ON LOCAL UNIQUENESS OF WEAK SOLUTIONS TO THE NAVIER-STOKES SYSTEM WITH BMO^{-1} INITIAL DATUM

IGOR KUKAVICA AND VLAD VICOL

ABSTRACT. We address the problem of local uniqueness of weak solutions to the Navier–Stokes system, with the initial datum in a subspace of $BMO^{-1}(\mathbb{R}^n)$. The existence and uniqueness of local mild solutions has been proven by Koch and Tataru [KT]. We present a necessary and sufficient condition for two weak solutions to evolve from the same initial datum, and for weak solutions to be mild.

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1. INTRODUCTION

In this paper we address the uniqueness of solutions of the initial value problem for the Navier–Stokes equations in $\mathbb{R}^n \times [0, \infty)$

$$\partial_t u_k - \Delta u_k + u_j \partial_j u_k + \partial_k p = 0, \quad k = 1, \dots, n, \tag{1.1}$$

$$\partial_j u_j = 0, \tag{1.2}$$

with the initial datum

$$u_k(\cdot, 0) = u_{0k}, \quad k = 1, \dots, n. \tag{1.3}$$

The unknowns are the velocity vector field $u(x,t) \in \mathbb{R}^n$ and the pressure $p(x,t) \in \mathbb{R}$, with $(x,t) \in \mathbb{R}^n \times [0, \infty)$. The initial datum is divergence free and belongs to a space that we detail upon in Section 2.

Fabes, Jones, and Riviere have proven in [FJR] that the Navier–Stokes system (1.1)–(1.3) with $u_0 \in L^p(\mathbb{R}^n)$ has a unique local mild solution in $L^q(0,T;L^p(\mathbb{R}^n))$ if $n < p \le \infty$ and $n/p + 2/q < 1$. Moreover, they showed that if $2 \le p < \infty$ and $2 \le q \le \infty$, then u is weak solution of (1.1)–(1.3) if and only if u is a mild solution, i.e.,

$$u(\cdot, t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u \otimes u) ds, \tag{1.4}$$

where $\mathbb{P}$ denotes the Hopf–Leray projector, and $e^{t\Delta} f$ denotes the solution of the heat equation with an initial datum f.

The first author of the present paper considered in [Ku] the case $p = \infty$. It is well–known (cf. [S]) that uniqueness of weak solutions with bounded initial datum does not hold. In [Ku] it was shown that two weak solutions u and $\widetilde{u}$ evolve from the same initial datum if and only if $\widetilde{u}$ is obtained from u by means of a transform $\widetilde{u}(x,t) = u(x - \Phi(t), t) + \phi(t)$, where $\phi \in L^\infty([0,T))$, with $\lim_{t\to 0} \phi(t) = 0$ and $\Phi(t) = \int_0^t \phi(s) ds$. In particular, uniqueness holds when one imposes a growth restriction on the pressure (cf. [GIKM, K, Ku]).

We recall the scaling invariance of the Navier–Stokes equations: If $u(x,t)$ and $p(x,t)$ solve (1.1)–(1.3) with an initial datum $u_0(x)$, then $\lambda u(\lambda x, \lambda^2 t)$ and $\lambda^2 p(\lambda x, \lambda^2 t)$ also solve the equations, but with the initial datum $\lambda u_0(\lambda x)$. Spaces that are invariant under the above transformations are called critical spaces for the Navier–Stokes equations.

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The existence and uniqueness of local mild solutions of the Navier–Stokes equations in the critical space $L^n(\mathbb{R}^n)$ was proven by Kato in [Ka]. (For other existence results for the initial value problem also cf. [CF, FK, L, W].) The existence and uniqueness in a critical Morrey space was obtained by Taylor in [T] (also see Giga and Miyakawa [GM]). Koch and Tataru established in [KT] the existence and uniqueness of mild solutions on $[0,T)$ for initial datum in the critical space BMO_T^{-1}, extending those previous results. They showed that there exists $\varepsilon > 0$ such that for all $T > 0$, if

$$\|u_0\|_{BMO_T^{-1}} = \sup_{0<t<T} \sup_{x\in\mathbb{R}^n} \left(t^{-n/2} \int_0^t \int_{|y-x|<\sqrt{t}} |e^{s\Delta} u_0(y)|^2 dy ds \right)^{1/2} < \varepsilon,$$

and if u_0 is divergence free, there exists a unique local mild solution $u \in \mathcal{E}_T$ of (1.1)–(1.3) on $\mathbb{R}^n \times [0,T)$. Here $\mathcal{E}_T$ is defined as $u \in L^2_{uloc,x} L^2_t(\mathbb{R}^n \times [0,T))$ such that

$$\|u\|_{\mathcal{E}_T} = \sup_{0<t<T} \sqrt{t} \|u(\cdot,t)\|_{L^\infty} + \sup_{x\in\mathbb{R}^n} \sup_{0<t<T} \left(t^{-n/2} \int_0^t \int_{|y-x|<\sqrt{t}} |u(y,s)|^2 dy ds \right)^{1/2} < \infty.$$

Moreover, if u_0 belongs to the closure of $\mathcal{S}(\mathbb{R}^n)$ in BMO^{-1}, denoted by VMO^{-1}, there exists $T > 0$ and a mild solution $u \in \mathcal{C}_T$ of (1.1)–(1.3) on $\mathbb{R}^n \times [0,T)$, where

$$\mathcal{C}_T = \{ v \in \mathcal{E}_T \ : \ \lim_{\tau\to 0} \|v\|_{\mathcal{E}_\tau} = 0 \}.$$

For further results on solutions evolving from BMO_T^{-1} initial datum see for instance [ADT, GPS, L] and references therein.

In the present paper, we prove that if $u \in \mathcal{E}_T$ is a weak solution of (1.1)–(1.3), then there exists $\phi(t) \in L^\infty((0,T))$, with $\lim_{t\to 0} \phi(t) = 0$, such that $\widetilde{u}(x,t) = u(x - \Phi(t), t) + \phi(t) \in \mathcal{E}_T$ is a mild solution, where $\Phi(t) = \int_0^t \phi(s) ds$. Our main result states that all weak solutions $u \in \mathcal{C}_T$ are obtained from the unique $\mathcal{C}_T$-mild solution by the transgalilean transformation $u \mapsto \widetilde{u}$ described above. We also give a natural necessary and sufficient condition for a weak solution to be mild, and hence for weak solutions to be unique, in terms of the pressure. This generalizes the results in [GIKM, K, Ku].

