# A direct approach to Gevrey regularity on the half-space 

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#### Abstract

We consider the inhomogeneous heat and Stokes equations on the half space and prove an instantaneous space-time analytic regularization result, uniformly up to the boundary of the half space. February 19, 2018


## 1. Introduction

We consider the Dirichlet problem for the inhomogeneous heat equation

$$
\begin{align*}
\partial_{t} u-\Delta u & =f, & & \text { in } \Omega,  \tag{1.1}\\
u & =0, & & \text { on } \partial \Omega, \tag{1.2}
\end{align*}
$$

with the initial condition

$$
\begin{equation*}
u(x, 0)=u_{0}(x), \quad \text { in } \Omega, \tag{1.3}
\end{equation*}
$$

posed in the half space

$$
\begin{equation*}
\Omega=\left\{x=\left(x_{1}, \ldots, x_{d}\right) \in \mathbb{R}^{d}: x_{d}>0\right\} . \tag{1.4}
\end{equation*}
$$

Throughout the paper we denote by $\bar{\partial}$ the vector of tangential derivatives $\bar{\partial}=\left(\partial_{1}, \ldots, \partial_{d-1}\right)$.
In this paper we prove that from an initial datum of finite Sobolev regularity, the solution to (1.1)-(1.4) instantly becomes real-analytic, jointly in space and time, with the analyticity radius which is uniform up to the boundary of $\partial \Omega$. The result holds under the assumption that the force is real-analytic in space-time. In order to state the main result of the paper, we introduce first some notation. For $r \geq 1$, define the index sets

$$
\begin{equation*}
B=\left\{(i, j, k): i, j, k \in \mathbb{N}_{0}, i+j+k \geq r\right\}, \quad B^{c}=\mathbb{N}_{0}^{3} \backslash B . \tag{1.5}
\end{equation*}
$$

Fix $T>0$, let $0<\tilde{\epsilon}, \bar{\epsilon}, \epsilon \leq 1$, and define the sum

$$
\begin{align*}
\phi(u) & =\sum_{B} \frac{(i+j+k)^{r} \epsilon^{i} \epsilon^{j} \bar{\epsilon}^{k}}{(i+j+k)!}\left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}([0, T] \times \Omega)}+\sum_{B^{c}}\left\|\partial_{t}^{i} \partial_{d}^{j} \bar{\partial}^{k}\right\|_{L_{x, t}^{2}([0, T] \times \Omega)}  \tag{1.6}\\
& =\bar{\phi}(u)+\phi_{0}(u) .
\end{align*}
$$

We note that $\phi_{0}(u)$ is the $H^{r-1}([0, T] \times \Omega)$ norm. In particular, it is well known (see e.g. [20,5]) that for smooth and compatible initial datum which vanishes on $\partial \Omega$, for instance $u_{0} \in H_{0}^{1}(\Omega) \cap H^{2(r-1)}(\Omega)$ is sufficient, and force $f \in H_{t, x}^{2(r-2)_{+}}((0, T) \times \Omega)$ we have that

$$
\begin{equation*}
\phi_{0}(u) \lesssim\left\|u_{0}\right\|_{H^{2(r-1)}(\Omega)}+\|f\|_{H^{2(r-2)}+((0, T) \times \Omega)} . \tag{1.7}
\end{equation*}
$$

The following is our main result.
THEOREM 1.1. Let $T>0$ and $r \geq 1$. Then there exist $\epsilon, \tilde{\epsilon}, \bar{\epsilon} \in(0,1]$, which only depend on $T, r$, and $d$, such that for any $u_{0} \in H_{0}^{1}(\Omega) \cap H^{2(r-1)}(\Omega)$ which satisfies the compatibility conditions, and $f$ sufficiently

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smooth, the solution $u$ of (1.1)-(1.3) satisfies the estimate

$$
\begin{align*}
\phi(u) \lesssim \phi_{0}(u) & +\sum_{i+j+k \geq(r-2)_{+}} \frac{(i+j+k+2)^{r} \epsilon^{i} \tilde{\epsilon}^{j+2} \bar{\epsilon}^{k}}{(i+j+k+2)!}\left\|t^{i+j+k+2-r} \partial_{t}^{i} \partial_{d}^{j} \bar{\partial}^{k} f\right\|_{L_{x, t}^{2}((0, T) \times \Omega)} \\
& +\sum_{i+k \geq(r-2)_{+}} \frac{(i+k+2)^{r} \epsilon^{i} \bar{\epsilon}^{k+2}}{(i+k+2)!}\left\|t^{i+k+2-r} \partial_{t}^{i} \bar{\partial}^{k} f\right\|_{L_{x, t}^{2}((0, T) \times \Omega)} \\
& +\sum_{i \geq r-1} \frac{(i+1)^{r} \epsilon^{i+1}}{(i+1)!}\left\|t^{i+1-r} \partial_{t}^{i} f\right\|_{L_{x, t}^{2}((0, T) \times \Omega)} . \tag{1.8}
\end{align*}
$$

Here we assume on $f$ that the norms in (1.7) above and (1.8) below are finite.
REMARK 1.2. It is clear from the proof of Theorem 1.1 that the Gevrey-s regularization, jointly in space and time, also follows. For this purpose, simply replace $(i+j+k)$ ! with $((i+j+k)!)^{s}$, where $s>1$, throughout the paper.

Combining (1.7) and (1.8), we conclude that for an initial datum $u_{0}$ of finite Sobolev regularity, and real-analytic force $f$, the solution $u$ of (1.1)-(1.3) becomes real-analytic, jointly in space and time, for any $t \in(0, T]$, with an analyticity radius that is uniform up to $\partial \Omega$. This latter fact concerning the analyticity radius which does not vanish as one approaches $\partial \Omega$ is the main result of the paper. For a direct method for proving analyticity in the interior, see $[\mathbf{1 , 1 1 ]}$.

The method used to prove Theorem 1.1 extends easily to the case of the inhomogeneous Stokes system on the half space

$$
\begin{align*}
\partial_{t} u-\Delta u+\nabla p & =f, & & \text { in } \Omega \\
\nabla \cdot u & =0, & & \text { in } \Omega \tag{1.9}
\end{align*}
$$

with the Dirichlet boundary condition

$$
\begin{equation*}
u=0, \quad \text { on } \partial \Omega \tag{1.10}
\end{equation*}
$$

For $T>0$, smooth and compatible initial datum $u_{0}$, and sufficiently smooth force $f$, it is known (see e.g. [25, Chapter III]) that $\phi_{0}(u)$ is a priori bounded in terms of $u_{0}$ and $f$, similarly to (1.7). With the notation of Theorem 1.1 we have that

THEOREM 1.3. Let $T>0$ and $r \geq 1$. Then there exist $\epsilon, \tilde{\epsilon}, \bar{\epsilon} \in(0,1]$, which only depend on $T$, $r$, and $d$, such that for any sufficiently smooth $u_{0}$ which satisfies compatibility conditions, and sufficiently smooth $f$, the solution $u$ of the Cauchy problem for (1.9)-(1.10) satisfies the estimate (1.8).

The motivation for treating the inhomogeneous linear heat (1.1) and Stokes (1.9) equations comes from the study of nonlinear semi-linear problems (i.e. $f=f(t, x, u, \nabla u)$ ) for the heat and Stokes equations on domains with boundary, with Dirichlet boundary conditions. These nonlinear problems will be addressed in our forthcoming work [4].

