

for all $k, n \geq 1$. Furthermore, using $n! \geq (n/C)^n$ for a sufficiently large C, we obtain

$$c_n \leq n^2 \exp\left(n \log\left(C k^{4-\beta} n^{\beta-3}\right)\right) \tag{3.23}$$

for all $\beta \in (1, 2]$ and $n, k \geq 1$. In particular, for any $\tau > 0$, we see that for

$$n \geq n(\tau, k, \beta) := C e^{4\tau} k^{(4-\beta)/(3-\beta)}, \tag{3.24}$$

where C is sufficiently large, it holds that

$$c_n \leq (n^2 + k^2) \exp\left(-2\tau (n^2 + k^2)^{1/2}\right).$$

Using (3.1), we see that for $\ell = (\ell_1, \ell_2) \in \mathbb{Z}^2$, we have

$$|\hat{\theta}_k(\ell)|^2 = \begin{cases} c_n^2, & \ell_1 = \pm k, \ell_2 = \pm n, \\ 0, & |\ell_1| \neq k. \end{cases}$$

Therefore,

$$\begin{aligned} \|\theta_k\|^2_{G^1_\tau} &= \sum_{n\geq 1} c_n^2 (k^2 + n^2)^2 \exp\left(2\tau (k^2+n^2)^{1/2}\right) \\ &\leq \sum_{n=1}^{n(\tau,k,\beta)} (k^2 + n^2)^4 \exp\left(2\tau (k^2+n^2)^{1/2} + 2n \log\left(\frac{C k^{4-\beta}}{n^{3-\beta}}\right)\right) \\ &\quad + \sum_{n > n(\tau,k,\beta)} (n^2 + k^2)^4 \exp\left(-2\tau (n^2+k^2)^{1/2}\right) < \infty \end{aligned}$$

which shows that the function θ_k whose Fourier series n-th coefficients is c_n, is in fact entire real-analytic.

Next, we provide a proof of (2.8). For $s \geq 0$, recalling (3.24) we estimate

$$\begin{aligned} \|\theta_k\|^2_{H^s} &= \sum_{n\geq 1} (n^2 + k^2)^s c_n^2 \\ &\leq 2^s n(1,k,\beta)^{2s} \sum_{n=1}^{n(1,k,\beta)} c_n^2 + \sum_{n > n(1,k,\beta)} (n^2+k^2)^{s+2} \exp\left(-2(n^2+k^2)^{1/2}\right) \\ &\leq C_s k^{2s(4-\beta)/(3-\beta)} \|\theta_k\|^2_{L^2} + C_s \exp\left(-k^{(4-\beta)/(3-\beta)}\right) \\ &\leq 2 C_s k^{2s(4-\beta)/(3-\beta)} \|\theta_k\|^2_{L^2} \end{aligned} \tag{3.25}$$

where $C_s > 0$ is a sufficiently large constant that depends only on s, and is independent of $\beta \in (1,2]$; we have also used $\|\theta_k\|_{L^2}^2 \geq 1$, which follows from $c_1 = 1$. Since the equation $L\theta_k = \lambda_k \theta_k$ is linear, we may renormalize θ_k to have the unit H^s norm. Then, upon multiplying both sides of (3.25) with this constant, we obtain (2.8).

The proof of (2.9) is similar. Given $s \geq 1$ and $\tau > 0$ we have

$$\begin{aligned}\|\theta_k\|_{G^s_\tau}^2 &= \sum_{n\geq 1} c_n^2 (k^2+n^2)^2 \exp\left(2\tau (k^2+n^2)^{1/2s}\right) \\ &\leq n(2\tau,k,\beta)^4 \exp\left(4\tau (n(2\tau,k,\beta))^{1/s}\right) \sum_{n=1}^{n(2\tau,k,\beta)} c_n^2 \\ &\quad + \sum_{n>n(2\tau,k,\beta)} (k^2+n^2)^4 \exp\left(2\tau(k^2+n^2)^{1/(2s)} - 4\tau(n^2+k^2)^{1/2}\right) \\ &\leq 2C_{s,\tau} \exp\left(C_{s,\tau} k^{(4-\beta)/(s(3-\beta))}\right) \|\theta_k\|_{L^2}^2 \end{aligned} \tag{3.26}$$

where we have also used that $c_1 = 1$.

In order to conclude the proof we renormalize θ_k to have unit G^s_τ norm, and then multiply both sides of (3.26) with this constant we obtain (2.9). $\square$

4. Proof of nonlinear Lipschitz ill-posedness

PROOF FOR THEOREM 2.1. The proof of the above theorem follows directly from Theorem 2.3 and a classical perturbative argument (see, e.g. [**25, 26, 32, 29, 30**]), so we only give here a sketch of these details.