Even though weak solutions in $\mathcal{E}_T$ are bounded for positive time, the results in [Ku] do not apply here since we are considering the uniqueness at $t = 0$. We note that the criticality of the problem creates several difficulties which do not arise in the L^∞ case [Ku], including the definition of a weak solution and the behavior of weak solutions as $t \to 0$. A consequence of Lemma 3.1 is that the class of weak solutions we consider is contained in the class of weak solutions in [Ku]. Thus, even when restricting the class of weak solutions, we prove that uniqueness still fails.

The paper is organized as follows. In Section 2 we give the definition of a weak solution and state the main theorems. Section 3 contains the proof that a weak solution attains the initial value in the sense of distributions, and that every mild solution is a weak solution. Lemma 3.3 is the main ingredient of the proofs of our main theorems. Both, Theorem 2.1 and Theorem 2.2, are then proven at the end of Section 3.

2. Main Results

The summation convention on repeated indices will be used throughout. We shall use the same notation for scalars and vectors. In the following, C denotes a sufficiently large, positive generic constant depending only on n, while the additional dependence on a quantity will be represented by a lower index. Lastly, $\lim_{t\to 0}$ denotes $\lim_{t\to 0+}$.

Throughout this paper, fix $T > 0$ and $u_0 \in BMO_T^{-1}$ such that $\nabla \cdot u_0 = 0$ in $\mathcal{D}'(\mathbb{R}^n)$.

Definition. *We say that $u \in \mathcal{E}_T$ is a weak solution of the initial value problem* (1.1)–(1.3) *on $\mathbb{R}^n \times [0,T)$ if it is weakly divergence free and there exists $p \in L^1_{loc}(\mathbb{R}^n \times (0,T))$ such that for all $\psi = (\psi_1, \ldots, \psi_n) \in C_0^\infty(\mathbb{R}^n \times [0,T))$:*

(i) *The following limit exists*

$$\int_{0+}^{T}\int p\partial_k\psi_k = \lim_{\varepsilon\to 0}\int_{\varepsilon}^{T}\int p\partial_k\psi_k. \tag{2.1}$$

(ii) *For any $k = 1,\dots,n$, we have*

$$\int_0^T\int u_k(\partial_t\psi_k + \Delta\psi_k + u_j\partial_j\psi_k) + \int_{0+}^{T}\int p\partial_k\psi_k = -\int u_{0k}\psi_k(\cdot,0). \tag{2.2}$$

Note that $|u|,\ |u|^2 \in L^1_{loc}(\mathbb{R}^n\times[0,T))$ for all $u\in\mathcal{E}_T$, which shows that the first integral on the left of (2.2) is well-defined. We shall refer to a weak/mild solution of (1.1)–(1.3) on $\mathbb{R}^n\times[0,T)$ with the initial datum u_0 simply as a weak/mild solution of $(NSE)_{u_0}$ on $[0,T)$. Note that as opposed to [FJR], we work in $\mathcal{D}'$ instead of $\mathcal{S}'$. We now state our main result.

Theorem 2.1. *A function $u\in\mathcal{E}_T$ is a weak solution of $(NSE)_{u_0}$ on $[0,T)$ if and only if there exists a mild solution $\widetilde{u}\in\mathcal{E}_T$ of $(NSE)_{u_0}$ on $[0,T)$, and a function $\phi=(\phi_1,\dots,\phi_n)\in L^\infty((0,T))$ with $\lim_{t\to0}\phi(t)=0$, such that*

$$u(x,t)=\widetilde{u}(x-\Phi(t),t)+\phi(t),$$

where $\Phi(t)=\int_0^t\phi(s)ds$.

Additionally, if $u^{(1)}$ and $u^{(2)}\in\mathcal{C}_T$ (or if $\|u^{(1)}\|_{\mathcal{E}_T}$ and $\|u^{(2)}\|_{\mathcal{E}_T}$ are sufficiently small) are two weak solutions of $(NSE)_{u_0}$ on $[0,T)$, then there exists ϕ as above such that $u^{(1)}(x,t)=u^{(2)}(x-\Phi(t),t)+\phi(t)$.

Note that by [KT] and Lemma 3.2 below, if $u_0\in VMO^{-1}$, the closure of $\mathcal{S}(\mathbb{R}^n)$ in BMO^{-1}, there exists $T>0$ and a weak solution $u\in\mathcal{C}_T$ of $(NSE)_{u_0}$ on $[0,T)$. The following natural condition guarantees the uniqueness of weak solutions.

Theorem 2.2. *There is at most one $u\in\mathcal{E}_T$ which is a is a weak solution of $(NSE)_{u_0}$ on $[0,T)$ and for which the associated pressure p satisfies*

$$p(x,t)=o(|x|),\ |x|\to\infty, \tag{2.3}$$

for a.e. $t\in(0,T)$.

The proof of Theorem 2.2 shows that (2.3) may be replaced by $\|p(\cdot,t)\|_{L^1(x+Q_1)}=o(|x|)$ as $|x|\to\infty$ for a.e. $t\in(0,T)$, where $x+Q_1$ represents the unit cube in $\mathbb{R}^n$ centered at x. This is a weaker condition on the pressure than in [GIKM, K, Ku].

We recall the definition of the j-th Riesz-transform R_j as the Fourier multiplier $i\xi_j/|\xi|$ (cf. [St, Chapter 3]). We shall denote the composition of the Riesz-transforms R_i and R_j by R_{ij}. It is well known that R_{ij} is a Calderón-Zygmund operator, and in particular it is bounded from L^∞ to BMO [F].

Remark 2.3. Several comments are in order to justify our definition of a weak solution of $(NSE)_{u_0}$ on $[0,T)$:

(a) In the case $u_0\in L^p$, where $1<p<\infty$, a weak solution of (1.1)–(1.3) is commonly defined, for instance in [FJR], as a divergence free vector field $u\in L^q(0,T;L^p(\mathbb{R}^n))$, such that for any $\psi\in\mathcal{S}(\mathbb{R}^n\times[0,T))$, with $\operatorname{div}(\psi(\cdot,t))=0$ for all $0\le t<T$, and for all $k=1,2,\dots,n$

$$\int_0^T\int u_k(\partial_t\psi_k+\Delta\psi_k+u_j\partial_j\psi_k)=-\int u_{0k}\psi_k(\cdot,0). \tag{2.4}$$

If we allow $u_0(x)$ and $u(x,t)$ not to decay at infinity, for instance $u_0(x)=1$ for all $x\in\mathbb{R}^n$, then letting $u_k(x,t)=2$ for all $(x,t)\in\mathbb{R}^n\times(0,T)$, we have that for any ψ as above

$$\int_0^T\int u_k(\partial_t\psi_k+\Delta\psi_k+u_j\partial_j\psi_k)=0=-\int u_{0k}\psi_k(\cdot,0).$$

This is because for any $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\operatorname{div}(\phi) = 0$, we have that $\int \phi = 0$ (cf. [MS]). Thus, (2.4) for divergence free test functions ψ would allow the constant function 2 to be a weak solution of (1.1)–(1.3) with the initial condition 1.