In the absence of boundaries, analytic and Gevrey-class regularization results are well-known, even in the context of nonlinear problems (see e.g. [8] for the Navier-Stokes equations and [16, 21, 23] for other models). On domains with boundary, analyticity up to the boundary was obtained in the fundamental works $[13,14]$ (see also $[9,15]$ ). These classical works achieve real-analyticity based on an anduction scheme on the number of derivatives. In comparison, our proof is based directly on classical energy inequalities for the heat and Laplace equations on $\Omega$. Moreover, Theorem 1.1 obtains the analyticity in the time variable concomitantly. We believe that this transparent proof is going to be useful in establishing realanalytic and Gevrey regularization results for PDEs with different types of boundary conditions for which the methods of [8] work in the absence of boundaries. In fact, our motivation for the present paper comes from the simplicity of the approach in [8] in the case when the boundaries are not present. For other works, using the Gevrey regularity method of Foias and Temam, see [2, 3, 7, 22] (cf. [10, 17] for an alternative
approach to analyticity in domains without the boundary, to $[\mathbf{9 , 1 2}, \mathbf{1 6}, \mathbf{1 8}, \mathbf{2 3}]$ for some other results on analyticity, and to [6] for some applications).

The main idea in the proof of (1.8) is to split the sum defining $\bar{\phi}(u)$ into several sub-sums, and on each one perform a derivative reduction estimate, as specified in Section 3. These estimates follow from the classical maximal regularity of Laplace's equation in $\Omega$ (cf. Section 2 ), and an energy estimate for solutions to (1.1)-(1.3). These reduction estimates are used in Section 4 to conclude the proof of Theorem 1.1.

## 2. Preliminaries

We use the following notational agreement: If the domain in the Sobolev space is not indicated, it is either $\Omega$ or $\Omega \times(0, T)$, where $T>0$ is a fixed parameter, depending on the context. Also, definition (1.6), the meaning of $\bar{\partial}^{k}$ for $k \in \mathbb{N}_{0}$, is that

$$
\left\|\partial_{t}^{i} \partial_{d}^{j} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}}=\sum_{\alpha \in \mathbb{N}_{0}^{d-1},|\alpha|=k}\left\|\partial_{t}^{i} \partial_{d}^{j} \partial^{\alpha} u\right\|_{L_{x, t}^{2}}
$$

We recall (cf. [19]) a simple statement on interpolation, which asserts that

$$
\begin{equation*}
\|\nabla u\|_{L^{2}(\Omega)} \lesssim\|u\|_{L^{2}(\Omega)}^{1 / 2}\|u\|_{\dot{H}^{2}(\Omega)}^{1 / 2}+\|u\|_{L^{2}(\Omega)} \tag{2.1}
\end{equation*}
$$

holds for $u \in H^{2}(\Omega)$. The proof uses a Sobolev extension operator and the interpolation inequality in $\mathbb{R}^{d}$.
Besides (2.1), we shall also use the $H^{2}$ regularity for Laplace's equation

$$
\begin{equation*}
\Delta u=g, \quad \text { in } \Omega . \tag{2.2}
\end{equation*}
$$

For the problem (2.2), we have

$$
\begin{equation*}
\|u\|_{\dot{H}^{2}(\Omega)} \lesssim\|g\|_{L^{2}(\Omega)}+\|\bar{\partial} u\|_{\dot{H}^{1}(\Omega)}+\|u\|_{L^{2}(\Omega)} . \tag{2.3}
\end{equation*}
$$

If, in addition, $\left.u\right|_{\partial \Omega}=0$, then we have

$$
\begin{equation*}
\|u\|_{\dot{H}^{2}(\Omega)} \lesssim\|g\|_{L^{2}(\Omega)} . \tag{2.4}
\end{equation*}
$$

The estimate (2.4) follows from the $H^{2}$ inequality for the odd extension of $u$ to $\mathbb{R}^{d}$, while the bound (2.3) follows from the $H^{2}$ regularity for the problem (2.2) (cf. [19]), the trace theorem, and the interpolation inequality (2.1) as follows

$$
\begin{aligned}
\|u\|_{H^{2}} & \lesssim\|g\|_{L^{2}}+\|u\|_{H^{3 / 2}(\partial \Omega)} \lesssim\|g\|_{L^{2}}+\|\bar{\partial} u\|_{H^{1 / 2}(\partial \Omega)} \lesssim\|g\|_{L^{2}}+\|\bar{\partial} u\|_{H^{1}} \\
& \lesssim\|g\|_{L^{2}}+\|\bar{\partial} u\|_{\dot{H}^{1}}+\|u\|_{H^{1}} \lesssim\|g\|_{L^{2}}+\|\bar{\partial} u\|_{\dot{H}^{1}}+\|u\|_{L^{2}}+\|u\|_{L^{2}}^{1 / 2}\|u\|_{\dot{H}^{2}}^{1 / 2} .
\end{aligned}
$$

The estimate (2.3) follows from the $\epsilon$-Young inequality.

## 3. Derivative reduction

In this section we give the proofs of the normal, tangential, and time derivative reduction estimates which are the main ingredients in the proof of Theorem 1.1.

Let $u$ be a smooth solution of (1.1)-(1.3). Throughout this section we require that the nonnegative integers $i, j, k$ obey $i+j+k \geq r$.
3.1. Normal derivative reduction. For $j \geq 2$ we claim that

$$
\begin{align*}
& \left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}} \\
& \quad \lesssim\left\|t^{i+j+k-r} \partial_{t}^{i+1} \partial_{d}^{j-2} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}}+\left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j-1} \bar{\partial}^{k+1} u\right\|_{L_{x, t}^{2}}+\left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j-2} \bar{\partial}^{k+2} u\right\|_{L_{x, t}^{2}} \\
& \quad+\left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j-2} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}}+\left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j-2} \bar{\partial}^{k} f\right\|_{L_{x, t}^{2}}^{2} \tag{3.1}
\end{align*}
$$

This inequality allows us to reduce the number of vertical derivatives ( $\partial_{d}$ ) in the Gevrey (analytic) norm. On the other hand, for $j=1$ and $k \geq 1$ we have

$$
\begin{equation*}
\left\|t^{i+1+k-r} \partial_{t}^{i} \partial_{d} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}} \lesssim\left\|t^{i+1+k-r} \partial_{t}^{i+1} \bar{\partial}^{k-1} u\right\|_{L_{x, t}^{2}}+\left\|t^{i+1+k-r} \partial_{t}^{i} \bar{\partial}^{k-1} f\right\|_{L_{x, t}^{2}}, \tag{3.2}
\end{equation*}
$$

while for $j=1$ and $k=0$, we claim

$$
\begin{equation*}
\left\|t^{i+1-r} \partial_{t}^{i} \nabla u\right\|_{L_{x, t}^{2}} \lesssim\left\|t^{i+1-r} \partial_{t}^{i} u\right\|_{L_{x, t}^{2}}^{1 / 2}\left\|t^{i+1-r} \partial_{t}^{i+1} u\right\|_{L_{x, t}^{2}}^{1 / 2}+\left\|t^{i+1-r} \partial_{t}^{i} u\right\|_{L^{2}}+\left\|t^{i+1-r} \partial_{t}^{i} f\right\|_{L_{x, t}^{2}} \tag{3.3}
\end{equation*}
$$

whenever $i \geq r$. The remainder of this subsection contains the proofs of (3.1)-(3.3).
Proof of (3.1). We first use (1.1) to compute

$$
\begin{align*}
\Delta\left(t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j-2} \bar{\partial}^{k} u\right) & =t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j-2} \bar{\partial}^{k} \Delta u \\
& =t^{i+j+k-r} \partial_{t}^{i+1} \partial_{d}^{j-2} \bar{\partial}^{k} u-t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j-2} \bar{\partial}^{k} f \tag{3.4}
\end{align*}
$$

Using the $H^{2}$-regularity estimate (2.3), we get

$$
\begin{aligned}
\left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j} \bar{\partial}^{k} u\right\|_{L^{2}} & \leq\left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j-2} \bar{\partial}^{k} u\right\|_{\dot{H}^{2}} \\
& \lesssim\left\|t^{i+j+k-r} \partial_{t}^{i+1} \partial_{d}^{j-2} \bar{\partial}^{k} u\right\|_{L^{2}}+\left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j-2} \bar{\partial}^{k} f\right\|_{L^{2}} \\
& +\left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j-2} \bar{\partial}^{k+1} u\right\|_{\dot{H}^{1}}+\left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j-2} \bar{\partial}^{k} u\right\|_{L^{2}}
\end{aligned}
$$

and (3.1) follows.

