MOMENTS FOR STRONG SOLUTIONS OF THE 2D STOCHASTIC NAVIER-STOKES EQUATIONS IN A BOUNDED DOMAIN

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ABSTRACT. We address a question posed by Glatt-Holtz and Ziane in [GHZ09, Remark 2.1 (ii)], regarding moments of strong pathwise solutions to the Navier-Stokes equations in a two-dimensional bounded domain $\mathcal{O}$. We prove that $\mathbb{E}\varphi(\|u(t)\|^2_{H^1(\mathcal{O})}) < \infty$ for any deterministic $t > 0$, where $\varphi(x) = \log(1 + \log(1 + x))$. Such moment bounds may be used to study statistical properties of the long time behavior of the equation. In addition, we obtain algebraic moment bounds on compact subdomains $\mathcal{O}_0$ of the form $\mathbb{E}\varphi_\varepsilon(\|u(t)\|^2_{H^1(\mathcal{O}_0)}) < \infty$, where $\varphi_\varepsilon(x) = (1 + x)^{(1-\varepsilon)/2}$, for any deterministic $t > 0$ and any $\varepsilon > 0$.

1. INTRODUCTION

We consider the Navier-Stokes equation in a smooth bounded domain $\mathcal{O} \subset \mathbb{R}^2$ with a multiplicative white noise stochastic body force

$$du + (u \cdot \nabla u + \nabla p - \nu \Delta u - f)dt = g(u)d\mathcal{W}, \tag{1.1}$$

$$\nabla \cdot u = 0, \tag{1.2}$$

$$u(0) = u_0 \tag{1.3}$$

and Dirichlet boundary condition

$$u = 0 \text{ on } [0, \infty) \times \partial\mathcal{O}. \tag{1.4}$$

Here $\nu > 0$ is the kinematic viscosity, f is a deterministic force, and $g(u)d\mathcal{W}$ is a cylindrical Brownian motion, formally written as $\sum_k g_k(u)dW_k$ with independent one dimensional Brownian motions W_k and Lipschitz coefficients $g_k(u)$.

The mathematical literature on the stochastic Navier-Stokes equation (SNSE) is vast (see e.g. the reviews [Kuk06, PR07, Fla08, KS12] and references therein). Starting with the work [BT73] this subject has attracted a lot of interest, in part due to a number of difficulties which arise in the well-posedness theory due to the stochastic term (such as a lack of compactness). Typically, two notions of solutions are considered: *martingale* solutions [CG94, Cru89, FG95, MR04, Vio76], for which the probabilistic basis is constructed along with the solution, and (probabilistically) strong solutions [BF00, Bre00, BP00, CP97, DPD03, DPZ92, DGHT11, FR02, GHZ09, MR05], for which the probability space is given in advance (we follow the convention of reffering to the later as *path-wise* solutions). In this paper we consider pathwise solutions. In terms of regularity in x and t, since here we are concerned with two-dimensional domains, it is natural to consider *strong* solutions (from the deterministic point of view), i.e., solutions which evolve continuously in V ($=$ divergence free $H^1_0(\mathcal{O})$) and are square integrable in time with values in $H^2(\mathcal{O}) \cap V$ (see the classical works [CF88, Lad92, Tem01] for details).

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In this paper we address the question posed by Glatt-Holtz and Ziane [GHZ09, Remark 2.1(ii)]. A strong pathwise solution (u, τ) of (1.1)–(1.4) (cf. Definition 2.1 below) obeys

$$\mathbb{E} \sup_{t\in[0,\tau]} \|u(t)\|_V^2 < \infty$$

where τ is a maximal stopping time, which in the two-dimensional case satisfies $\mathbb{P}(\tau < \infty) = 0$. The question raised in [GHZ09] is whether we have

$$\mathbb{E}\|u(t)\|_V^2 < \infty \tag{1.5}$$

for any fixed *deterministic time* $t > 0$. That is, does the H^1 norm of the solution have a finite second moment for $t > 0$? We give a partial answer to this question, by showing that there exists a certain finite moment of the H^1 norm of the solution at any given deterministic time t.

Let $\varphi\colon [0,\infty) \to [0,\infty)$ be a strictly increasing C^2 function, with $\varphi(x) \to \infty$ as $x \to \infty$. The goal of this manuscript is to prove that there exits such function φ such that

$$\mathbb{E}\varphi(\|u(t)\|_V^2) < \infty \tag{1.6}$$

for any *deterministic* $t > 0$, where u is a strong pathwise solution of the SNSE. The moment bound (1.6) may be used to obtain higher regularity properties for the support of invariant measures to (1.1)–(1.4) on a bounded domain [GHKVZ12].

We note that moment bounds for *strong norms* of the solution are notoriously difficult to obtain in the case of a bounded domain due to a lack of cancellation in the nonlinear term in H^1. See e.g. [KS12]. On the contrary, in the case of a periodic domain we have that $\langle B(u,u), Au\rangle = 0$, and thus moment bounds in strong norms may be established. In fact, moments may be established for high Sobolev, Gevrey-class, and even analytic norms (c.f. [BKL00, Mat02, Shi03, Oda06, KS12] and references therein).

The main difficulty in obtaining moment bounds lies in estimating the nonlinear term $B(u,u)$. Note that as opposed to the periodic domain, on a bounded domain we do not have $\langle B(u,u), Au\rangle = 0$. This is the sole term preventing one to prove (1.6) with $\varphi(x) = x$ for any deterministic time $t > 0$. We overcome this difficulty by using an idea from [Kuk01], and the logarithmically corrected endpoint Sobolev inequality in 2D. Our observation is that moment bounds in the stochastic case may be obtained by treating the nonlinear term similarly to the way it is treated in the deterministic case when obtaining bounds on $\|\nabla u(t)\|_{L^2}$ in terms of the Grashof number, for solutions on the global attractor. These bounds in the deterministic case depend *super-exponentially* on the Grashof number when the norm on the whole domain is considered [FP67, CF88], and *algebraically* on the Grashof number when the norm on compact subdomains is considered [Kuk01]. In comparison, our main result for the whole domain is that (1.6) holds for $\varphi(x) = \log(1+\log(1+x))$. For compact subdomains $\mathcal{O}_0$ of $\mathcal{O}$, if we replace the norm $\|u\|_V^2 = \|u\|_{H_0^1(\mathcal{O})}^2$ with $\|\nabla u\|_{L^2(\mathcal{O}_0)}^2$, the estimate (1.6) holds for algebraic φ's, more precisely for $\varphi_\varepsilon(x) = (1+x)^{(1-\varepsilon)/2}$ for $\varepsilon > 0$. Obtaining algebraic moment bounds for $\|u(t)\|_V^2$ at deterministic times t remains open. The question posed in [GHZ09] appears to be related to a long standing open problem of Foias and Prodi [FP67] (c.f. also [FMT88]), of obtaining polynomial upper bounds in terms of the Grashof number for the H_0^1 norm of solutions in the Dirichlet case. We refer the reader to [CF85, CF88, FMT88, JT93] and references therein for further aspects of this problem in the deterministic case.

The article is organized as follows. In Section 2 we introduce the functional setting of the equations, state the definition of solutions, and present our main results, Theorem 2.2 and Theorem 2.3. Section 3 contain the proof of Theorem 2.2, while Section 4 contains the proof of Theorem 2.3.

2. FUNCTIONAL SETTING

For convenience, we recall the deterministic and probabilistic framework considered in [GHZ09].