(b) The solutions constructed by Koch and Tataru (cf. [KT]) for $u_0 \in BMO_T^{-1}$, have the property that the associated pressure is given by $p = R_{ij}(u_i u_j)$. Moreover, since $\sqrt{t}\, u(x,t) \in L^\infty(\mathbb{R}^n \times [0,T))$ and R_{ij} maps L^∞ to BMO, we obtain that $\sup_{0<t<T} t\, \|p(\cdot,t)\|_{BMO} < \infty$. But this does not guarantee that $p \in L^1_{loc}(\mathbb{R}^n \times [0,T))$, and therefore $\int_0^T\int p \cdot \nabla\psi$ is not assured to exist as a Lebesgue integral. This explains the requirement (2.1).

(c) The following example suggests that $\int_0^T\int p \cdot \nabla\psi$ might not always exist in the Lebesgue sense. Namely, $u_k(x,t) = t\sin(1/t)$ and $p(x,t) = -x_k(\sin(1/t) - (1/t)\cos(1/t))$ solve (1.1)–(1.3) with initial datum $u_0(x) = 0$. However, for $\psi \in C_0^\infty(\mathbb{R}^n \times [0,T))$ which does not vanish identically at $t = 0$, one can readily verify that $\int_0^T\int |p\partial_k\psi_k| = \infty$.

3. Proofs of Results

The following lemma shows how weak solutions behave as time converges to 0.

Lemma 3.1. *If $u \in \mathcal{E}_T$ is a weak solution of $(NSE)_{u_0}$ on $[0,T)$, then $u(\cdot,t) \to u_0$ in $\mathcal{D}'(\mathbb{R}^n)$ as $t \to 0$. Moreover, if the pressure associated to u is given by $p = R_{ij}(u_i u_j)$, then the convergence holds in $\mathcal{S}'(\mathbb{R}^n)$.*

Using the explicit representation of the kernel K_{ij} one can prove (cf. [L, Proposition 11.1]) that for $f \in \mathcal{S}(\mathbb{R}^n)$ and $i,j,k \in \{1,\ldots,n\}$ we have

$$|R_{ij}(\partial_k f)(x)| \le C_f \frac{1}{(1+|x|)^{n+1}}, \tag{3.1}$$

where C_f can be taken

$$C_f = \sum_{|\alpha|\le n+1} \|(1+|\xi|)\partial^\alpha \hat{f}(\xi)\|_{L^1} + \sum_{|\alpha|\le n+2} \|\partial^\alpha \hat{f}\|_{L^\infty}. \tag{3.2}$$

Proof of Lemma 3.1. Since we are interested in the behavior of $u(\cdot,t)$ as $t \to 0$, and since $BMO_{T_1} \subseteq BMO_{T_2}$ whenever $0 < T_2 \le T_1$, we may assume in this proof that $T \le 1$. First we prove the lemma for a general $p \in L^1_{loc}(\mathbb{R}^n \times (0,T))$. Let $\varphi = (\varphi_1,\ldots,\varphi_n) \in \mathcal{D}(\mathbb{R}^n)$ be a fixed test function. Since $\sqrt{t}\, u_k(x,t) \in L^\infty(\mathbb{R}^n \times [0,T))$, we have that $\int u_k(\cdot,t)\varphi_k \in L^1([0,T))$.

Fix $\tau \in (0,T/2)$ which is a point in the Lebesgue set of $\int u_k(\cdot,t)\varphi_k$. For each integer $m \ge 1/\tau$ we define a nonincreasing function $\alpha_m \in C_0^\infty([0,T))$ by $\alpha_m = 1$ on $[0,\tau]$ and $\alpha_m = 0$ on $[\tau + 1/m, 2\tau]$. Let the sequence $\{\alpha_m\}$ also satisfy $(1/m)\|\alpha_m'\|_{L^\infty} < C$, for all $m \ge 1/\tau$. Then $\alpha_m(t)\varphi(x) \in C_0^\infty(\mathbb{R}^n \times [0,T))$, and according to (2.2) with $\psi_k(x,t) = \alpha_m(t)\varphi_k(x)$, we have

$$\int_0^{2\tau}\int u_k\alpha_m'\varphi_k + \int_0^{2\tau}\int u_k\alpha_m(\Delta\varphi_k + u_j\partial_j\varphi_k) + \int_{0+}^{2\tau}\int p\alpha_m\partial_k\varphi_k = -\int u_{0k}\varphi_k. \tag{3.3}$$

Note that $\operatorname{supp}\alpha_m' \subset [\tau,\tau+1/m]$ and $\int_0^{2\tau}\alpha_m'(t)dt = \alpha_m(2\tau) - \alpha_m(0) = -1$. The Lebesgue Differentiation Theorem implies that

$$\lim_{m\to\infty}\int_0^{2\tau}\int u_k(x,t)\alpha_m'(t)\varphi_k(x)dxdt = -\int u_k(x,\tau)\varphi_k(x)dx. \tag{3.4}$$

Sending $m \to \infty$ in (3.3) we obtain

$$\int u_k(x,\tau)\varphi_k(x)dx - \int u_{0k}(x)\varphi_k(x)dx = \int_0^\tau\int u_k(\Delta\varphi_k + u_j\partial_j\varphi_k) + \int_{0+}^\tau\int p\partial_k\varphi_k. \tag{3.5}$$

To analyze the first term on the right of (3.5), note that

$$\int_0^\tau\int |u_k\Delta\varphi_k| \le \int_0^\tau \|u_k(\cdot,t)\|_{L^\infty}\|\Delta\varphi_k\|_{L^1}dt \le C_\varphi\sqrt{\tau}\,\|u\|_{\mathcal{E}_\tau}. \tag{3.6}$$