Proof of (3.2). Let $k \geq 1$. First, set $j=2$ in (3.4) and replace $k$ with $k-1$. We obtain

$$
\begin{aligned}
\Delta\left(t^{i+1+k-r} \partial_{t}^{i} \bar{\partial}^{k-1} u\right) & =t^{i+1+k-r} \partial_{t}^{i} \bar{\partial}^{k-1} \Delta u \\
& =t^{i+1+k-r} \partial_{t}^{i+1} \bar{\partial}^{k-1} u-t^{i+1+k-r} \partial_{t}^{i} \bar{\partial}^{k-1} f
\end{aligned}
$$

Since $\left.\partial_{t}^{i} \bar{\partial}^{k-1} u\right|_{\partial \Omega}=0$, we may apply (2.4) to this equation, leading to

$$
\begin{aligned}
\left\|t^{i+1+k-r} \partial_{t}^{i} \partial_{d} \bar{\partial}^{k} u\right\|_{L^{2}} & \leq\left\|t^{i+1+k-r} \partial_{t}^{i} \bar{\partial}^{k-1} u\right\|_{\dot{H}^{2}} \\
& \lesssim\left\|t^{i+1+k-r} \partial_{t}^{i+1} \bar{\partial}^{k-1} u\right\|_{L^{2}}+\left\|t^{i+1+k-r} \partial_{t}^{i} \bar{\partial}^{k-1} f\right\|_{L^{2}}
\end{aligned}
$$

and we obtain (3.2).

Proof of (3.3). First, we have

$$
\Delta\left(t^{i+1-r} \partial_{t}^{i} u\right)=t^{i+1-r} \partial_{t}^{i} \Delta u=t^{i+1-r} \partial_{t}^{i+1} u-t^{i+1-r} \partial_{t}^{i} f
$$

Using that the $H^{1}$ norm may be interpolated as in (2.1), and applying the $H^{2}$ regularity estimate (2.4), which may be used since $\left.\partial_{t}^{i} u\right|_{\partial \Omega}=0$, we get

$$
\begin{aligned}
\left\|t^{i+1-r} \partial_{t}^{i} u\right\|_{\dot{H}^{1}} & \lesssim\left\|t^{i+1-r} \partial_{t}^{i} u\right\|_{\dot{H}^{2}}^{1 / 2}\left\|t^{i+1-r} \partial_{t}^{i} u\right\|_{L^{2}}^{1 / 2}+\left\|t^{i+1-r} \partial_{t}^{i} u\right\|_{L^{2}} \\
& \lesssim\left(\left\|t^{i+1-r} \partial_{t}^{i+1} u\right\|_{L^{2}}+\left\|t^{i+1-r} \partial_{t}^{i} f\right\|_{L^{2}}\right)^{1 / 2}\left\|t^{i+1-r} \partial_{t}^{i} u\right\|_{L^{2}}^{1 / 2}+\left\|t^{i+1-r} \partial_{t}^{i} u\right\|_{L^{2}} \\
& \lesssim\left\|t^{i+1-r} \partial_{t}^{i+1} u\right\|_{L^{2}}^{1 / 2}\left\|t^{i+1-r} \partial_{t}^{i} u\right\|_{L^{2}}^{1 / 2}+\left\|t^{i+1-r} \partial_{t}^{i} f\right\|_{L^{2}}+\left\|t^{i+1-r} \partial_{t}^{i} u\right\|_{L^{2}}
\end{aligned}
$$

The inequality (3.3) then follows.
3.2. Tangential derivative reduction. In order to reduce the number of tangential derivatives, for $k \geq$ 2 we claim that

$$
\begin{equation*}
\left\|t^{i+k-r} \partial_{t}^{i} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}} \lesssim\left\|t^{i+k-r} \partial_{t}^{i+1} \bar{\partial}^{k-2} u\right\|_{L_{x, t}^{2}}+\left\|t^{i+k-r} \partial_{t}^{i} \bar{\partial}^{k-2} f\right\|_{L_{x, t}^{2}}, \tag{3.5}
\end{equation*}
$$

while for $k=1$, we use

$$
\begin{equation*}
\left\|t^{i+1-r} \partial_{t}^{i} \bar{\partial} u\right\|_{L_{x, t}^{2}} \lesssim\left\|t^{i+1-r} \partial_{t}^{i} u\right\|_{L_{x, t}^{2}}^{1 / 2}\left\|t^{i+1-r} \partial_{t}^{i+1} u\right\|_{L_{x, t}^{2}}^{1 / 2}+\left\|t^{i+1-r} \partial_{t}^{i} u\right\|_{L^{2}}+\left\|t^{i+1-r} \partial_{t}^{i} f\right\|_{L_{x, t}^{2}} \tag{3.6}
\end{equation*}
$$

for all $i \geq r$.
Proof of (3.5). We first set $j=2$ in (3.4) and obtain

$$
\begin{aligned}
\Delta\left(t^{i+2+k-r} \partial_{t}^{i} \bar{\partial}^{k} u\right) & =t^{i+2+k-r} \partial_{t}^{i} \bar{\partial}^{k} \Delta u \\
& =t^{i+2+k-r} \partial_{t}^{i+1} \bar{\partial}^{k} u-t^{i+2+k-r} \partial_{t}^{i} \bar{\partial}^{k} f .
\end{aligned}
$$

Since $t^{i+2+k-r} \partial_{t}^{i} \bar{\partial}^{k} u$ vanishes on $\partial \Omega$, we have by (2.4)

$$
\begin{aligned}
\left\|t^{i+2+k-r} \partial_{t}^{i} \bar{\partial}^{k+2} u\right\|_{L^{2}} & \leq\left\|t^{i+2+k-r} \partial_{t}^{i} \bar{\partial}^{k} u\right\|_{\dot{H}^{2}} \\
& \lesssim\left\|t^{i+2+k-r} \partial_{t}^{i+1} \bar{\partial}^{k} u\right\|_{L^{2}}+\left\|t^{i+2+k-r} \partial_{t}^{i} \bar{\partial}^{k} f\right\|_{L^{2}}
\end{aligned}
$$

for $k \geq 0$. The bound (3.5) follows upon replacing $k+2$ with $k$.