Fix $\theta_0^{(1)}(x_2) = \theta^{(1)}(x_2,t) = \cos(x_2)$ a steady state of the system (1.1)–(1.3) with (1.6). Clearly $\|\theta_0^{(1)}\|_{H^s} = 1$. For $\epsilon \in (0,1]$, let $\theta_0^{(2,\epsilon)}(x) = \theta_0^{(1)}(x_2) + \epsilon\psi_0(x)$ where $\psi_0(x)$ is a smooth function such that $\|\psi_0(x)\|_{H^s} = 1$, to be determined below. We have $\|\theta_0^{(2,\epsilon)}\|_{H^s} \leq 2$ for all $\epsilon \in (0,1]$.

If the system (1.1)–(1.3) with (1.6) would be locally Lipschitz well posed, then there would exist $T_0, K_0 > 0$ (uniform in ϵ), such that the family of solutions $\theta^{(2,\epsilon)}(x,t) \in L^\infty(0,T;H^r)$ of (1.1)–(1.3) with (1.6) with the initial datum $\theta_0^{(2,\epsilon)}(x)$ obey

$$\sup_{t\in[0,T_0]} \|\theta^{(2,\epsilon)}(t) - \theta_0^{(1)}\|_{H^r} \leq K_0 \|\theta_0^{(2,\epsilon)} - \theta_0^{(1)}\|_{H^s} = K_0\epsilon \tag{4.1}$$

for all $\epsilon \in (0,1]$. We define

$$\psi_\epsilon(x,t) = \epsilon^{-1}\left(\theta^{(2,\epsilon)}(x,t) - \theta_0^{(1)}\right),$$

to be the ϵ-perturbation of $\theta^{(2,\epsilon)}$ about $\theta_0^{(1)}$. In view of (4.1) we have that ψ_ϵ is uniformly bounded in $L^\infty(0,T;H^r)$, with the bound

$$\sup_{t\in[0,T_0]} \|\psi_\epsilon(t)\|_{H^r} \leq K_0. \tag{4.2}$$

The equation obeyed by ψ_ϵ is

$$\partial_t \psi_\epsilon = L\psi_\epsilon - \epsilon N[\psi_\epsilon]$$

so that by (2.5)–(2.6) we have that $\partial_t\psi_\epsilon$ is uniformly bounded in $L^\infty(0,T;L^2)$. Thus, Aubin–Lions lemma $\psi_\epsilon \to \psi$ in $L^2(0,T;L^2)$, where $\psi \in L^\infty(0,T;H^r)$ with a bound inherited from (4.2), solves the equation

$$\partial_t \psi = L\psi, \qquad \psi(0) = \psi_0.$$

In order to conclude the proof, let $\psi_0(x) = \theta_k(x)$, where θ_k is an eigenfunction of L as constructed in Theorem 2.3, with eigenvalue λ_k, and we choose k large enough such that

$$\frac{\exp\left(T_0 k^{\beta-1}/(2C_0)\right)}{C_s k^{s(4-\beta)/(3-\beta)}} \geq 2K_0$$

where C_0 is the constant from (2.7), and C_s is the constant from (2.8). Then using (2.8) we obtain that

$$\|\psi(T_0/2)\|_{L^2} = \exp(\lambda_k T_0/2)\|\psi_0\|_{L^2} \geq 2K_0$$

which contradicts (4.2) since $r > 0$, thereby concluding the proof. $\square$

PROOF OF THEOREM 2.2. The proof is the same as the proof of Theorem 2.1, except that the eigenfunction $\psi_0 = \theta_k$ is normalized to have a unit G^s_τ norm, and we pick k large enough so that

$$\frac{\exp(T_0 k^{\beta-1}/(2C_0))}{C_{s,\tau}\exp(C_{s,\tau} k^{(4-\beta)/(s(3-\beta))})} \geq 2K_0,$$

where C_0 is the constant from (2.7), and $C_{s,\tau}$ is the constant from (2.9). Finding such a values of k uses the restriction on s. Using (2.9) we then obtain that $\|\psi(T_0/2)\|_{L^2} \geq 2K_0$ which yields a contradiction. $\square$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CA 90089
E-mail address: kukavica@usc.edu

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ 08544
E-mail address: vvicol@math.princeton.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CA 90089
E-mail address: wang828@usc.edu