Letting $q = 1/\sqrt{n}$ and $M \geq q$ be the smallest number such that $\operatorname{supp} \varphi \subseteq [-M+q, M-q]^n$, we get

$$\begin{aligned}\int_0^\tau \int |u_k u_j \partial_j \varphi_k| &\leq \sum_{k\in\mathbb{Z}^n, kq\sqrt{\tau}\in[-M,M]^n} \int_0^\tau \int_{|x-kq\sqrt{\tau}|\leq\sqrt{\tau}} |u_k|\, |u_j|\, |\partial_j \varphi_k| \\ &\leq C \, \|\nabla\varphi\|_{L^\infty}\, \tau^{n/2} \left(\frac{M}{\sqrt{\tau}}\right)^n \|u\|^2_{\mathcal{E}_\tau} \leq C_\varphi \, \|u\|^2_{\mathcal{E}_\tau}. \end{aligned} \tag{3.7}$$

By the Dominated Convergence Theorem we also obtain that

$$\lim_{\tau\to 0} \int_0^\tau \int u_k u_j \partial_j \varphi_k = 0, \tag{3.8}$$

without making the assumption that $\lim_{\tau\to 0} \|u\|_{\mathcal{E}_\tau} = 0$. Observe that the condition (i) in the definition of a weak solution implies that for any $\psi \in C_0^\infty(\mathbb{R}^n \times [0,T))$

$$\lim_{\tau\to 0} \int_{0+}^\tau \int p \partial_k \psi_k = 0. \tag{3.9}$$

Indeed, by the definition of the limit in (2.1), for any $\varepsilon > 0$, there exists $\delta_0 > 0$ such that for any $0 < \delta_2 < \delta_1 < \delta_0$, we have $\left|\int_{\delta_2}^{\delta_1}\int p\partial_k\psi_k\right| < 2\varepsilon$, which proves (3.9). Now collecting (3.5)–(3.9) we obtain the desired $\mathcal{D}'$ convergence, along the Lebesgue set of $\int u_k(\cdot,t)\varphi_k$; namely,

$$\begin{aligned}&\lim_{\tau\to 0} \int \left(u_k(x,\tau) - u_{0k}(x)\right) \varphi_k(x) dx \\ &= \lim_{\tau\to 0} \int_0^\tau \int u_k(\Delta\varphi_k + u_j \partial_j \varphi_k) + \lim_{\tau\to 0} \int_{0+}^\tau \int p \partial_k \varphi_k = 0, \end{aligned} \tag{3.10}$$

where all the limits in τ are taken on the Lebesgue set. Choosing another suitable sequence $\alpha_m(t) \in C_0^\infty([0,T))$ and proceeding as above, we prove the weak continuity of u_k in time, i.e., the continuity in t of $\int u_k(\cdot,t)\varphi_k$. This shows that the limit in (3.10) can be taken along any sequence $\tau \to 0$ (not necessarily along the Lebesgue set).

Now, assume that $p = R_{ij}(u_i u_j)$, and let $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathcal{S}(\mathbb{R}^n)$. Fix a smooth radial function $\theta \in \mathcal{D}(\mathbb{R}^n)$ that is identically 1 in a neighborhood of the origin, with $\int \theta = 1$. For any $R \in \mathbb{N}$ and $x \in \mathbb{R}^n$, let $\theta_R(x) = \theta(x/R)$. Also fix $\tau \in (0, T/2)$, an element in the intersection of the Lebesgue sets of $\int u_k(\cdot,t)\varphi_k$ and $\int u_k(\cdot,t)\theta_R\varphi_k$, for $R \in \mathbb{N}$. Define $\alpha_m(t)$ be as above, and substitute $\alpha_m(t)\theta_R(x)\varphi_k(x)$ in (2.2) to obtain

$$\begin{aligned}&\int_0^{2\tau} \int u_k \alpha_m' \theta_R \varphi_k + \int u_{0k} \theta_R \varphi_k \\ &\quad = \int_0^{2\tau} \int u_k \alpha_m \left(\Delta(\theta_R\varphi_k) + u_j \partial_j(\theta_R\varphi_k)\right) + \int_{0+}^{2\tau} \int R_{ij}(u_i u_j) \alpha_m \partial_k(\theta_R\varphi_k) \\ &\quad = \int_0^{2\tau} \int u_k \alpha_m \left(\Delta(\theta_R\varphi_k) + u_j \partial_j(\theta_R\varphi_k)\right) + \int_0^{2\tau} \int u_i u_j \alpha_m R_{ij} \left(\partial_k(\theta_R\varphi_k)\right). \end{aligned} \tag{3.11}$$

As in (3.5) we send $m \to \infty$ and use the Lebesgue Differentiation Theorem to get

$$\begin{aligned}&\int u_{0k}\theta_R\varphi_k - \int u_k(\cdot,\tau)\theta_R\varphi_k \\ &\qquad = \int_0^\tau \int u_k \left(\Delta(\theta_R\varphi_k) + u_j\partial_j(\theta_R\varphi_k)\right) + \int_0^\tau \int u_i u_j R_{ij} \left(\partial_k(\theta_R\varphi_k)\right). \end{aligned} \tag{3.12}$$

For $R \in \mathbb{N}$, similarly to (3.6), we have

$$\int_0^\tau \int |u_k \Delta((1-\theta_R)\varphi_k)| \leq C\sqrt{\tau} \|u\|_{\mathcal{E}_\tau} \|\Delta((1-\theta_R)\varphi_k)\|_{L^1}, \tag{3.13}$$

which converges to 0 as $R \to \infty$. We also have

$$\lim_{R\to\infty} \int_0^\tau \int u_k u_j \partial_j ((1-\theta_R)\varphi_k) = 0. \tag{3.14}$$

The proof is similar to the pressure term , treated next. To bound the last term on the right of (3.12), we split the space integral into cubes and use the decay rate given by (3.1) to obtain that for $R \in \mathbb{N}$ we have

$$\int_0^\tau \int |u_i u_j R_{ij} \partial_k ((1-\theta_R)\varphi_k)| \leq C_{(1-\theta_R)\varphi} \, \|u\|_{\mathcal{E}_\tau}^2 \sum_{k\in\mathbb{Z}^n} \frac{(\sqrt{\tau})^n}{(1+|k|q\sqrt{\tau})^{n+1}} \leq C_{(1-\theta_R)\varphi} \, \|u\|_{\mathcal{E}_\tau}^2, \tag{3.15}$$

where $C_{(1-\theta_R)\varphi}$ is given by (3.2). Using that $\|f\|_{L^1} \leq C \|f\|_{L^2}^{1/2} \, \||x|^n f\|_{L^2}^{1/2}$ holds for $f \in \mathcal{S}(\mathbb{R}^n)$, and the Hausdorff-Young Inequality, it is then easy to show that $\varphi \in \mathcal{S}(\mathbb{R}^n)$ implies $C_{(1-\theta_R)\varphi} \to 0$ as $R \to 0$. In (3.15) we also used that

$$\sup_{0<\tau<1} \sum_{k\in\mathbb{Z}^n} \frac{(\sqrt{\tau})^n}{(1+|k|q\sqrt{\tau})^{n+1}} \leq C, \tag{3.16}$$

where $q = 1/\sqrt{n}$ as earlier. Let us now analyze the left side of (3.12). We first show that

$$\lim_{R\to\infty} \int u_k(x,\tau)(1-\theta_R(x))\varphi_k(x)dx = 0.$$

The above follows from the Dominated Convergence Theorem and

$$\int |u_k(x,\tau)\varphi_k(x)| dx dt \leq C_\varphi \|u\|_{\mathcal{E}_\tau} \frac{1}{\sqrt{\tau}}.$$