2.1. **Deterministic framework.** As in [CF88, Tem01] we consider the classical spaces $\mathcal{V} = \{u \in C_0^\infty(\mathcal{O})\colon \operatorname{div} u = 0\}$, H is the closure of $\mathcal{V}$ in $L^2(\mathcal{O})$, and V is the closure of $\mathcal{V}$ in $H^1(\mathcal{O})$, which may be identified as

$$H = \{u \in L^2(\mathcal{O})\colon \nabla \cdot u = 0, u \cdot n|_{\partial\mathcal{O}} = 0\}, \qquad V = \{u \in H_0^1(\mathcal{O})\colon \nabla \cdot u = 0\}.$$

since the boundary of the bounded domain $\mathcal{O}$ is smooth throughout this work. We denote the inner product on H by $\langle\cdot,\cdot\rangle$ and the norm by $\|\cdot\|_H$, while the norm on V is denoted by $\|\cdot\|_V$. Let $\mathcal{P}_H$ be the Leray-Hopf projector of $L^2(\mathcal{O})$ onto H. We recall that for $u \in L^2(\mathcal{O})$, $\mathcal{P}_H u = (1 - \mathcal{Q}_H)u$ where $\mathcal{Q}_H u = -\nabla\pi$, and $\pi \in H^1(\mathcal{O})$ is a solution of the elliptic Neumann problem

$$-\Delta\pi = \nabla \cdot u \text{ in } \mathcal{O}, \qquad \nabla\pi \cdot n = -u \cdot n \text{ on } \partial\mathcal{O}.$$

Let

$$A = -\mathcal{P}_H \Delta$$

be the Stokes operator, with domain $\mathcal{D}(A) = V \cap H^2(\mathcal{O})$. The dual of $V = \mathcal{D}(A^{1/2})$ with respect to H is denoted by $V' = \mathcal{D}(A^{-1/2})$. In view of the Dirichlet boundary condition, there exists $\lambda_* > 0$ such that the Poincaré inequality $\|u\|_H \le \lambda_* \|u\|_V$ holds. At last, denote the nonlinear term as the bilinear operator mapping $V \times V$ to V' via

$$B(u, v) = \mathcal{P}_H(u \cdot \nabla v)$$

and recall that the cancellation property $\langle B(u, v), v\rangle = 0$ holds for $u, v \in V$. The deterministic force f is assumed to be bounded in time with values in H.

2.2. **Stochastic framework.** We briefly recall some aspects of the stochastic analysis in infinite dimensions needed in this note cf. [DPZ92] (see also [DGHT11, GHZ09, Fla08, PR07]). Fix a stochastic basis $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge 0}, \mathcal{W})$, i.e., a complete probability space equipped with a complete right-continuous filtration and a cylindrical Brownian motion $\mathcal{W}$, defined on an auxiliary separable Hilbert space U, adapted to this filtration. Fixing an orthonormal basis $\{e_k\}_{k\ge 1}$ for U, we may formally write $\mathcal{W}(t, \omega) = \sum_k e_k W_k(t, \omega)$, where $\{W_k\}$ is a sequence of independent one-dimensional Brownian motions. This sum does not converge on U so one classically considers a larger Hilbert space $U_0 = \{u = \sum_k \alpha_k e_k \colon \|u\|_{U_0} < \infty\}$, where $\|u\|_{U_0}^2 = \sum_k k^{-2}\alpha_k^2$; note that the embedding $U \subset U_0$ is Hilbert-Schmidt. Then, $\mathcal{W} \in C([0, \infty); U_0)$ almost surely c.f. [DPZ92].

Given a separable Hilbert space X, we denote by $L_2(U, X)$ the space of Hilbert-Schmidt operators from U to X, equipped with the norm $\|G\|_{L_2(U,X)}^2 = \sum_k \|G_k\|_X^2$. Here and throughout the paper we denote $G_k = ge_k$. For an X-values predictable process $G \in L^2(\Omega; L^2_{\mathrm{loc}}([0, \infty); L_2(U, X))$ one may define the Itō stochastic integral

$$\int_0^t G d\mathcal{W} := \sum_k \int_0^t G_k dW_k, \tag{2.1}$$

which lies in the space $\mathcal{O}_X$ of X-valued square integrable martingales. In particular, we shall make use of the Burkholder-Davis-Gundy inequality: For any $p \ge 1$

$$\mathbb{E}\left(\sup_{t\in[0,T]} \left\|\int_0^t G d\mathcal{W}\right\|_X^p\right) \le C\mathbb{E}\left(\int_0^t \|G\|_{L_2(U,X)}^2\right)^{p/2} \tag{2.2}$$

for some $C = C(p) > 0$.

2.3. **Conditions on the noise.** Given a pair of Banach spaces X and Y we denote by $\mathrm{Lip_u}(X, Y)$ the collection of continuous functions $h\colon [0,\infty) \times X \to Y$ which are sublinear

$$\|h(t,x)\|_Y \leq K_Y(1 + \|x\|_X) \text{ for all } t \geq 0, x \in X \tag{2.3}$$

and Lipschitz

$$\|h(t,x) - h(t,y)\|_Y \leq K_Y \|x - y\|_X \text{ for all } t \geq 0,\ x, y \in X \tag{2.4}$$

and some constant $K_Y > 0$ which is independent of t.

The noise term $g(u)d\mathcal{W}$ considered is as in [GHZ09], and is defined in terms of the map

$$g = \{g_k\}_{k\geq 1} \colon [0,\infty) \times H \to L_2(U, H) \tag{2.5}$$

such that

$$g \in \mathrm{Lip_u}(H, L_2(U,H)) \cap \mathrm{Lip_u}(V, L_2(U,V)) \cap \mathrm{Lip_u}(\mathcal{D}(A), L_2(U, \mathcal{D}(A))). \tag{2.6}$$

Assumption (2.6) may be written explicitly as

$$\begin{aligned}
\|g(t,x)\|_{L_2(U,\mathcal{D}(A^{j/2}))} &\leq K_j(1 + \|x\|_{\mathcal{D}(A^{j/2})}), \text{ for } j \in \{0,1,2\},\\
\|g(t,x) - g(t,y)\|_{L_2(U,\mathcal{D}(A^{j/2}))} &\leq K_j \|x - y\|_{\mathcal{D}(A^{j/2})}, \text{ for } j \in \{0,1,2\}.
\end{aligned}$$

For convenience we shall denote the norm on $L_2(U,H)$ by $\|\cdot\|_{\mathbb{H}}$, and the one on $L_2(U,V)$ by $\|\cdot\|_{\mathbb{V}}$.

Given $u \in L^2(\Omega; L^2(0,T;H))$ and g as above, define as in (2.1) the stochastic integral $\int_0^t g(u) d\mathcal{W}$, which is a well-defined H-valued Itō stochastic integral, that is predictable, and

$$\left\langle \int_0^t g(u) d\mathcal{W}, v \right\rangle = \sum_k \int_0^t \langle g_k(u), v\rangle dW_k$$

holds for any $v \in H$.

Assumption (2.6) on the noise is quite general, and includes a wide class of examples, such as additive noise, linear multiplicative noise, Nemytskii operators, and stochastically forced functionals of the solution c.f. [GHZ09, GHV12].

2.4. **Notion of solution.** In this note we consider *strong pathwise solutions*, which are *strong* from the PDE point of view, i.e., bounded in time with values in V and square integrable in time with values in $\mathcal{D}(A)$, and *strong* from the probabilistic point of view (henceforth called *pathwise* to avoid confusion), i.e., the driving noise and the filtration are given.