PROOF OF (3.6). This inequality is obtain by replacing $\nabla$ with $\bar{\partial}$ in the inequality (3.3).
3.3. The time derivative reduction. Here we claim that for all $i \geq r+1$ we have

$$
\begin{equation*}
\left\|t^{i-r} \partial_{t}^{i} u\right\|_{L_{x, t}^{2}} \lesssim(i-r)\left\|t^{i-1-r} \partial_{t}^{i-1} u\right\|_{L_{x, t}^{2}}+\left\|t^{i-r} \partial_{t}^{i-1} f\right\|_{L_{x, t}^{2}} . \tag{3.7}
\end{equation*}
$$

Proof of (3.7). For the system (1.1), we have the energy inequality

$$
\begin{equation*}
\left\|\partial_{t} u\right\|_{L_{x, t}^{2}}+\left.\left.\|\nabla u\|_{L_{x}^{2}}\right|_{t=T} \lesssim\|\nabla u\|_{L_{x}^{2}}\right|_{t=0}+\|f\|_{L_{x, t}^{2}}, \tag{3.8}
\end{equation*}
$$

which is obtained by testing (1.1) with $\partial_{t} u$ (which obeys the homogeneous Dirichlet boundary condition (1.3)). We apply the estimate (3.8) to the equation

$$
\partial_{t}\left(t^{i-r} \partial_{t}^{i-1} u\right)-\Delta\left(t^{i-r} \partial_{t}^{i-1} u\right)=(i-r) t^{i-1-r} \partial_{t}^{i-1} u+t^{i-r} \partial_{t}^{i-1} f
$$

If $i \geq r+1$, then $\left.t^{i-r} \partial_{t}^{i-1} u\right|_{t=0}=0$, so that the initial value term in (3.8) vanishes, and we obtain

$$
\left\|\partial_{t}\left(t^{i-r} \partial_{t}^{i-1} u\right)\right\|_{L^{2}} \lesssim\left\|(i-r) t^{i-1-r} \partial_{t}^{i-1} u\right\|_{L^{2}}+\left\|t^{i-r} \partial_{t}^{i-1} f\right\|_{L^{2}}
$$

The inequality (3.7) now follows from the product rule.

## 4. Proof of Theorem 1.1

With the notation of (1.6), our goal is to establish an inequality of the type

$$
\begin{equation*}
\bar{\phi}(u) \leq C \phi_{0}(u)+\frac{1}{2} \phi(u)+C\|f\| \tag{4.1}
\end{equation*}
$$

where $\|f\|$ denotes a suitable (semi)-norm of $f$. Adding $\phi_{0}(u)$ to both sides of the above estimate, and absorbing the $\phi(u) / 2$ term on the left side yields

$$
\phi(u) \lesssim \phi_{0}(u)+\|f\|
$$

which concludes the proof, in view of (1.7) and the assumed regularity of the force $f$.

We split the sum $\bar{\phi}(u)$ according to the values of $i, j$, and $k$, so that the inequalities in Section 3 may be applied. Recall the definition of $B$ in (1.5). We split the first sum in (1.6) as

$$
\bar{\phi}(u)=\sum_{\ell=1}^{6} S_{\ell} \quad \text { where } \quad S_{\ell}=\sum_{B_{\ell}} \frac{(i+j+k)^{r} \epsilon^{i} \tilde{\epsilon}^{j} \bar{\epsilon}^{k}}{(i+j+k)!}\left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}}, \quad \ell=1, \ldots, 6
$$

and

$$
\begin{aligned}
& B_{1}=\{(i, j, k) \in B: j \geq 2\} \\
& B_{2}=\{(i, j, k) \in B: j=1, k \geq 1\} \\
& B_{3}=\{(i, j, k) \in B: j=1, k=0\} \\
& B_{4}=\{(i, j, k) \in B: j=0, k \geq 2\} \\
& B_{5}=\{(i, j, k) \in B: j=0, k=1\} \\
& B_{6}=\{(i, j, k) \in B: j=0, k=0\} .
\end{aligned}
$$

The above sums are bounded according to (3.1), (3.2), (3.3), (3.5), (3.6), and (3.7), respectively.
4.1. The $S_{1}$ term. We start with $S_{1}$, which may be estimated using (3.1) as

$$
\begin{aligned}
& S_{1} \lesssim \sum_{B_{1}} \\
& \frac{(i+j+k)^{r} \epsilon^{i} \tilde{\epsilon}^{j} \bar{\epsilon}^{k}}{(i+j+k)!}\left\|t^{i+j+k-r} \partial_{t}^{i+1} \partial_{d}^{j-2} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}} \\
& \quad+\sum_{B_{1}} \frac{(i+j+k)^{r} \epsilon^{i} \tilde{\epsilon}^{j} \bar{\epsilon}^{k}}{(i+j+k)!}\left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j-1} \bar{\partial}^{k+1} u\right\|_{L_{x, t}^{2}} \\
& \quad+\sum_{B_{1}} \frac{(i+j+k)^{r} \epsilon^{i} \tilde{\epsilon}^{j} \bar{\epsilon}^{k}}{(i+j+k)!}\left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j-2} \bar{\partial}^{k+2} u\right\|_{L_{x, t}^{2}} \\
& \quad+\sum_{B_{1}} \frac{(i+j+k)^{r} \epsilon^{i} \tilde{\epsilon}^{j} \bar{\epsilon}^{k}}{(i+j+k)!}\left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j-2} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}} \\
& \quad+\sum_{B_{1}} \frac{(i+j+k)^{r} \epsilon^{i} \tilde{\epsilon}^{j} \bar{\epsilon}^{k}}{(i+j+k)!}\left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j-2} \bar{\partial}^{k} f\right\|_{L_{x, t}^{2}} .
\end{aligned}
$$

By relabeling, we obtain

$$
\begin{aligned}
S_{1} \lesssim & \sum_{(i-1, j+2, k) \in B_{1}} \frac{(i+j+k+1)^{r} \epsilon^{i-1} \tilde{\epsilon}^{j+2} \bar{\epsilon}^{k}}{(i+j+k+1)!}\left\|t^{i+j+k+1-r} \partial_{t}^{i} \partial_{d}^{j} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}} \\
& +\sum_{(i, j+1, k-1) \in B_{1}} \frac{(i+j+k)^{r} \epsilon^{i} \epsilon^{j+1} \bar{\epsilon}^{k-1}}{(i+j+k)!}\left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}} \\
& +\sum_{(i, j+2, k-2) \in B_{1}} \frac{(i+j+k)^{r} \epsilon^{i} \epsilon^{j+2} \bar{\epsilon}^{k-2}}{(i+j+k)!}\left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}} \\
& +\sum_{(i, j+2, k) \in B_{1}} \frac{(i+j+k+2)^{r} \epsilon^{i} \tilde{\epsilon}^{j+2} \bar{\epsilon}^{k}}{(i+j+k+2)!}\left\|t^{i+j+k+2-r} \partial_{t}^{i} \partial_{d}^{j} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}} \\
& +\sum_{(i, j+2, k) \in B_{1}} \frac{(i+j+k+2)^{r} \epsilon^{i} \tilde{\epsilon}^{j+2} \bar{\epsilon}^{k}}{(i+j+k+2)!}\left\|t^{i+j+k+2-r} \partial_{t}^{i} \partial_{d}^{j} \bar{\partial}^{k} f\right\|_{L_{x, t}^{2}} .
\end{aligned}
$$