We now prove that

$$\lim_{R\to\infty} \int u_{0k}(x)(1-\theta_R(x))\varphi_k(x)dx = 0. \tag{3.17}$$

This follows by writing $u_0 = v_0 + \partial_i v_i$, for appropriate functions $v_i \in BMO_T$ for all $i = 0, 1, \ldots, n$ (cf. [KT, L]). Since $v_0 \in BMO_T$, we also have $|v_0| \in BMO_T$; hence,

$$\begin{aligned} \int |v_{0j}(x)| \, |\varphi_j(x)| dx &\leq \|\varphi_j\|_{L^\infty} \sum_{k\in\mathbb{Z}^n} \int_{kT/2+[0,T/2]^n} |v_{0j}(x)| dx \\ &\leq C_{T,\varphi} \sum_{k\in\mathbb{Z}^n} (1+|k|) \, \|v_0\|_{BMO_T} \frac{1}{(1+|k|)^{n+2}} \leq C_{\varphi,T} \|v_0\|_{BMO_T}. \end{aligned}$$

In the above we used that $\varphi \in \mathcal{S}$ and that the difference between the averages of $|v_0|$ on two cubes Q and Q' of volume T, but whose centers are at distance $|k|$ apart, is proportional to $(1+|k|) \, \|v_0\|_{BMO_T}$, with an implicit constant depending on T. Similarly one can show that the terms with $\partial_i v_i$ have an R-independent L^1 bound, and hence (3.17) follows from the Dominated Convergence Theorem.

Collecting (3.12)-(3.17), we obtain by letting $R \to \infty$ that

$$\int (u_k(x,\tau) - u_{0k}(x)) \, \varphi_k(x) dx = \int_0^\tau \int u_k \left(\Delta\varphi_k + u_j \partial_j \varphi_k\right) + \int_0^\tau \int u_i u_j R_{ij} \left(\partial_k \varphi_k\right). \tag{3.18}$$

Using similar bounds as before we note that

$$\int_0^\tau \int |u_k \left(\Delta\varphi_k + u_j \partial_j \varphi_k\right)| + \int_0^\tau \int |u_i u_j R_{ij} \left(\partial_k \varphi_k\right)| \leq C_\varphi (\sqrt{\tau} + \|u\|_{\mathcal{E}_\tau}) \|u\|_{\mathcal{E}_\tau},$$

and hence the Dominated Convergence Theorem implies that the right of (3.18) vanishes as $\tau \to 0$, along sequences in the above mentioned intersection of Lebesgue sets. We again prove the continuity of $\int u_k(\cdot,t)\varphi_k$ in the variable t and obtain the desired $\mathcal{S}'$ convergence along any sequence $\tau \to 0$, completing the proof of the lemma. $\square$

We recall that the Hopf–Leray projector is given explicitly by $(\mathbb{P}f)_k = f_k + R_{kj}f_j$. This allows us to write (1.4) in terms of the heat kernel $G(x,t) = (4\pi t)^{-n/2}e^{-|x|^2/4t}$ for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$. Thus $u \in \mathcal{E}_T$ is a mild solution of $(NSE)_{u_0}$ on $[0,T)$ if

$$\begin{aligned} u_k(x,t) &= -\int_0^t\int (\delta_{ik} + R_{jk})\partial_j G(x-y,t-s)u_i(y,s)u_j(y,s)dyds + \int G(x-y,t)u_{0k}(y)dy \\ &= -\int_0^t\int \partial_j G(x-y,t-s)u_j(y,s)u_k(y,s)dyds \\ &\quad -\int_{0+}^t\int \partial_k G(x-y,t-s)p(y,s)dyds + \int G(x-y,t)u_{0k}(y)dy \end{aligned} \tag{3.19}$$

for a.e. $(x,t) \in \mathbb{R}^n \times [0,T)$, where $p = R_{ij}(u_iu_j)$. In Lemma 3.2, using the decay rate of $\partial_k(R_{ij}G(x,t))$ at infinity (cf. [FJR]), we prove that every mild solution of $(NSE)_{u_0}$ on $[0,T)$ is also a weak solution, as defined above, with the same initial datum.

Note that the function $u(x,t) = t$, for all $(x,t) \in \mathbb{R} \times [0,T)$, is a weak solution of $(NSE)_0$ on $[0,T)$, but that the unique mild solution with $u_0(x) = 0$ is $u(x,t) = 0$. Hence, a weak solution of $(NSE)_{u_0}$ on $[0,T)$ need not be mild.

Lemma 3.2. *If $u \in \mathcal{E}_T$ is a mild solution of $(NSE)_{u_0}$ on $[0,T)$, then u is also a weak solution of $(NSE)_{u_0}$ on $[0,T)$.*

Proof of Lemma 3.2. Let $p = R_{ij}(u_iu_j)$. Since $\sup_{0<t<T}\sqrt{t}\,\|u_k(\cdot,t)\|_{L^\infty} < \infty$, and since R_{ij} maps L^∞ to BMO, we have $\sup_{0<t<T} t\,\|p(\cdot,t)\|_{BMO} < \infty$, and hence $p \in L^1_{loc}(\mathbb{R}^n \times (0,T))$. We now need to check that p satisfies the condition (i) in the definition of a weak solution. For all $0 < \tau_1 < \tau_2 < \tau \le T$ and $\psi \in C_0^\infty(\mathbb{R}^n \times [0,T))$, by letting $q = 1/\sqrt{n}$, we have

$$\begin{aligned} \left|\int_{\tau_1}^{\tau_2}\int p(x,t)\partial_k\psi_k(x,t)dxdt\right| &\le \int_{\tau_1}^{\tau_2}\int |u_i(x,t)u_j(x,t)R_{ij}(\partial_k\psi_k)(x,t)dxdt| \end{aligned} \tag{3.20}$$

$$\begin{aligned} &\le C_\psi \sum_{k\in\mathbb{Z}^n}\tau^{n/2}\left(\tau^{-n/2}\int_0^\tau\int_{|x-kq\sqrt{\tau}|<\sqrt{\tau}}|u(x,t)|^2dxdt\right)\sup_{|x-kq\sqrt{\tau}|<\sqrt{\tau}}\frac{1}{(1+|x|)^{n+1}} \\ &\le C_\psi\|u\|^2_{\mathcal{E}_\tau}\sum_{k\in\mathbb{Z}^n}\frac{(\sqrt{\tau})^n}{(1+|k|q\sqrt{\tau})^{n+1}}. \end{aligned}$$