Definition 2.1. *Fix a stochastic basis $\mathcal{S}$, let g as in (2.6) be predictable, $f \in L^4(\Omega; L^\infty([0,\infty); H))$ be predictable, and assume that the initial data $u_0 \in L^4(\Omega; V)$ is $\mathcal{F}_0$ measurable. The pair (u, τ) is called a* pathwise strong solution *of (1.1)–(1.4) if τ is a strictly positive stopping time, $u(\cdot \wedge \tau)$ is a predictable process in H such that*

$$\begin{aligned}
&u(\cdot \wedge \tau) \in L^2(\Omega; C([0,\infty); V))\\
&u \mathbf{1}_{t\leq\tau} \in L^2(\Omega; L^2_{\mathrm{loc}}([0,\infty); \mathcal{D}(A)))
\end{aligned}$$

and

$$\langle u(t\wedge\tau), v\rangle + \int_0^{t\wedge\tau} \langle \nu A u + B(u,u) - f, v\rangle dt = \langle u_0, v\rangle + \sum_k \int_0^{t\wedge\tau} \langle g_k(u), v\rangle dW_k \tag{2.7}$$

holds for every $v \in H$*. Additionally,* (u, ξ) *is called a* maximal *pathwise strong solution if* ξ *is a strictly positive stopping time, and there exits* $\tau_n \to \xi$ *increasing, such that* (u, τ_n) *is a local strong solution and*

$$\sup_{t\in[0,\tau_n]} \|u\|_V^2 + \int_0^{\tau_n} \|Au\|_H^2 dt \geq n$$

on the set $\{\xi < \infty\}$*. Such a solution is called* global *if* $\mathbb{P}(\xi < \infty) = 0$*.*

In [GHZ09] it is shown that under the above hypotheses on g, f, and u_0, there exists a maximal pathwise strong solution (u, ξ) of (1.1)–(1.4), in both two and three dimensions. In addition, in the two dimensional case the maximal solution (u, ξ) is global.

2.5. **Results.** Our main result regarding the whole domain concerns doubly-logarithmic moments.

Theorem 2.2. *Fix a stochastic basis* $\mathcal{S}$*, and let* g, f, u_0 *be as Definition 2.1. Given a pathwise strong global solution* (u, τ) *of* (1.1)–(1.4) *we have*

$$\mathbb{E}\left(\sup_{t\in[0,T]} \varphi(\|u(\cdot,t)\|_V^2)\right) \leq M(T, \nu, u_0, f, g)$$

for any deterministic time $T > 0$*, where*

$$\varphi(x) = \log(1 + \log(1 + x))$$

and M *is a function that depends on* T, ν, u_0, f*, and* g *explicitly (cf.* (3.21) *below).*

Note that $\|u\|_V = \|u\|_{H^1(\mathcal{O})}$ since u obeys the homogenous Dirichlet boundary condition. On the other hand, once we restrict to a compact subdomain we may allow φ to be algebraic.

Theorem 2.3. *Fix a stochastic basis* $\mathcal{S}$*, and let* g, f, u_0 *be as Definition 2.1. Let* $\mathcal{O}_0 \subset \mathcal{O}$ *be a closed set, with* $0 < d_0 = \mathrm{dist}(\mathcal{O}_0, \partial\mathcal{O})$*, and let* $\varepsilon \in (0, 1/4)$ *be arbitrary. Given a pathwise strong global solution* (u, τ) *of* (1.1)–(1.4) *we have*

$$\mathbb{E}\left(\sup_{t\in[0,T]} \varphi_\varepsilon(\|\nabla u(\cdot,t)\|_{L^2(\mathcal{O}_0)}^2)\right) \leq M_\varepsilon(T, d_0, \nu, u_0, f, g)$$

for any deterministic time $T > 0$*, where*

$$\varphi_\varepsilon(x) = (1 + x)^{\frac{1-\varepsilon}{2}}$$

and M_ε *is a function that depends on* $T, \nu, d_0, \varepsilon, u_0, f$*, and* g *explicitly (cf.* (4.29) *below).*

3. DOUBLE-LOGARITHMIC MOMENTS FOR THE WHOLE DOMAIN

In this section we give the proof of Theorem 2.2. We consider a maximal pathwise strong solution (u, ξ) of (1.1)–(1.4), and let $\tau < \xi$ be a stopping time. First we shall consider a series of a priori estimates on this solution, which we then make rigorous, and in the process pass from τ to an arbitrary deterministic stopping time t.

3.1. **A priori estimates.** From the infinite dimensional analogue of the Itō formula in H we obtain

$$d\|u\|_H^2 + 2\nu\|u\|_V^2 dt = \left(2\langle f, u\rangle + \|g(u)\|_{\mathbb{H}}^2\right) dt + 2\langle g(u), u\rangle d\mathcal{W} \tag{3.1}$$

where we also used the cancellation property $\langle B(u,u), u\rangle = 0$. Here $\langle g(u), u\rangle d\mathcal{W}$ is short hand notation for $\sum_k \langle g_k(u), u\rangle dW_k$. Integrating (3.1) from τ_a to t and taking a supremum over $t \in [\tau_a, \tau_b]$, where τ_a and τ_b are stopping times, and then taking expected values, we obtain

$$\begin{aligned}\mathbb{E}\sup_{[\tau_a,\tau_b]} \|u\|_H^2 + 2\nu\mathbb{E}\int_{\tau_a}^{\tau_b} \|u\|_V^2 dt \leq \mathbb{E}\|u(\tau_a)\|_H^2 + 2\mathbb{E}\int_{\tau_a}^{\tau_b} \left(\|f\|_{V'}\|u\|_V + \|g(u)\|_{\mathbb{H}}^2\right) dt \\ + 2\mathbb{E}\sup_{t\in[\tau_a,\tau_b]}\left|\int_{\tau_a}^t \langle g(u), u\rangle d\mathcal{W}\right|. \end{aligned}\tag{3.2}$$

Applying the Burkholder-Davis-Gundy inequality, we bound the last term on the right of (3.2) as

$$\begin{aligned}\mathbb{E}\sup_{t\in[\tau_a,\tau_b]}\left|\int_{\tau_a}^t \langle g(u), u\rangle d\mathcal{W}\right| &\leq C\mathbb{E}\left(\int_{\tau_a}^{\tau_b} \|g(u)\|_{\mathbb{H}}^2\|u\|_H^2 dt\right)^{1/2} \\ &\leq \frac{1}{4}\mathbb{E}\sup_{[\tau_a,\tau_b]}\|u\|_H^2 + C\mathbb{E}\int_{\tau_a}^{\tau_b}\|g(u)\|_{\mathbb{H}}^2 dt. \end{aligned}\tag{3.3}$$