Therefore, since $t \leq T$, we get

$$
\begin{aligned}
S_{1} \lesssim & \frac{T \tilde{\epsilon}^{2}}{\epsilon} \sum_{(i-1, j+2, k) \in B_{1}} \frac{(i+j+k+1)^{r} \epsilon^{i} \tilde{\epsilon}^{j} \bar{\epsilon}^{k}}{(i+j+k+1)!}\left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}} \\
& +\frac{\tilde{\epsilon}}{\epsilon} \sum_{(i, j+1, k-1) \in B_{1}} \frac{(i+j+k)^{r} \epsilon^{i} \tilde{\epsilon}^{j} \bar{\epsilon}^{k}}{(i+j+k)!}\left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}} \\
& +\frac{\tilde{\epsilon}^{2}}{\epsilon^{2}} \sum_{(i, j+2, k-2) \in B_{1}} \frac{(i+j+k)^{r} \epsilon^{i} \tilde{\epsilon}^{j} \bar{\epsilon}^{k}}{(i+j+k)!}\left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}} \\
& +T^{2} \tilde{\epsilon}^{2} \sum_{(i, j+2, k) \in B_{1}} \frac{(i+j+k+2)^{r} \epsilon^{i} \tilde{\epsilon}^{j} \bar{\epsilon}^{k}}{(i+j+k+2)!}\left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}} \\
& +\tilde{\epsilon}^{2} \sum_{(i, j+2, k) \in B_{1}} \frac{(i+j+k+2)^{r} \epsilon^{i} \tilde{\epsilon}^{j} \bar{\epsilon}^{k}}{(i+j+k+2)!}\left\|t^{i+j+k+2-r} \partial_{t}^{i} \partial_{d}^{j} \bar{\partial}^{k} f\right\|_{L_{x, t}^{2}},
\end{aligned}
$$

and thus

$$
\begin{align*}
& S_{1} \lesssim\left(\frac{T \tilde{\epsilon}^{2}}{\epsilon}+\frac{\tilde{\epsilon}}{\epsilon}+\frac{\tilde{\epsilon}^{2}}{\epsilon^{2}}+T^{2} \tilde{\epsilon}^{2}\right) \phi(u) \\
&+\tilde{\epsilon}^{2} \sum_{(i, j+2, k) \in B_{1}} \frac{(i+j+k+2)^{r} \epsilon^{i} \tilde{\epsilon}^{j} \bar{\epsilon}^{k}}{(i+j+k+2)!}\left\|t^{i+j+k+2-r} \partial_{t}^{i} \partial_{d}^{j} \bar{\partial}^{k} f\right\|_{L_{x, t}^{2}} \tag{4.2}
\end{align*}
$$

4.2. The $S_{2}$ term. Next, we use (3.2) to treat $S_{2}$ (note that $j=1$ and $k \geq 1$ ) and write

$$
S_{2} \lesssim \sum_{B_{2}} \frac{(i+1+k)^{r} \epsilon^{i} \tilde{\epsilon}^{k}}{(i+1+k)!}\left\|t^{i+1+k-r} \partial_{t}^{i+1} \bar{\partial}^{k-1} u\right\|_{L_{x, t}^{2}}+\sum_{B_{2}} \frac{(i+1+k)^{r} \epsilon^{i} \tilde{\epsilon} \bar{\epsilon}^{k}}{(i+1+k)!}\left\|t^{i+1+k-r} \partial_{t}^{i} \bar{\partial}^{k-1} f\right\|_{L_{x, t}^{2}}
$$

and then, by relabeling,

$$
\begin{aligned}
S_{2} \lesssim & \sum_{(i-1, j, k+1) \in B_{2}} \frac{(i+1+k)^{r} \epsilon^{i-1} \tilde{\epsilon} \bar{\epsilon}^{k+1}}{(i+1+k)!}\left\|t^{i+1+k-r} \partial_{t}^{i} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}} \\
& +\sum_{(i, j, k+1) \in B_{2}} \frac{(i+2+k)^{r} \epsilon^{i} \tilde{\epsilon} \bar{\epsilon}^{k+1}}{(i+2+k)!}\left\|t^{i+2+k-r} \partial_{t}^{i} \bar{\partial}^{k} f\right\|_{L_{x, t}^{2}}
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
S_{2} \lesssim & \frac{T \bar{\epsilon} \tilde{\epsilon}}{\epsilon} \sum_{(i-1, j, k+1) \in B_{2}} \frac{(i+1+k)^{r} \epsilon^{i} \bar{\epsilon}^{k}}{(i+1+k)!}\left\|t^{i+k-r} \partial_{t}^{i} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}} \\
& +\tilde{\epsilon} \bar{\epsilon} \sum_{(i, j, k+1) \in B_{2}} \frac{(i+k+2)^{r} \epsilon^{i} \bar{\epsilon}^{k}}{(i+k+2)!}\left\|t^{i+k+2-r} \partial_{t}^{i} \bar{\partial}^{k} f\right\|_{L_{x, t}^{2}}
\end{aligned}
$$

from where

$$
\begin{equation*}
S_{2} \lesssim \frac{T \bar{\epsilon} \tilde{\epsilon}}{\epsilon} \phi(u)+\tilde{\epsilon} \bar{\epsilon} \sum_{(i, j, k+1) \in B_{2}} \frac{(i+k+2)^{r} \epsilon^{i} \bar{\epsilon}^{k}}{(i+k+2)!}\left\|t^{i+k+2-r} \partial_{t}^{i} \bar{\partial}^{k} f\right\|_{L_{x, t}^{2}} \tag{4.3}
\end{equation*}
$$

4.3. The $S_{3}$ term. For $S_{3}$, we use (3.3) and write (note that $j=1$ and $k=0$ )

$$
\begin{aligned}
S_{3} \lesssim & \sum_{(i, j, k) \in B_{3}} \frac{(i+j+k)^{r} \epsilon^{i} \tilde{\epsilon}^{j} \bar{\epsilon}^{k}}{(i+j+k)!}\left\|t^{i+1-r} \partial_{t}^{i} u\right\|_{L_{x, t}^{2}}^{1 / 2}\left\|t^{i+1-r} \partial_{t}^{i+1} u\right\|_{L_{x, t}^{2}}^{1 / 2} \\
& +\sum_{(i, j, k) \in B_{3}} \frac{(i+j+k)^{r} \epsilon^{i} \tilde{\epsilon}^{j} \bar{\epsilon}^{k}}{(i+j+k)!}\left\|t^{i+1-r} \partial_{t}^{i} u\right\|_{L_{x, t}^{2}}+\sum_{(i, j, k) \in B_{3}} \frac{(i+j+k)^{r} \epsilon^{i} \epsilon^{j} \bar{\epsilon}^{k}}{(i+j+k)!}\left\|t^{i+1-r} \partial_{t}^{i} f\right\|_{L_{x, t}^{2}}
\end{aligned}
$$

which implies by the Cauchy-Schwartz inequality that

$$
\begin{aligned}
S_{3} \lesssim & \frac{T^{1 / 2} \tilde{\epsilon}}{\epsilon^{1 / 2}}\left(\sum_{(i-1, j, k) \in B_{3}} \frac{i^{r} \epsilon^{i}}{i!}\left\|t^{i-r} \partial_{t}^{i} u\right\|_{L_{x, t}^{2}}\right)^{1 / 2}\left(\sum_{(i, j, k) \in B_{3}} \frac{(i+1)^{r} \epsilon^{i}}{(i+1)!}\left\|t^{i-r} \partial_{t}^{i} u\right\|_{L_{x, t}^{2}}\right)^{1 / 2} \\
& +T \tilde{\epsilon} \sum_{(i, j, k) \in B_{3}} \frac{(i+1)^{r} \epsilon^{i}}{(i+1)!}\left\|t^{i-r} \partial_{t}^{i} u\right\|_{L_{x, t}^{2}}+\tilde{\epsilon} \sum_{(i, j, k) \in B_{3}} \frac{(i+1)^{r} \epsilon^{i}}{(i+1)!}\left\|t^{i+1-r} \partial_{t}^{i} f\right\|_{L_{x, t}^{2}}
\end{aligned}
$$