As earlier, the last sum in the above inequality is finite for any $0 < \tau \le T$, and (3.16) holds. Since the above is satisfied for all $0 < \tau_1 < \tau_2 < \tau$, the Dominated Convergence Theorem shows that the right term of (3.20) converges to 0 as $\tau \to 0$, and hence (2.1) holds. It remains to be shown that u and p satisfy (2.2). For this purpose, let $\psi \in C_0^\infty(\mathbb{R}^n \times [0,T))$. Using that u_k is given by (3.19), we can calculate $\partial_t u_k - \Delta u_k$ in $\mathcal{D}'(\varepsilon,T)$. For any $0 < \varepsilon < T$ we have

$$\int_\varepsilon^T\int \big(u_k(\partial_t\psi_k + \Delta\psi_k + u_j\partial_j\psi_k) + p\partial_k\psi_k\big) = -\int u_k(\cdot,\varepsilon)\psi_k(\cdot,\varepsilon).$$

By $u_k \in \mathcal{E}_T$, and by the previous discussion, the left side of the above equation is well–defined for $\varepsilon = 0$. Thus it remains to be shown that $\lim_{\varepsilon\to 0}\int u_k(\cdot,\varepsilon)\psi_k(\cdot,\varepsilon) = \int u_{0k}\psi_k(\cdot,0)$. We write

$$\begin{aligned} &\left|\int u_k(\cdot,\varepsilon)\psi_k(\cdot,\varepsilon) - \int u_{0k}\psi_k(\cdot,0)\right| \\ &\le \int |u_k(\cdot,\varepsilon)|\,|\psi_k(\cdot,\varepsilon)-\psi_k(\cdot,0)| + \left|\int \big(u_k(\cdot,\varepsilon)-u_{0k}(\cdot)\big)\,\psi_k(\cdot,0)\right| \\ &\le \|u(\cdot,\varepsilon)\|_{\mathcal{E}_\varepsilon}\,\varepsilon^{-1/2}\|\psi_k(\cdot,\varepsilon)-\psi_k(\cdot,0)\|_{L^1} + \left|\int \big(u_k(\cdot,\varepsilon)-u_{0k}(\cdot)\big)\,\psi_k(\cdot,0)\right|. \end{aligned} \tag{3.21}$$

Since $\psi_k \in C_0^\infty(\mathbb{R}^n \times [0,T))$, the mean value theorem implies that the first term on the far right side of (3.21) converges to 0 as $\varepsilon \to 0$. To approximate the second term, use the definition of $u_k(x,t)$ in (3.19), to obtain

$$
\begin{aligned}
u_k(x,\varepsilon) - u_{0k}(x) &= \int G(x-y,\varepsilon)u_{0k}(y)dy - u_{0k}(x) \\
&+ \int_{0+}^{\varepsilon}\int \Big(\partial_k G(x-y,\varepsilon-s)p(y,s) + \partial_j G(x-y,\varepsilon-s)u_j(y,s)u_k(y,s)\Big)dyds.
\end{aligned} \tag{3.22}
$$

The proof of the lemma is complete if the right side of the above converges to 0 in $\mathcal{D}'(\mathbb{R}^n)$ as $\varepsilon \to 0$. Since $p = R_{ij}(u_i u_j)$ and $\max\{|\partial_j G(y,s)|, |\partial_k R_{ij} G(y,s)|\} \le C/(|y| + \sqrt{s})^{n+1}$ (cf. [FJR]), we may split the space integral into cubes and obtain that the second term in (3.22) converges in to 0 as $\varepsilon \to 0$. Moreover

$$
\left| \int \left(\int G(x-y,\varepsilon)u_{0k}(y)dy - u_{0k}(x) \right) \psi_k(x,0)dx \right| \to 0 \text{ as } \varepsilon \to 0,
$$

since the heat semigroup $e^{\varepsilon\Delta}$ converges to the identity in $\mathcal{S}'$ as $\varepsilon \to 0$, proving the lemma. □

The following lemma proves the first part of Theorem 2.1.

Lemma 3.3. *If $u \in \mathcal{E}_T$ is a weak solution of $(NSE)_{u_0}$ on $[0,T)$, then there exists $\widetilde{u} \in \mathcal{E}_T$, a mild solution of $(NSE)_{u_0}$ on $[0,T)$, and a function $\phi = (\phi_1, \ldots, \phi_n) \in L^\infty((0,T))$, with $\lim_{t\to 0}\phi(t) = 0$, such that*

$$
u(x,t) = \widetilde{u}(x - \Phi(t), t) + \phi(t), \qquad (x,t) \in \mathbb{R}^n \times [0,T)
$$

where $\Phi(t) = \int_0^t \phi(s)ds$.

Moreover, if $u \in \mathcal{C}_T$ then $\widetilde{u} \in \mathcal{C}_T$. A weak solution $u \in \mathcal{C}_T$ of $(NSE)_{u_0}$ on $[0,T)$ is a mild solution if and only if the associated pressure satisfies $p = R_{ij}(u_i u_j)$.

Proof of Lemma 3.3. As in [Ku], we first prove that there exists a function $\phi \in L^\infty((0,T),\mathbb{R}^n)$, with $\lim_{t\to 0}\phi(t) = 0$, such that

$$
\partial_t u_k - \Delta u_k + \partial_j(u_j u_k) + \partial_k \pi + \phi_k'(t) = 0 \tag{3.23}
$$

holds in $\mathcal{D}'(\mathbb{R}^n \times (0,T))$, where $\pi = R_{ij}(u_i u_j)$.