Combining estimates (3.2)–(3.3), and using assumption (2.6) on g, we arrive at

$$\begin{aligned}&\mathbb{E}\sup_{[\tau_a,\tau_b]}\|u\|_H^2 + \nu\mathbb{E}\int_{\tau_a}^{\tau_b}\|u\|_V^2 dt \\ &\quad\leq 2\mathbb{E}\|u(\tau_a)\|_H^2 + 2\nu^{-1}\mathbb{E}\int_{\tau_a}^{\tau_b}\|f\|_{V'}^2 dt + C\mathbb{E}\int_{\tau_a}^{\tau_b}\|g(u)\|_{\mathbb{H}}^2 dt \\ &\quad\leq 2\mathbb{E}\|u(\tau_a)\|_H^2 + 2\nu^{-1}\mathbb{E}\int_{\tau_a}^{\tau_b}\|f\|_{V'}^2 dt + CK_0\mathbb{E}\int_{\tau_a}^{\tau_b}(1 + \sup_{[\tau_a,t]}\|u\|_H^2) dt. \end{aligned}\tag{3.4}$$

for any pair of stopping times $\tau_a < \tau_b$. The usual Grönwall inequality (or, equivalently Lemma 5.3 in [GHZ09]) then implies

$$\mathbb{E}\sup_{[0,t]}\|u\|_H^2 + \nu\mathbb{E}\int_0^t\|u\|_V^2 d\tau \leq \left(2\mathbb{E}\|u_0\|_H^2 + \mathbb{E}\int_0^t (CK_0 + 2\nu^{-1}\|f\|_{V'}^2)\right) e^{CK_0 t} \tag{3.5}$$

for any time t. Estimate (3.5) will be used when choosing a specific φ. In (3.5) we can replace $\nu^{-1}\|f\|_{V'}^2$ with $\|f\|_H^2$ if we wish to avoid ν-dependence of the right side in this estimate.

Remark 3.1. If $\|g(u)\|_H^2 \leq K_0(1 + \|g(u)\|_H^{2\alpha})$ with either $\alpha < 1$, or for $\alpha = 1$ but $K_0 \ll 1$, then we can use the Poincaré inequality in (3.4) and obtain a bound for the right side of (3.5) which is bounded as $t \to \infty$. For example this is the case for additive noise.

We now turn to estimate $\varphi(\|u\|_V^2)$. The infinite dimensional version of the Itō formula yields

$$d\|u\|_V^2 + 2\nu\|Au\|_H^2 dt = \left(2\langle f, Au\rangle - 2\langle B(u,u), Au\rangle + \|g(u)\|_{\mathbb{V}}^2\right) dt + 2\langle g(u), Au\rangle d\mathcal{W}. \tag{3.6}$$

Using again the Itō formula, we obtain from (3.6) that

$$\begin{aligned}d\varphi(\|u\|_V^2) + 2\nu\varphi'(\|u\|_V^2)\|Au\|_H^2 dt = \varphi'(\|u\|_V^2)\left(2\langle f, Au\rangle - 2\langle B(u,u), Au\rangle + \|g(u)\|_{\mathbb{V}}^2\right) dt \\ + 2\varphi''(\|u\|_V^2)\sum_k \langle g_k(u), Au\rangle^2 dt + T_0 d\mathcal{W} \end{aligned}\tag{3.7}$$

as long as φ is at least C^2 in a neighborhood of $[0,\infty)$, where we have denoted

$$T_0 = 2\varphi'(\|u\|_V^2)\langle g(u), Au\rangle. \tag{3.8}$$

Integrating (3.7) from 0 to τ, taking a supremum for $\tau \in [0,t]$, where $t > 0$ is arbitrary, and taking expected values, we arrive at

$$\begin{aligned}&\mathbb{E}\sup_{[0,t]}\varphi(\|u\|_V^2) + 2\nu\mathbb{E}\int_0^t \varphi'(\|u\|_V^2)\|Au\|_H^2 d\tau \\ &\qquad \le \mathbb{E}\varphi(\|u_0\|_V^2) + \mathbb{E}\int_0^t (T_1+T_2+T_3+T_4)d\tau + 2\mathbb{E}\sup_{\tau\in[0,t]}\left|\int_0^\tau T_0 d\mathcal{W}\right|\end{aligned} \tag{3.9}$$

where, using (2.6) we have

$$T_1 = 2\varphi'(\|u\|_V^2)|\langle B(u,u), Au\rangle| \tag{3.10}$$

$$T_2 = 2\varphi'(\|u\|_V^2)|\langle f, Au\rangle| \le 2\varphi'(\|u\|_V^2)\|f\|_H\|Au\|_H \tag{3.11}$$

$$T_3 = \varphi'(\|u\|_V^2)\|g(u)\|_{\mathbb{V}}^2 \le K_1\varphi'(\|u\|_V^2)(1+\|u\|_V^2) \tag{3.12}$$

$$T_4 = 2|\varphi''(\|u\|_V^2)|\sum_k \langle g_k(u), Au\rangle^2 \le 2K_1|\varphi''(\|u\|_V^2)|\,\|u\|_V^2(1+\|u\|_V^2). \tag{3.13}$$

Combining (3.9)–(3.8) and applying the Burkholder-Davis-Gundy inequality on the T_0 term

$$\mathbb{E}\sup_{\tau\in[0,t]}\left|\int_0^\tau T_0 d\mathcal{W}\right| \le C\mathbb{E}\left(\int_0^t \varphi'(\|u\|_V^2)^2\|g(u)\|_{\mathbb{V}}^2\|u\|_V^2 d\tau\right)^{1/2}$$

we obtain

$$\begin{aligned}&\mathbb{E}\sup_{[0,t]}\varphi(\|u\|_V^2) + \nu\mathbb{E}\int_0^t \varphi'(\|u\|_V^2)\|Au\|_H^2 d\tau \\ &\quad \le \mathbb{E}\varphi(\|u_0\|_V^2) + \mathbb{E}\int_0^t T_1 d\tau + \nu^{-1}\mathbb{E}\int_0^t \varphi'(\|u\|_V^2)\|f\|_H^2 d\tau \\ &\qquad + 2K_1\mathbb{E}\int_0^t \left(\varphi'(\|u\|_V^2) + \|u\|_V^2\varphi''(\|u\|_V^2)\right)(1+\|u\|_V^2)d\tau \\ &\qquad + CK_1\mathbb{E}\left(\int_0^t \left(\varphi'(\|u\|_V^2)(1+\|u\|_V^2)\right)^2 d\tau\right)^{1/2}.\end{aligned} \tag{3.14}$$

It remains to estimate T_1, the leading term in (3.14). Using the Hölder inequality and the Brézis-Gallouët-Waigner inequality (see, e.g. [CF88, Tem01]) we may bound

$$\begin{aligned}T_1 &\le 2\varphi'(\|u\|_V^2)\|Au\|_H\|u\|_V\|u\|_{L^\infty} \\ &\le C\varphi'(\|u\|_V^2)\|Au\|_H\|u\|_V^2\left(1+\log\frac{\|Au\|_H^2}{\lambda_*\|u\|_V^2}\right)^{1/2}.\end{aligned} \tag{3.15}$$