Thus we get

$$
\begin{equation*}
S_{3} \lesssim\left(\frac{T^{1 / 2} \tilde{\epsilon}}{\epsilon^{1 / 2}}+T \tilde{\epsilon}\right) \phi(u)+\tilde{\epsilon} \sum_{(i, j, k) \in B_{3}} \frac{(i+1)^{r} \epsilon^{i}}{(i+1)!}\left\|t^{i+1-r} \partial_{t}^{i} f\right\|_{L_{x, t}^{2}} \tag{4.4}
\end{equation*}
$$

4.4. The $S_{4}$ term. Using (3.5) for $S_{4}$ (note that $j=0$ and $k \geq 2$ ), we have

$$
\begin{aligned}
S_{4} \lesssim & \sum_{(i, j, k) \in B_{4}} \frac{(i+j+k)^{r} \epsilon^{i} \tilde{\epsilon}^{j} \bar{\epsilon}^{k}}{(i+j+k)!}\left\|t^{i+k-r} \partial_{t}^{i+1} \bar{\partial}^{k-2} u\right\|_{L_{x, t}^{2}} \\
& +\sum_{(i, j, k) \in B_{4}} \frac{(i+j+k)^{r} \epsilon^{i} \tilde{\epsilon}^{j} \bar{\epsilon}^{k}}{(i+j+k)!}\left\|t^{i+k-r} \partial_{t}^{i} \bar{\partial}^{k-2} f\right\|_{L_{x, t}^{2}} .
\end{aligned}
$$

By relabeling, we have

$$
\begin{aligned}
S_{4} \lesssim & \sum_{(i-1, j, k+2) \in B_{4}} \frac{(i+j+k+1)^{r} \epsilon^{i-1} \bar{\epsilon}^{k+2}}{(i+j+k+1)!}\left\|t^{i+k+1-r} \partial_{t}^{i} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}} \\
& +\sum_{(i, j, k+2) \in B_{4}} \frac{(i+j+k+2)^{r} \epsilon^{i} \epsilon^{k+2}}{(i+j+k+2)!}\left\|t^{i+k+2-r} \partial_{t}^{i} \bar{\partial}^{k} f\right\|_{L_{x, t}^{2}}
\end{aligned}
$$

and thus

$$
\begin{aligned}
S_{4} \lesssim & \frac{\bar{\epsilon}^{2} T}{\epsilon} \sum_{(i-1, j, k+2) \in B_{4}} \frac{(i+j+k+1)^{r} \epsilon^{i} \bar{\epsilon}^{k}}{(i+j+k+1)!}\left\|t^{i+k-r} \partial_{t}^{i} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}} \\
& +\bar{\epsilon}^{2} \sum_{(i, j, k+2) \in B_{4}} \frac{(i+j+k+2)^{r} \epsilon^{i} \bar{\epsilon}^{k}}{(i+j+k+2)!}\left\|t^{i+k+2-r} \partial_{t}^{i} \bar{\partial}^{k} f\right\|_{L_{x, t}^{2}} .
\end{aligned}
$$

We obtain

$$
\begin{equation*}
S_{4} \lesssim \frac{\bar{\epsilon}^{2} T}{\epsilon} \phi(u)+\bar{\epsilon}^{2} \sum_{(i, j, k+2) \in B_{4}} \frac{(i+j+k+2)^{r} \epsilon^{i} \bar{\epsilon}^{k}}{(i+j+k+2)!}\left\|t^{i+k+2-r} \partial_{t}^{i} \bar{\partial}^{k} f\right\|_{L_{x, t}^{2}} . \tag{4.5}
\end{equation*}
$$

4.5. The $S_{5}$ term. For $S_{5}$ (note that $j=0$ and $k=1$ ), we use (3.6) and write

$$
\begin{aligned}
S_{5} \lesssim & \sum_{(i, j, k) \in B_{5}} \frac{(i+j+k)^{r} \epsilon^{i} \tilde{\epsilon}^{j} \bar{\epsilon}^{k}}{(i+j+k)!}\left\|t^{i+1-r} \partial_{t}^{i} u\right\|_{L_{x, t}^{2}}^{1 / 2}\left\|t^{i+1-r} \partial_{t}^{i+1} u\right\|_{L_{x, t}^{2}}^{1 / 2} \\
& +\sum_{(i, j, k) \in B_{5}} \frac{(i+j+k)^{r} \epsilon^{i} \tilde{\epsilon}^{j} \bar{\epsilon}^{k}}{(i+j+k)!}\left\|t^{i+1-r} \partial_{t}^{i} u\right\|_{L_{x, t}^{2}}+\sum_{(i, j, k) \in B_{5}} \frac{(i+j+k)^{r} \epsilon^{i} \tilde{\epsilon}^{j} \bar{\epsilon}^{k}}{(i+j+k)!}\left\|t^{i+1-r} \partial_{t}^{i} f\right\|_{L_{x, t}^{2}} .
\end{aligned}
$$

We again relabel and use the Cauchy-Schwartz inequality to deduce

$$
\begin{aligned}
S_{5} \lesssim & \frac{T^{1 / 2} \bar{\epsilon}}{\epsilon^{1 / 2}}\left(\sum_{(i-1, j, k) \in B_{5}} \frac{i^{r} \epsilon^{i}}{i!}\left\|t^{i-r} \partial_{t}^{i} u\right\|_{L_{x, t}^{2}}\right)^{1 / 2}\left(\sum_{(i, j, k) \in B_{5}} \frac{(i+1)^{r} \epsilon^{i}}{(i+1)!}\left\|t^{i-r} \partial_{t}^{i} u\right\|_{L_{x, t}^{2}}\right)^{1 / 2} \\
& +T \bar{\epsilon} \sum_{(i, j, k) \in B_{5}} \frac{(i+1)^{r} \epsilon^{i}}{(i+1)!}\left\|t^{i-r} \partial_{t}^{i} u\right\|_{L_{x, t}^{2}}+\bar{\epsilon} \sum_{(i, j, k) \in B_{5}} \frac{(i+1)^{r} \epsilon^{i}}{(i+1)!}\left\|t^{i+1-r} \partial_{t}^{i} f\right\|_{L_{x, t}^{2}} .
\end{aligned}
$$

We conclude

$$
\begin{equation*}
S_{5} \lesssim\left(\frac{T^{1 / 2} \bar{\epsilon}}{\epsilon^{1 / 2}}+T \bar{\epsilon}\right) \phi(u)+\bar{\epsilon} \sum_{(i, j, k) \in B_{5}} \frac{(i+1)^{r} \epsilon^{i}}{(i+1)!}\left\|t^{i+1-r} \partial_{t}^{i} f\right\|_{L_{x, t}^{2}} . \tag{4.6}
\end{equation*}
$$

4.6. The $S_{6}$ term. Lastly, for $S_{6}$ (note that $j=k=0$ ), we use (3.7) and write

$$
\begin{aligned}
S_{6} & \lesssim \sum_{(i, j, k) \in B_{6}} \frac{(i-r)(i+j+k)^{r} \epsilon^{i} \tilde{\epsilon}^{j} \epsilon^{k}}{(i+j+k)!}\left\|t^{i-1-r} \partial_{t}^{i-1} u\right\|_{L_{x, t}^{2}}+\sum_{(i, j, k) \in B_{6}} \frac{(i+j+k)^{r} \epsilon^{i} \tilde{\epsilon}^{j} \epsilon^{k}}{(i+j+k)!}\left\|t^{i-r} \partial_{t}^{i-1} f\right\|_{L_{x, t}^{2}} \\
& =\sum_{(i+1, j, k) \in B_{6}} \frac{i^{r}(i-r+1) \epsilon^{i+1}}{(i+1)!}\left\|t^{i-r} \partial_{t}^{i} u\right\|_{L_{x, t}^{2}}+\sum_{(i+1, j, k) \in B_{6}} \frac{(i+1)^{r} \epsilon^{i+1}}{(i+1)!}\left\|t^{i+1-r} \partial_{t}^{i} f\right\|_{L_{x, t}^{2}}
\end{aligned}
$$

whence

$$
\begin{equation*}
S_{6} \lesssim \epsilon \phi(u)+\epsilon \sum_{(i+1, j, k) \in B_{6}} \frac{(i+1)^{r} \epsilon^{i}}{(i+1)!}\left\|t^{i+1-r} \partial_{t}^{i} f\right\|_{L_{x, t}^{2}} . \tag{4.7}
\end{equation*}
$$