Since u is a weak solution of $(NSE)_{u_0}$ on $[0,T)$, there exists $p \in L^1_{loc}(\mathbb{R}^n \times (0,T))$ such that the conditions (i) and (ii) hold. In particular, taking the divergence of (2.2), and using that both u and u_0 are divergence-free, we obtain that $\Delta p = -\partial_j\partial_k(u_j u_k) = \Delta\pi$ in $\mathcal{D}'(\mathbb{R}^n \times (0,T))$. Let $p_h = p - \pi$ be the harmonic component of the pressure. Following [Ku] one can show that for $1 \le k \le n$, $\partial_k p_h$ is a distribution depending only on t, and hence (3.23) holds in $\mathcal{D}'(\mathbb{R}^n \times (0,T))$ with $\partial_k p_h$ instead of ϕ_k'. We therefore define

$$
\begin{aligned}
\phi_k(t) &= \int u_k(x,t)\beta(x)dx - \int u_{0k}(x)\beta(x)dx - \int_0^t\int u_k(x,s)\Delta\beta(x)dxds \\
&- \int_0^t\int u_k(x,s)u_j(x,s)\partial_j\beta(x)dxds - \int_{0+}^t\int \pi(x,s)\partial_k\beta(x)dxds, \ t > 0
\end{aligned} \tag{3.24}
$$

where $\beta \in \mathcal{D}(\mathbb{R}^n)$ is fixed, with $\int \beta = 1$. We need to check that $\lim_{t\to 0+}\phi_k(t) = 0$. As is in the proof of Lemma 3.1, the last three terms on the right hand side of (3.24) are bounded as

$$
|\phi_k(t)| \le \left| \int u_k(x,t)\beta(x)dx - \int u_{0k}(x)\beta(x)dx \right| + C_\beta(\sqrt{t} + \|u\|_{\mathcal{E}_t})\|u\|_{\mathcal{E}_t}.
$$

Using the fact that $u(\cdot,t) \to u_0$ in $\mathcal{D}'(\mathbb{R}^n)$ as $t \to 0$, and the Dominated Convergence Theorem as in (3.8), we obtain the desired convergence of ϕ.

Define $\widetilde{u}(x,t) = u(x-\Phi(t),t)+\phi(t)$ and $\widetilde{p}(x,t) = \pi(x-\Phi(t),t)$. We claim that $\widetilde{u}$ is a weak solution with the associated pressure $\widetilde{p}$. The fact that $\widetilde{u} \in \mathcal{E}_T$ follows from

$$\|\widetilde{u}\|_{\mathcal{E}_\tau} \leq \|u\|_{\mathcal{E}_\tau} + C(\sqrt{\tau}+\tau)\|\phi\|_{L^\infty} + C\|u\|^2_{\mathcal{E}_\tau} \sup_{0<t<\tau} N(t), \tag{3.25}$$

where $N(t) \leq C\left(1+\sqrt{t}\|\phi\|_{L^\infty}\right)^n$ is the number of balls of radius $\sqrt{t}$ necessary to cover the ball of radius $\sqrt{t}+t\|\phi(t)\|_{L^\infty}$. Note that if $u \in \mathcal{C}_T$ all terms on the right of (3.25) vanish as $\tau \to 0$, proving that $\widetilde{u} \in \mathcal{C}_T$. After a short verification, we check that $\widetilde{u}$ and $\widetilde{p}$ satisfy condition (ii) of the definition of a weak solution. Condition (i) is satisfied by $\widetilde{p} = R_{ij}(\widetilde{u}_i\widetilde{u}_j)$ due to Lemma 3.2. This proves our claim, and by Lemma 3.1 we can conclude that $\widetilde{u}$ attains the initial value in $\mathcal{S}'$.

Following the arguments in [FJR] and [Ku] we now show that $\widetilde{u}$ is actually a mild solution, proving the first claim in the lemma. Let $\theta \in \mathcal{D}(\mathbb{R}^n)$ be identically 1 in a neighborhood of 0. For $R>0$ and $y \in \mathbb{R}^n$ let $\theta_R(y) = \theta(y/R)$. Also fix $\alpha \in C^\infty(\mathbb{R})$ such that $0 \leq \alpha \leq 1,\ \alpha' \geq 0,\ \alpha(s) = 0$ for $s \leq 1$ and $\alpha(s) = 1$ for $2 \leq s$. For a fixed $(x,t) \in \mathbb{R}^n \times [0,T)$ and for any $\varepsilon > 0$, define

$$\psi(y,s) = \alpha\left(\frac{s}{\varepsilon}\right)\alpha\left(\frac{t-s}{\varepsilon}\right)\theta_R(y)G(x-y,t-s),$$

for any $(y,s) \in \mathbb{R}^n \times [0,T)$. Since $\operatorname{supp}\ (\alpha(s/\varepsilon)\alpha((t-s)/\varepsilon)) \in (\varepsilon, t-\varepsilon)$ and $\alpha(s/\varepsilon)\alpha((t-s)/\varepsilon) = 1$ for $s \in (2\varepsilon, t-2\varepsilon)$, we have $\psi \in \mathcal{D}(\mathbb{R}^n \times (0,T))$. Moreover, letting $R \to \infty$, we obtain $\theta_R(y) \to 1$ for every $y \in \mathbb{R}^n$; using ψ as a test function in (2.2), we have

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_0^T\!\!\int \widetilde{u}_k(y,s)\alpha'\left(\frac{s}{\varepsilon}\right)\alpha\left(\frac{t-s}{\varepsilon}\right)G(x-y,t-s)dyds\\
&\quad - \frac{1}{\varepsilon}\int_0^T\!\!\int \widetilde{u}_k(y,s)\alpha\left(\frac{s}{\varepsilon}\right)\alpha'\left(\frac{t-s}{\varepsilon}\right)G(x-y,t-s)dyds\\
&\quad - \int_0^T\!\!\int \widetilde{u}_k(y,s)\widetilde{u}_j(y,s)\alpha\left(\frac{s}{\varepsilon}\right)\alpha\left(\frac{t-s}{\varepsilon}\right)\partial_j G(x-y,t-s)dyds\\
&\quad - \int_{0+}^T\!\!\int \widetilde{p}_k(y,s)\alpha\left(\frac{s}{\varepsilon}\right)\alpha\left(\frac{t-s}{\varepsilon}\right)\partial_k G(x-y,t-s)dyds = 0.
\end{aligned} \tag{3.26}$$

We analyze the behavior of the first term on the left side of (3.26). For $t \geq 4\varepsilon$ this term equals

$$(1/\varepsilon)\int_\varepsilon^{2\varepsilon}\!\!\int \widetilde{u}_k(y,s)\alpha'(s/\varepsilon)G(x-y,t-s)dyds.$$

Since $(1/\varepsilon)\int_\varepsilon^{2\varepsilon}\alpha'(s/\varepsilon)ds = 1$, we re-write this term as