Moreover the inequality

$$a\mu\left(1+\log\frac{\mu^2}{b^2}\right)^{1/2} \le \varepsilon\mu^2 + \frac{a^2}{\varepsilon}\log\frac{2a}{\varepsilon b}, \tag{3.16}$$

which holds when $a,\varepsilon > 0$ and $\mu \ge b$ (see, e.g. [FMT88, Kuk96]), when applied to (3.15) implies

$$T_1 \le \frac{\nu}{2}\varphi'(\|u\|_V^2)\|Au\|_H^2 + C\nu^{-1}\varphi'(\|u\|_V^2)\|u\|_V^4\left(1+\log C\nu^{-1}\|u\|_V\right) \tag{3.17}$$

for some positive constant C (that depends on λ_*). Finally, combining (3.14) and (3.17) yields

$$
\begin{aligned}
&\mathbb{E}\sup_{[0,t]}\varphi(\|u\|_V^2) + \frac{\nu}{2}\mathbb{E}\int_0^t \varphi'(\|u\|_V^2)\|Au\|_H^2 d\tau \\
&\quad \le \mathbb{E}\varphi(\|u_0\|_V^2) + \nu^{-1}\mathbb{E}\int_0^t \varphi'(\|u\|_V^2)\|f\|_H^2 d\tau \\
&\qquad + 2K_1\mathbb{E}\int_0^t \left(\varphi'(\|u\|_V^2) + \|u\|_V^2\varphi''(\|u\|_V^2)\right)(1+\|u\|_V^2)d\tau \\
&\qquad + CK_1\mathbb{E}\left(\int_0^t \left(\varphi'(\|u\|_V^2)(1+\|u\|_V^2)\right)^2 d\tau\right)^{1/2} \\
&\qquad + C\nu^{-1}\mathbb{E}\int_0^t \varphi'(\|u\|_V^2)\|u\|_V^4\left(1+\log C\nu^{-1}\|u\|_V\right)d\tau.
\end{aligned}
\tag{3.18}
$$

In order to control the last term in (3.18), recall that c.f. (3.5) we have a bound on $\nu\mathbb{E}\int_0^t \|u\|_V^2 d\tau$, and hence we need φ to satisfy

$$
\varphi'(x) \le \frac{1}{x(1+\log x)}
\tag{3.19}
$$

for x large. Therefore, we define $\varphi\colon [0,\infty) \to [0,\infty)$ by

$$
\varphi(x) = \log(1+\log(1+x)),
\tag{3.20}
$$

which satisfies

$$
0 \le \varphi'(x) = \frac{1}{(1+x)(1+\log(1+x))}.
$$

Note that this function is smooth in a neighborhood of $[0,\infty)$. Upon bounding

$$
|\varphi''(x)| \le \frac{2}{(1+x)^2(1+\log(1+x))}
$$

and using $1+\log C\nu^{-1}x \le (1+\log C\nu^{-1})(1+\log(1+x))$, we obtain from (3.18) and (3.5) that

$$
\begin{aligned}
\mathbb{E}\sup_{[0,t]}\varphi(\|u\|_V^2) &\le \mathbb{E}\varphi(\|u_0\|_V^2) + \nu^{-1}\mathbb{E}\int_0^t \|f\|_H^2 d\tau + CK_1\mathbb{E}(t+t^{1/2}) \\
&\quad + C\nu^{-1}(1+\log C\nu^{-1})\left(2\mathbb{E}\|u_0\|_H^2 + \mathbb{E}\int_0^t (CK_0 + 2\nu^{-1}\|f\|_{V'}^2)\right)e^{CK_0t} \\
&=: M(t,\nu,u_0,f,g)
\end{aligned}
\tag{3.21}
$$

for any time $t>0$, where the dependence on g is through the constants K_0 and K_1 in (2.6).

Remark 3.2. Similarly to Remark 3.1, under stronger conditions on the growth of g, one may obtain a t-independent bound for the right-side of (3.21). In particular, this is the case of additive noise.

3.2. **Rigorous justification.** We now turn to justifying the a priori estimates from Section 3.1. From Theorem 4.2 in [GHZ09], we have that the maximal pathwise strong solution (u,ξ) is global in time, i.e., $\mathbb{P}(\xi<\infty)=0$.

Let $\xi_n \to \xi$ be an increasing sequence of stopping times which announces the maximal time ξ, and let $t>0$ be arbitrary. Since $\varphi(x)\le x$ for all $x\ge 0$, the estimate (3.21) — with t replaced by

$t_n = \xi_n \wedge t$ — is fully justified for every $n \geq 1$. Moreover, for any $n \geq 1$, in view of (3.21) we have $M(t_n, \nu, u_0, f, g) \leq M(t, \nu, u_0, f, g)$, so that we obtain an n-independent upper bound

$$\sup_{n\geq 1} \left(\mathbb{E} \sup_{[0,\xi_n \wedge t]} \varphi(\|u\|_V^2) \right) \leq M(t, \nu, u_0, f, g). \tag{3.22}$$

Note that since $\mathbb{P}(\xi < \infty) = 0$, we have $\xi_n \wedge t \to t$ a.s. as $n \to \infty$.

Lastly, define an increasing sequence of sets $\Omega_n \subset \Omega$ by

$$\Omega_n = \{\omega \in \Omega \colon \xi_n(\omega) > t\}$$

so that

$$\lim_{n\to\infty} |\Omega - \Omega_n| \to 0. \tag{3.23}$$

From (3.22) we obtain

$$\mathbb{E} \left(\mathbf{1}_{\Omega_n} \sup_{[0,\xi_n \wedge t]} \varphi(\|u\|_V^2) \right) \leq \mathbb{E} \left(\sup_{[0,\xi_n \wedge t]} \varphi(\|u\|_V^2) \right) \leq M(t, \nu, u_0, f, g) \tag{3.24}$$

for any $n \geq 1$, and in view of (3.23) and the monotone convergence theorem we have

$$\mathbb{E} \left(\mathbf{1}_{\Omega_n} \sup_{[0,\xi_n \wedge t]} \varphi(\|u\|_V^2) \right) = \mathbb{E} \left(\mathbf{1}_{\Omega_n} \sup_{[0,t]} \varphi(\|u\|_V^2) \right) \to \mathbb{E} \left(\sup_{[0,t]} \varphi(\|u\|_V^2) \right) \tag{3.25}$$

as $n \to \infty$. We conclude the proof of Theorem 2.2 by combining (3.24) and (3.25) to obtain

$$\mathbb{E}\varphi(\|u(t)\|_V^2) \leq M(t, \nu, u_0, f, g) \tag{3.26}$$

for any deterministic time $t > 0$, with M defined by (3.21).

4. Algebraic moments for compact subdomains

In this section we give the proof of Theorem 2.3. Let $\mathcal{O}_0$ be an arbitrary compact subset of $\mathcal{O}$ (no regularity assumption on $\mathcal{O}_0$), and denote $d_0 = \operatorname{dist}(\mathcal{O}_0, \partial\mathcal{O}) > 0$. Our goal is to estimate

$$\mathbb{E} \sup_{t\in[0,T]} \varphi \left(\|\nabla u(\cdot, t)\|_{L^2(\mathcal{O}_0)}^2 \right)$$

in terms of ν, T, u_0, f, g, and d_0, for any deterministic time $T > 0$, where φ is an algebraic, increasing function, which is C^2 in a neighborhood of $[0, \infty)$.

Let $\eta(x)$ be a cutoff function adapted to $(\mathcal{O}_0, \mathcal{O})$, i.e., $\eta \in C_0^\infty(\mathbb{R}^2)$ such that

$$0 \leq \eta \leq 1, \quad \operatorname{supp}(\eta) \subset \mathcal{O}, \quad \eta \equiv 1 \text{ on } \mathcal{O}_0, \quad |\nabla \eta| \leq C d_0^{-1}$$

where $C > 0$ is some constant. First, we estimate moments of the vorticity $w = \nabla^\perp \cdot u$ on $\mathcal{O}_0$, i.e.,

$$\mathbb{E} \sup_{[0,T]} \left(\varphi(\|\eta^2 w\|_{L^2(\mathcal{O})}^2) \right)$$

for some C^2 smooth, positive, increasing, unbounded (at infinity) function φ, to be chosen below.