4.7. Conclusion of the proof. Combining the $\bar{\phi}$ bounds (4.2), (4.3), (4.4), (4.5), (4.6), and (4.7), we conclude

$$
\begin{align*}
\bar{\phi}(u) \lesssim & \left(\frac{T\left(\tilde{\epsilon}^{2}+\bar{\epsilon}^{2}\right)}{\epsilon}+\frac{\tilde{\epsilon}}{\epsilon}+\frac{\tilde{\epsilon}^{2}}{\epsilon^{2}}+T^{2} \tilde{\epsilon}^{2}+\frac{T \bar{\epsilon} \tilde{\epsilon}}{\epsilon}+\frac{T^{1 / 2}(\tilde{\epsilon}+\bar{\epsilon})}{\epsilon^{1 / 2}}+T(\tilde{\epsilon}+\bar{\epsilon})+\epsilon\right) \phi(u)+\phi_{0}(u) \\
& +\sum_{(i, j+2, k) \in B_{1}} \frac{(i+j+k+2)^{r} \epsilon^{i} \tilde{\epsilon}^{j+2} \bar{\epsilon}^{k}}{(i+j+k+2)!}\left\|t^{i+j+k+2-r} \partial_{t}^{i} \partial_{d}^{j} \bar{\partial}^{k} f\right\|_{L_{x, t}^{2}} \\
& +\sum_{(i, j, k+1) \in B_{2}} \frac{(i+k+2)^{r} \epsilon^{i} \tilde{\epsilon} \bar{\epsilon}^{k+1}}{(i+k+2)!}\left\|t^{i+2+k-r} \partial_{t}^{i} \bar{\partial}^{k} f\right\|_{L_{x, t}^{2}} \\
& +\sum_{(i, j, k) \in B_{3}} \frac{(i+1)^{r} \epsilon^{i} \tilde{\epsilon}^{2}}{(i+1)!}\left\|t^{i+1-r} \partial_{t}^{i} f\right\|_{L_{x, t}^{2}}+\sum_{(i, j, k+2) \in B_{4}} \frac{(i+k+2)^{r} \epsilon^{i} \bar{\epsilon}^{k+2}}{(i+k+2)!}\left\|t^{i+k+2-r} \partial_{t}^{i} \bar{\partial}^{k} f\right\|_{L_{x, t}^{2}} \\
& +\sum_{(i, j, k) \in B_{5}} \frac{(i+1)^{r} \epsilon^{i} \bar{\epsilon}^{2}}{(i+1)!}\left\|t^{i+1-r} \partial_{t}^{i} f\right\|_{L_{x, t}^{2}}+\sum_{(i+1, j, k) \in B_{6}} \frac{(i+1)^{r} \epsilon^{i+1}}{(i+1)!}\left\|t^{i+1-r} \partial_{t}^{i} f\right\|_{L_{x, t}^{2}} . \tag{4.8}
\end{align*}
$$

Denote by $C$ the implicit dimensional constant in the $\lesssim$ symbol in (4.8). Let $T>0$ be arbitrary. First, choose a sufficiently small $\epsilon=\epsilon(C)>0$ such that

$$
\begin{equation*}
\epsilon \leq \frac{1}{6 C} \tag{4.9}
\end{equation*}
$$

Next, we choose a sufficiently small $\bar{\epsilon}=\bar{\epsilon}(\epsilon, T, C)=\bar{\epsilon}(T, C)>0$ such that

$$
\begin{equation*}
\frac{T \bar{\epsilon}^{2}}{\epsilon}+\frac{T^{1 / 2} \bar{\epsilon}}{\epsilon^{1 / 2}}+T \bar{\epsilon} \leq \frac{1}{6 C} \tag{4.10}
\end{equation*}
$$

Finally, choose $\tilde{\epsilon}=\tilde{\epsilon}(\bar{\epsilon}, \epsilon, T, C)=\tilde{\epsilon}(T, C)>0$ such that

$$
\begin{equation*}
\frac{T \tilde{\epsilon}^{2}}{\epsilon}+\frac{\tilde{\epsilon}}{\epsilon}+\frac{\tilde{\epsilon}^{2}}{\epsilon^{2}}+T^{2} \tilde{\epsilon}^{2}+\frac{T \bar{\epsilon} \tilde{\epsilon}}{\epsilon}+\frac{T^{1 / 2} \tilde{\epsilon}}{\epsilon^{1 / 2}}+T \tilde{\epsilon} \leq \frac{1}{6 C} \tag{4.11}
\end{equation*}
$$

For convenience, in addition to (4.9)-(4.11) we may also ensure that

$$
\begin{equation*}
\tilde{\epsilon} \leq \bar{\epsilon} \leq \epsilon \leq 1 \text {. } \tag{4.12}
\end{equation*}
$$

Upon reorganizing the terms in (4.8), the choices (4.9)-(4.12) imply that

$$
\begin{align*}
& \bar{\phi}(u) \leq \frac{1}{2} \phi(u)+C \phi_{0}(u)+C \sum_{(i, j+2, k) \in B_{1}} \frac{(i+j+k+2)^{r} \epsilon^{i} \tilde{\epsilon}^{j+2} \bar{\epsilon}^{k}}{(i+j+k+2)!}\left\|t^{i+j+k+2-r} \partial_{t}^{i} \partial_{d}^{j} \bar{\partial}^{k} f\right\|_{L_{x, t}^{2}} \\
&+C\left(\tilde{\epsilon} \bar{\epsilon} \sum_{(i, j, k+1) \in B_{2}}+\bar{\epsilon}^{2} \sum_{(i, j, k+2) \in B_{4}}\right)\left(\frac{(i+k+2)^{r} \epsilon^{i} \bar{\epsilon}^{k}}{(i+k+2)!}\left\|t^{i+2+k-r} \partial_{t}^{i} \bar{\partial}^{k} f\right\|_{L_{x, t}^{2}}\right) \\
&+C\left(\tilde{\epsilon} \sum_{(i, j, k) \in B_{3}}+\bar{\epsilon} \sum_{(i, j, k) \in B_{5}}+\epsilon \sum_{(i+1, j, k) \in B_{6}}\right)\left(\frac{(i+1)^{r} \epsilon^{i}}{(i+1)!}\left\|t^{i+1-r} \partial_{t}^{i} f\right\|_{L_{x, t}^{2}}\right) \\
&+C \sum_{(i, j+2, k) \in B_{1}} \frac{\left.(i+j+k+2)^{r} \epsilon^{i} \tilde{\epsilon}\right)^{+2} \bar{\epsilon}^{k}}{(i+j+k+2)!}\left\|t^{i+j+k+2-r} \partial_{t}^{i} \partial_{d}^{j} \bar{\partial}^{k} f\right\|_{L_{x, t}^{2}} \\
& \leq \frac{1}{2} \phi(u)+C \phi_{0}(u)+2 C \sum_{i+j+k \geq(r-2)_{+}} \frac{(i+j+k+2)^{r} \epsilon^{i} \tilde{\epsilon}^{j+2} \bar{\epsilon}^{k}}{(i+j+k+2)!}\left\|t^{i+j+k+2-r} \partial_{t}^{i} \partial_{d}^{j} \bar{\partial}^{k} f\right\|_{L_{x, t}^{2}} \\
&+2 C \sum_{i+k \geq(r-2)_{+}} \frac{(i+k+2)^{r} \epsilon^{i} \epsilon^{k+2}}{(i+k+2)!}\left\|t^{i+k+2-r} \partial_{t}^{i} \bar{\partial}^{k} f\right\|_{L_{x, t}^{2}} \\
&+3 C \sum_{i \geq(r-1)_{+}} \frac{(i+1)^{r} \epsilon^{i+1}}{(i+1)!}\left\|t^{i+1-r} \partial_{t}^{i} f\right\|_{L_{x, t}^{2}} . \tag{4.13}
\end{align*}
$$

The estimate (4.13) is precisely the desired bound (4.1), thereby completing the proof of the a priori estimates. These a priori estimate may be made rigorous by working with truncated sums in $\bar{\phi}(u)$, such that $i+2 j+2 k \leq N$, which is allowed by parabolic regularity, and passing $N \rightarrow \infty$. We omit further details.

## 5. Derivative reduction for the Stokes problem and the proof of Theorem 1.3

The proof of Theorem 1.3 is identical to the proof of Theorem 1.1, except that the derivative reduction estimates in Section 3 need to be adapted to the case of the Stokes system (1.9). To avoid redundancy, for the Stokes system we only present the arguments for these reduction estimates.

The reductions are mainly based on two $H^{2}$ inequalities. Namely, for the stationary Stokes system

$$
\begin{align*}
-\Delta u+\nabla p & =g \\
\nabla \cdot u & =0, \tag{5.1}
\end{align*}
$$

we have

$$
\begin{equation*}
\|u\|_{\dot{H}^{2}(\Omega)}+\|p\|_{\dot{H}^{1}} \lesssim\|g\|_{L^{2}(\Omega)}+\|\bar{\partial} u\|_{\dot{H}^{1}(\Omega)}+\|u\|_{L^{2}(\Omega)} \tag{5.2}
\end{equation*}
$$

If also (1.10) holds, i.e. $\left.u\right|_{\partial \Omega}=0$, then the above estimate becomes

$$
\begin{equation*}
\|u\|_{\dot{H}^{2}(\Omega)} \lesssim\|g\|_{L^{2}(\Omega)} \tag{5.3}
\end{equation*}
$$

5.1. Normal derivative reduction for the Stokes operator. In all the inequalities next, we require $i+j+k \geq r$. We first claim

$$
\begin{align*}
& \left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}}+\left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j-1} \bar{\partial}^{k} p\right\|_{L^{2}} \\
& \quad \lesssim\left\|t^{i+j+k-r} \partial_{t}^{i+1} \partial_{d}^{j-2} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}}+\left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j-1} \bar{\partial}^{k+1} u\right\|_{L_{x, t}^{2}}+\left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j-2} \bar{\partial}^{k+2} u\right\|_{L_{x, t}^{2}} \\
& \quad+\left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j-2} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}}+\left\|t^{i+j+k-r} \partial_{t}^{i} \partial_{d}^{j-2} \bar{\partial}^{k} f\right\|_{L_{x, t}^{2}}, \quad j \geq 2 \tag{5.4}
\end{align*}
$$

which allows us to reduce the number of vertical derivatives $\left(\partial_{d}\right)$ in the Gevrey (analytic) norm. On the other hand, for $j=1$, we have

$$
\begin{align*}
& \left\|t^{i+1+k-r} \partial_{t}^{i} \partial_{d} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}}+\left\|t^{i+1+k-r} \partial_{t}^{i} \bar{\partial}^{k} p\right\|_{L_{x, t}^{2}} \\
& \quad \lesssim\left\|t^{i+1+k-r} \partial_{t}^{i+1} \bar{\partial}^{k-1} u\right\|_{L_{x, t}^{2}}+\left\|t^{i+1+k-r} \partial_{t}^{i} \bar{\partial}^{k-1} f\right\|_{L_{x, t}^{2}}, \quad k \geq 1 . \tag{5.5}
\end{align*}
$$

For $j=1$ and $k=0$, we claim

$$
\begin{equation*}
\left\|t^{i+1-r} \partial_{t}^{i} \nabla u\right\|_{L_{x, t}^{2}} \lesssim\left\|t^{i+1-r} \partial_{t}^{i} u\right\|_{L_{x, t}^{2}}^{1 / 2}\left\|t^{i+1-r} \partial_{t}^{i+1} u\right\|_{L_{x, t}^{2}}^{1 / 2}+\left\|t^{i+1-r} \partial_{t}^{i} u\right\|_{L^{2}}+\left\|t^{i+1-r} \partial_{t}^{i} f\right\|_{L_{x, t}^{2}}, \quad i \geq r . \tag{5.6}
\end{equation*}
$$

5.2. Tangential derivative reduction for the Stokes operator. In order to reduce the number of tangential derivatives, we claim

$$
\begin{align*}
& \left\|t^{i+k-r} \partial_{t}^{i} \bar{\partial}^{k} u\right\|_{L_{x, t}^{2}}+\left\|t^{i+k-r} \partial_{t}^{i} \bar{\partial}^{k-1} p\right\|_{L_{x, t}^{2}} \\
& \quad \lesssim\left\|t^{i+k-r} \partial_{t}^{i+1} \bar{\partial}^{k-2} u\right\|_{L_{x, t}^{2}}+\left\|t^{i+k-r} \partial_{t}^{i} \bar{\partial}^{k-2} f\right\|_{L_{x, t}^{2}}, \quad k \geq 2 \tag{5.7}
\end{align*}
$$

while for $k=1$, we use a special case (replace $\nabla$ with $\bar{\partial}$ ) of the inequality (5.6):

$$
\begin{equation*}
\left\|t^{i+1-r} \partial_{t}^{i} \bar{\partial} u\right\|_{L_{x, t}^{2}} \lesssim\left\|t^{i+1-r} \partial_{t}^{i} u\right\|_{L_{x, t}^{2}}^{1 / 2}\left\|t^{i+1-r} \partial_{t}^{i+1} u\right\|_{L_{x, t}^{2}}^{1 / 2}+\left\|t^{i+1-r} \partial_{t}^{i} u\right\|_{L^{2}}+\left\|t^{i+1-r} \partial_{t}^{i} f\right\|_{L_{x, t}^{2}}, \quad i \geq r . \tag{5.8}
\end{equation*}
$$

5.3. Time derivative reduction for the Stokes operator. Here we have

$$
\begin{equation*}
\left\|t^{i-r} \partial_{t}^{i} u\right\|_{L_{x, t}^{2}} \lesssim(i-r)\left\|t^{i-1-r} \partial_{t}^{i-1} u\right\|_{L_{x, t}^{2}}+\left\|t^{i-r} \partial_{t}^{i-1} f\right\|_{L_{x, t}^{2}}, \quad i-1 \geq r \tag{5.9}
\end{equation*}
$$

The proofs of the inequalities (5.4)-(5.9) are the same as for those in Section 3, except that instead of appealing to (2.3)-(2.4), we use (5.2)-(5.3). Moreover, the energy inequality (3.8) is the same for the Stokes system, since $\nabla \cdot u=0$. Further details are omitted.

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