Recall that the unique maximal pathwise strong solution (u, ξ) (c.f. Definition 2.1) constructed in [GHZ09] obeys

$$\mathbb{E} \sup_{t\in[0,\tau]} \|u\|_V^2 + \mathbb{E} \int_0^\tau \|Au\|_H^2 dt < \infty \tag{4.1}$$

for any $\tau < \xi$, and the solution is global $\mathbb{P}(\xi < \infty) = 0$. Let $\tau_n \to \xi$ be a sequence of stopping times announcing the maximal (stopping) time of existence, and let $\tau = T \wedge \tau_n$. We have by (4.1) that

$$\mathbf{1}_{t\leq\tau}\eta^2 w \in L^2(\Omega; C([0,\infty); H)) \cap L^2(\Omega; L^2([0,\infty); V)). \tag{4.2}$$

In order to study the vorticity formulation of (1.1), observe that

$$\nabla^\perp \cdot \mathcal{P}_H v = \nabla^\perp \cdot v - \nabla^\perp \cdot \mathcal{Q}_H v = \nabla^\perp \cdot v, \tag{4.3}$$

and therefore $\nabla^\perp \cdot Au = -\Delta w$ and $\nabla^\perp \cdot B(u,u) = u \cdot \nabla w$ in the two-dimensional case. Here we use the standard notation for the 2D curl operator $\nabla^\perp \cdot v = (-\partial_2, \partial_1) \cdot (v_1, v_2) = \partial_1 v_2 - \partial_2 v_1$. Upon applying the curl operator to the Navier-Stokes equation we thus formally obtain that

$$dw + (u \cdot \nabla w - \nu\Delta w - F)dt = G(u)d\mathcal{W} \tag{4.4}$$

holds, where we denoted

$$F = \nabla^\perp \cdot f$$

and

$$G(u) = \nabla^\perp \cdot g(u).$$

Then, using the Itō product rule we have

$$d(\eta^2 w) + \eta^2(u \cdot \nabla w - \nu\Delta w - F)dt = \eta^2 G(u)d\mathcal{W} \tag{4.5}$$

which is an equation in V'. Equivalently, in order to prove (4.5) consider $v = \nabla^\perp(\eta^2\widetilde{v})$ with $\widetilde{v} \in H^1$, in the weak formulation (2.7), integrate by parts, and use (4.3). If (4.5) were an equation in H, by the infinite dimensional Itō lemma we could directly conclude

$$\begin{aligned} d\|\eta^2 w\|_{L^2}^2 &= -2\langle \eta^2 w, \eta^2(u\cdot\nabla w - \nu\Delta w - F)\rangle dt \\ &\quad + \|\eta^2 G(u)\|_{L_2(U,L^2)}^2 dt + 2\langle \eta^2 w, \eta^2 G(u)\rangle d\mathcal{W}. \end{aligned} \tag{4.6}$$

Instead, in order to justify (4.6) we proceed as follows. Let P_n be the n^{th} Galerking projection operator, onto the space spanned by the first n eigenfunction of A. Then, from (4.5) we obtain that

$$dP_n(\eta^2 w) + P_n\left(\eta^2(u\cdot\nabla w - \nu\Delta w - F)\right)dt = P_n(\eta^2 G(u))d\mathcal{W}$$

holds in H, and we may apply the Itō lemma to conclude

$$\begin{aligned} d\|P_n(\eta^2 w)\|_{L^2}^2 &= -2\langle P_n(\eta^2 w), P_n\left(\eta^2(u\cdot\nabla w - \nu\Delta w - F)\right)\rangle dt \\ &\quad + \|P_n\left(\eta^2 G(u)\right)\|_{L_2(U,L^2)}^2 dt + 2\langle P_n\left(\eta^2 w\right), P_n(\eta^2 G(u))\rangle d\mathcal{W}, \end{aligned} \tag{4.7}$$

which is understood as usual in the time-integrated sense. In view of the a priori regularity (4.2), we may now pass $n \to \infty$ in each term of the time-integrated version of (4.7) and obtain (4.6).

Upon integrating by parts, we have

$$-2\langle \eta^2 w, \eta^2(u\cdot\nabla w)\rangle = 4\int_{\mathcal{O}} \eta^3 u \cdot \nabla\eta w^2 \tag{4.8}$$

and as usual

$$-2\langle \eta^2 w, -\nu\eta^2\Delta w\rangle = -2\nu\|\eta^2\nabla w\|_{L^2}^2 - 8\nu\int_{\mathcal{O}} \eta^2 w\eta\nabla\eta\cdot\nabla w. \tag{4.9}$$

Then, in view of (4.8) and (4.9), the identity (4.6) may be written as

$$\begin{aligned} d\|\eta^2 w\|_{L^2}^2 &= -2\nu\|\eta^2\nabla w\|_{L^2}^2 dt + 2\langle \eta^2 w, F + 2\eta w u\cdot\nabla\eta - 4\nu\eta\nabla\eta\cdot\nabla w\rangle dt \\ &\quad + \|\eta^2 G(u)\|_{L_2(U,L^2)}^2 dt + 2\langle \eta^2 w, \eta^2 G(u)\rangle d\mathcal{W}. \end{aligned} \tag{4.10}$$

Since φ is assumed to be C^2 in a neighborhood of $[0,\infty)$, we may apply the Itō lemma one more time to obtain

$$\begin{aligned} &d\varphi(\|\eta^2 w\|_{L^2}^2) \\ &= -2\nu\varphi'(\|\eta^2 w\|_{L^2}^2)\|\eta^2\nabla w\|_{L^2}^2 dt \\ &\quad + 2\varphi'(\|\eta^2 w\|_{L^2}^2)\left(\langle \eta^2 w, F\rangle + 2\langle \eta^2 w, \eta w u\cdot\nabla\eta\rangle - 4\nu\langle \eta^2 w, \eta\nabla\eta\cdot\nabla w\rangle\right) dt \\ &\quad + \varphi'(\|\eta^2 w\|_{L^2}^2)\|\eta^2 G(u)\|_{L_2(U,L^2)}^2 dt + 8\varphi''(\|\eta^2 w\|_{L^2}^2)\sum_k \langle \eta^2 w, \eta^2\nabla^\perp\cdot g_k(u)\rangle^2 dt \\ &\quad + 2\varphi'(\|\eta^2 w\|_{L^2}^2)\langle \eta^2 w, \eta^2 G(u)\rangle d\mathcal{W}. \end{aligned} \tag{4.11}$$

Let $0 \le \tau_a < \tau_b$. We integrate (4.11) from τ_a to t, take a supremum over $t \in (\tau_a, \tau_b)$, and take expected values; we obtain

$$\begin{aligned} &\mathbb{E}\sup_{[\tau_a,\tau_b]}\varphi(\|\eta^2 w\|_{L^2}^2) + 2\nu\mathbb{E}\int_{\tau_a}^{\tau_b}\varphi'(\|\eta^2 w\|_{L^2}^2)\|\eta^2\nabla w\|_{L^2}^2 dt \\ &\quad \le \mathbb{E}\varphi(\|\eta w(\tau_a)\|_{L^2}^2) + 2\mathbb{E}\int_{\tau_a}^{\tau_b}(T_1+T_2+T_3+T_4+T_5)\,dt + \mathbb{E}\sup_{t\in[\tau_a,\tau_b]}\left|\int_{\tau_a}^{t} T_6\, d\mathcal{W}\right|, \end{aligned} \tag{4.12}$$

where we denoted

$$\begin{aligned} T_1 &= 2\varphi'(\|\eta^2 w\|_{L^2}^2)|\langle \eta^2 w, F\rangle| \\ T_2 &= 4\varphi'(\|\eta^2 w\|_{L^2}^2)|\langle \eta^2 w, \eta w u\cdot\nabla\eta\rangle| \\ T_3 &= 8\nu\varphi'(\|\eta^2 w\|_{L^2}^2)|\langle \eta^2 w, \eta\nabla\eta\cdot\nabla w\rangle| \\ T_4 &= \varphi'(\|\eta^2 w\|_{L^2}^2)\|\eta^2 G(u)\|_{L_2(U,L^2)}^2 \\ T_5 &= 8\varphi''(\|\eta^2 w\|_{L^2}^2)\sum_k \langle \eta^2 w, \eta^2\nabla\cdot g_k(u)\rangle^2 \\ T_6 &= 2\varphi'(\|\eta^2 w\|_{L^2}^2)\langle \eta^2 w, \eta^2 G(u)\rangle. \end{aligned}$$

Let $\varepsilon \in (0, 1/4)$ be arbitrary. We now choose φ such that

$$x^{1+\varepsilon}\varphi'(x^2) \le 1 \tag{4.13}$$

for large x, a condition which is clearly satisfied by the function

$$\varphi(x) = (1+x)^{\frac{1-\varepsilon}{2}} \tag{4.14}$$

which is also C^2 smooth on $(-1,\infty)$. The above chosen φ additionally satisfies

$$0 < \varphi'(x^2) \le 1, \qquad x^2|\varphi''(x^2)| \le 1, \tag{4.15}$$

bounds which will come in handy later. Having chosen φ, we turn to estimating the terms on the right side of (4.12).

We first bound the most interesting term T_2, which corresponds to the nonlinear term. From Hölder and (4.13) we have

$$
\begin{aligned}
T_2 &\le C d_0^{-1} \varphi'(\|\eta^2 w\|_{L^2}^2)\|\eta^2 w\|_{L^2}\|\eta w u\|_{L^2}\\
&\le C d_0^{-1} \left(\|\eta^2 w\|_{L^2}^{1+\varepsilon}\varphi'(\|\eta^2 w\|_{L^2}^2)\right)^{\frac{1}{1+\varepsilon}} \left(\varphi'(\|\eta^2 w\|_{L^2}^2)\right)^{\frac{\varepsilon}{1+\varepsilon}} \left(\int_{\mathcal{O}} \eta^2 w^2 u^2\right)^{1/2}\\
&\le C d_0^{-1} \left(\varphi'(\|\eta^2 w\|_{L^2}^2)\right)^{\frac{\varepsilon}{1+\varepsilon}} \|\eta^2 w\|_{L^{\frac{2(1+\varepsilon)}{1-3\varepsilon}}}^{\frac12} \|w\|_{L^2}^{\frac12} \|u^2\|_{L^{\frac{1+\varepsilon}{2\varepsilon}}}^{\frac12}.
\end{aligned}
\tag{4.16}
$$

We now use the Gagliardo-Nirenberg inequality

$$
\|h\|_{L^p} \le C_{\mathcal{O},p} \|h\|_{L^2}^{\frac{2}{p}} \|\nabla h\|_{L^2}^{\frac{p-2}{p}} \tag{4.17}
$$

which holds in two dimensions for all $p \in [2,\infty)$, and $h \in H_0^1(\mathcal{O})$. Since both u and $\eta^2 w$ belong to $H_0^1(\mathcal{O})$, using (4.17) we may bound

$$
\|u\|_{L^{\frac{1+\varepsilon}{\varepsilon}}} \le C\|u\|_{L^2}^{\frac{2\varepsilon}{1+\varepsilon}} \|\nabla u\|_{L^2}^{\frac{1-\varepsilon}{1+\varepsilon}} \tag{4.18}
$$

and

$$
\|\eta^2 w\|_{L^{\frac{2(1+\varepsilon)}{1-3\varepsilon}}} \le C\|\eta^2 w\|_{L^2}^{\frac{1-3\varepsilon}{1+\varepsilon}} \left(\|\eta^2 \nabla w\|_{L^2}^{\frac{4\varepsilon}{1+\varepsilon}} + d_0^{-\frac{4\varepsilon}{1+\varepsilon}} \|\nabla u\|_{L^2}^{\frac{4\varepsilon}{1+\varepsilon}}\right) \tag{4.19}
$$

where the constant C depends on ε (and blows up as $\varepsilon \to 0$), but is independent of ν and d_0. Inserting the estimates (4.18) and (4.19) into the bound (4.16) yields

$$
\begin{aligned}
T_2 &\le C d_0^{-1} \left(\varphi'(\|\eta^2 w\|_{L^2}^2)\|\eta^2 \nabla w\|_{L^2}^2\right)^{\frac{\varepsilon}{1+\varepsilon}} \|u\|_{L^2}^{\frac{2\varepsilon}{1+\varepsilon}} \|\nabla u\|_{L^2}^{\frac{2(1-\varepsilon)}{1+\varepsilon}} + C d_0^{-\frac{1+3\varepsilon}{1+\varepsilon}} \|u\|_{L^2}^{\frac{2\varepsilon}{1+\varepsilon}} \|\nabla u\|_{L^2}^{\frac{2}{1+\varepsilon}}\\
&\le \frac{\nu}{2}\varphi'(\|\eta^2 w\|_{L^2}^2)\|\eta^2 \nabla w\|_{L^2}^2 + C d_0^{-1-\varepsilon}\nu^{-\varepsilon}\|u\|_{L^2}^{2\varepsilon}\|\nabla u\|_{L^2}^{2(1-\varepsilon)} + C d_0^{-\frac{1+3\varepsilon}{1+\varepsilon}} \|u\|_{L^2}^{\frac{2\varepsilon}{1+\varepsilon}} \|\nabla u\|_{L^2}^{\frac{2}{1+\varepsilon}}\\
&\le \frac{\nu}{2}\varphi'(\|\eta^2 w\|_{L^2}^2)\|\eta^2 \nabla w\|_{L^2}^2 + \nu d_0^{-1}\|\nabla u\|_{L^2}^2 + C d_0^{-3}\nu^{-\frac{1}{\varepsilon}}\|u\|_{L^2}^2.
\end{aligned}
\tag{4.20}
$$

Bounding the remaining terms on the right side of (4.12) is direct. Upon integrating by parts and applying the Hölder inequality we have

$$
\begin{aligned}
T_1 &\le C\varphi'(\|\eta^2 w\|_{L^2}^2)\|f\|_{L^2}\left(\|\eta^2\nabla w\|_{L^2} + d_0^{-1}\|w\|_{L^2}\right)\\
&\le \frac{\nu}{2}\varphi'(\|\eta^2 w\|_{L^2}^2)\|\eta^2\nabla w\|_{L^2}^2 + C\nu^{-1}(1+d_0^{-2})\|f\|_{L^2}^2 + \nu\|\nabla u\|_{L^2}^2
\end{aligned}
\tag{4.21}
$$

where $C > 0$ is a constant which is independent of d_0 and ν. Similarly,

$$
\begin{aligned}
T_3 &\le C d_0^{-1}\nu\varphi'(\|\eta^2 w\|_{L^2}^2)\|\eta^2\nabla w\|_{L^2}\|\eta w\|_{L^2}\\
&\le \frac{\nu}{2}\varphi'(\|\eta^2 w\|_{L^2}^2)\|\eta^2\nabla w\|_{L^2}^2 + C d_0^{-2}\nu\|\nabla u\|_{L^2}^2.
\end{aligned}
\tag{4.22}
$$

Note that we have $\|\eta^2 G(u)\|_{L_2(U,L^2)}^2 \le C_\eta \|g(u)\|_{\mathbb{V}}^2$. From (2.3) with $k = 1$, we then estimate

$$
T_4 \le C\|g(u)\|_{\mathbb{V}}^2 \le K_1(1 + \|\nabla u\|_{L^2}^2). \tag{4.23}
$$

Next, using (4.15) we also bound

$$
T_5 \le C\|\eta^2 w\|_{L^2}^2 \varphi''(\|\eta^2 w\|_{L^2}^2)\|g(u)\|_{\mathbb{V}}^2 \le C K_1(1 + \|\nabla u\|_{L^2}^2). \tag{4.24}
$$

Lastly, the Burkholder-Davis-Gundy inequality, (2.3), and (4.13) imply

$$
\begin{aligned}
2\mathbb{E} \sup_{t\in[\tau_a,\tau_b]} \left| \int_{\tau_a}^{t} T_6 d\mathcal{W} \right| &\le C\mathbb{E} \left(\int_{\tau_a}^{\tau_b} \varphi'(\|\eta^2 w\|_{L^2}^2)^2 \|\eta^2 w\|_{L^2}^2 \|\eta^2 G(u)\|_{L_2(U,L^2)}^2 dt \right)^{1/2} \\
&\le C\mathbb{E} \left(\int_{\tau_a}^{\tau_b} K_1(1+\|\nabla u\|_{L^2}^2) dt \right)^{1/2}. \qquad (4.25)
\end{aligned}
$$

Thus, by combining (4.12) with (4.20)–(4.25) we obtain

$$
\begin{aligned}
&\mathbb{E} \sup_{[\tau_a,\tau_b]} \varphi(\|\eta^2 w\|_{L^2}^2) + \frac{\nu}{2} \mathbb{E} \int_{\tau_a}^{\tau_b} \varphi'(\|\eta^2 w\|_{L^2}^2) \|\eta \nabla w\|_{L^2}^2 dt \\
&\quad \le \mathbb{E}\varphi(\|\eta^2 w(\tau_a)\|_{L^2}^2) + C\nu(1+d_0^{-2})\mathbb{E}\int_{\tau_a}^{\tau_b} \|\nabla u\|_{L^2}^2 dt \\
&\qquad + C\nu^{-1/\varepsilon} d_0^{-3} \mathbb{E} \int_{\tau_a}^{\tau_b} \|u\|_{L^2}^2 dt + C\nu^{-1}(1+d_0^{-2})\mathbb{E}\int_{\tau_a}^{\tau_b} \|f\|_{L^2}^2 dt \\
&\qquad + C\mathbb{E} \left(\int_{\tau_a}^{\tau_b} K_1(1+\|\nabla u\|_{L^2}^2) dt \right)^{1/2} + CK_1 \mathbb{E} \int_{\tau_a}^{\tau_b} (1+\|\nabla u\|_{L^2}^2) dt \qquad (4.26)
\end{aligned}
$$

for a suitable constant $C > 0$, which may depend on $\mathcal{O}$ and ε, but is independent of ν and d_0.

It is left to combine (4.26) with (3.5), which gives

$$
\mathbb{E} \sup_{[\tau_a,\tau_b]} \|u\|_H^2 + 2\nu \mathbb{E} \int_{\tau_a}^{\tau_b} \|u\|_V^2 \le \left(2\mathbb{E}\|u(\tau_a)\|_H^2 + \mathbb{E}\int_{\tau_a}^{\tau_b} (CK_0 + 2\nu^{-1}\|f\|_{V'}^2) \right) e^{CK_0(\tau_b-\tau_a)}.
$$

Therefore, recalling that $\eta \equiv 1$ on $\mathcal{O}_0$ and setting $\tau_a = 0, \tau_b = T$, we get

$$
\begin{aligned}
&\mathbb{E} \sup_{[0,T]} \varphi(\|w\|_{L^2(\mathcal{O}_0)}^2) + \frac{\nu}{2}\mathbb{E}\int_0^T \varphi'(\|\eta^2 w\|_{L^2}^2) \|\eta\nabla w\|_{L^2}^2 dt \\
&\quad \le 2\mathbb{E}\varphi(\|u_0\|_V^2) + C\nu^{-1}(1+d_0^{-2})\mathbb{E}\int_0^T \|f\|_H^2 dt + CK_1(1+T) \\
&\qquad + C\left(d_0^{-2} + \nu^{-1/\varepsilon} d_0^{-3} T + CK_1\nu^{-1}\right) \left(2\mathbb{E}\|u_0\|_H^2 + \mathbb{E}\int_0^T (CK_0 + 2\nu^{-1}\|f\|_{V'}^2) \right) e^{CK_0 T} \\
&\quad =: \widetilde{M}_\varepsilon(T, d_0^{-1}, \nu, u_0, f, g). \qquad (4.27)
\end{aligned}
$$

Bounds on $\varphi(\|\nabla u\|_{L^2(\mathcal{O}_0)}^2)$ now follow from the concavity of φ, and the next lemma.

Lemma 4.1. *Let $\mathcal{O}_0 \subset \mathcal{O}$ be a compact subset and let $d_0 = \operatorname{dist}(\mathcal{O}_0, \partial\mathcal{O})$. Let $\mathcal{O}_1$ be open such that $\mathcal{O}_0 \subset \mathcal{O}_1 \subset \mathcal{O}$, $\operatorname{dist}(\mathcal{O}_0, \partial\mathcal{O}_1) \ge d_0/4$, and $\operatorname{dist}(\mathcal{O}_1, \partial\mathcal{O}) \ge d_0/4$. Then we have*

$$
\|\nabla u\|_{L^2(\mathcal{O}_0)}^2 \le C\|w\|_{L^2(\mathcal{O}_1)}^2 + Cd_0^{-2}\|u\|_{L^2(\mathcal{O}_1)}^2 \qquad (4.28)
$$

for some positive universal constant C.

This lemma follows from a covering argument and the classical interior H^2 estimate for $-\Delta w = \psi$ in a ball, where $u = \nabla^\perp \psi$. Note that no smoothness is assumed on the set $\mathcal{O}_0$. We refer to [Kuk01, Lemma 4.1] for a proof.

At last, since φ is concave, $\varphi(x) \leq Cx$, and $\varphi(0) \geq 0$, from (4.27) and (4.28) we have that

$$\begin{aligned}\mathbb{E} \sup_{[0,T]} \varphi(\|\nabla u\|^2_{L^2(\mathcal{O}_0)}) &\leq C\mathbb{E} \sup_{[0,T]} \varphi(\|w\|^2_{L^2(\mathcal{O}_1)}) + C\mathbb{E} \sup_{[0,T]} \varphi(Cd_0^{-2}\|u\|^2_{L^2(\mathcal{O}_1)}) \\ &\leq C\widetilde{M}_\varepsilon(T, d_0^{-1}, \nu, u_0, f, g) \\ &\quad + Cd_0^{-2}\left(2\mathbb{E}\|u_0\|^2_H + \mathbb{E}\int_0^T (CK_0 + 2\nu^{-1}\|f\|^2_{V'})\right) e^{CK_0T} \\ &=: M_\varepsilon(T, d_0^{-1}, \nu, u_0, f, g) \end{aligned} \tag{4.29}$$

which concludes the proof of Theorem 2.3, modulo an argument similar to that in Section 3.2.

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