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_\varepsilon^{2\varepsilon}\alpha'\left(\frac{s}{\varepsilon}\right)\int \widetilde{u}_k(y,s)\left(G(x-y,t-s)-G(x-y,t)\right)dyds\\
&\quad + \frac{1}{\varepsilon}\int_\varepsilon^{2\varepsilon}\alpha'\left(\frac{s}{\varepsilon}\right)\int\left(\widetilde{u}_k(y,s)-u_{0k}(y)\right)G(x-y,t)dyds + \int u_{0k}(y)G(x-y,t)dy.
\end{aligned} \tag{3.27}$$

The first term in the above expression can be bounded as

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_\varepsilon^{2\varepsilon}\alpha'\left(\frac{s}{\varepsilon}\right)\|\widetilde{u}_k(\cdot,s)\|_{L^\infty}\|G(x-\cdot,t-s)-G(x-\cdot,t)\|_{L^1}ds\\
&\qquad \leq \frac{1}{\varepsilon}\int_\varepsilon^{2\varepsilon}\alpha'\left(\frac{s}{\varepsilon}\right)\|\widetilde{u}\|_{\mathcal{C}_{2\varepsilon}}\frac{1}{\sqrt{s}}\|G(\cdot,t-s)-G(\cdot,t)\|_{L^1}ds.
\end{aligned}$$

Using the mean value theorem we check that $\lim_{s\to 0}(1/\sqrt{s})\|G(\cdot,t-s)-G(\cdot,t)\|_{L^1} = 0$. The Dominated Convergence Theorem implies that right side of the last inequality converges to 0 as $\varepsilon \to 0$. The second term in (3.27) vanishes as $\varepsilon \to 0$ since by Lemma 3.1 $\widetilde{u}(\cdot,s) \to u_0$ in $\mathcal{S}'(\mathbb{R}^n)$, while clearly $G(\cdot,t) \in \mathcal{S}(\mathbb{R}^n)$ for all $t > 0$.

Note that as $\varepsilon \to 0$, we have $(1/\varepsilon)\alpha(s/\varepsilon)\alpha'((t-s)/\varepsilon) \to \delta_t$ and $\alpha(s/\varepsilon)\alpha((t-s)/\varepsilon) \to \chi_{(0,t)}$ in $\mathcal{D}'(\mathbb{R})$. By letting $\varepsilon \to 0$ in (3.26), for $k = 1, \dots, n$, and a.e. $(x,t) \in \mathbb{R}^n \times [0,T)$, we have

$$\widetilde{u}_k(x,t) + \int_0^t \int \widetilde{u}_k(y,s)\widetilde{u}_j(y,s)\partial_j G(x-y,t-s)dyds \\ + \int_{0+}^t \int \widetilde{p}_k(y,s)\partial_k G(x-y,t-s)dyds - \int u_{0k}(y)G(x-y,t)dy = 0.$$

We showed earlier that $\widetilde{p} = R_{ij}(\widetilde{u}_i\widetilde{u}_j)$; thus $\widetilde{u}$ is a mild solution. Mild solutions are unique in $\mathcal{C}_T$ (cf. [KT], [L]); hence a weak solution $u \in \mathcal{C}_T$ is mild if and only if the associated pressure p satisfies $p = R_{ij}(u_iu_j)$, proving the lemma. $\square$

Proof of Theorem 2.1. Let $u \in \mathcal{E}_T$ be a weak solution of $(NSE)_{u_0}$ on $[0,T)$. By Lemma 3.3, there exists a mild solution $\widetilde{u} \in \mathcal{E}_T$ given by $\widetilde{u}(x,t) = u(x+\Phi(t),t) - \phi(t)$, for a suitable $\phi \in L^\infty((0,T))$. Conversely, if $\widetilde{u} \in \mathcal{E}_T$ is a mild solution of $(NSE)_{u_0}$ on $[0,T)$, then by Lemma 3.2, this is a weak solution, and the function $u(x,t) = \widetilde{u}(x-\Phi(t),t) + \phi(t) \in \mathcal{E}_T$ is also a weak solution of $(NSE)_{u_0}$ on $[0,T)$, proving the first statement of the theorem.

Let $u^{(1)}$ and $u^{(2)} \in \mathcal{C}_T$ be two weak solutions of $(NSE)_{u_0}$ on $[0,T)$. By Lemma 3.3 there exist mild solutions $\widetilde{u}^{(i)} \in \mathcal{C}_T$ of $(NSE)_{u_0}$ on $[0,T)$, and functions $\phi^{(i)} \in L^\infty((0,T))$ with $\lim_{t\to 0}\phi^{(i)} = 0$, such that $u^{(i)}(x,t) = \widetilde{u}^{(i)}(x-\Phi^{(i)}(t),t) + \phi^{(i)}(t)$, for $i = 1,2$.

We claim that $\widetilde{u}^{(1)} = \widetilde{u}^{(2)}$ a.e. in $\mathbb{R}^n \times [0,T)$. This is known (cf. [ADT]), but we sketch the proof to emphasize the necessity of $\widetilde{u}^{(i)} \in \mathcal{C}_T$ (or that $\|u^{(i)}\|_{\mathcal{E}_T}$ is sufficiently small) for $i = 1,2$. Denote $B(u,v) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes v)ds$. Then $\widetilde{u}^{(1)}(\cdot,t) - \widetilde{u}^{(2)}(\cdot,t) = B(\widetilde{u}^{(1)}, \widetilde{u}^{(1)} - \widetilde{u}^{(2)}) + B(\widetilde{u}^{(1)} - \widetilde{u}^{(2)}, \widetilde{u}^{(2)})$. For $\tau \in (0,T)$, by [KT] we have that $\|B(u,v)\|_{\mathcal{E}_\tau} \le C_0\|u\|_{\mathcal{E}_\tau}\|v\|_{\mathcal{E}_\tau}$, and hence

$$\|\widetilde{u}^{(1)} - \widetilde{u}^{(2)}\|_{\mathcal{E}_\tau} \le C_0\left(\|\widetilde{u}^{(1)}\|_{\mathcal{E}_\tau} + \|\widetilde{u}^{(2)}\|_{\mathcal{E}_\tau}\right)\|\widetilde{u}^{(1)} - \widetilde{u}^{(2)}\|_{\mathcal{E}_\tau}. \tag{3.28}$$

Since $\lim_{\tau\to 0}\|\widetilde{u}^{(i)}\|_{\mathcal{E}_\tau} = 0$, for $i = 1,2$, we can fix τ such that $C_0\left(\|\widetilde{u}^{(1)}\|_{\mathcal{E}_\tau} + \|\widetilde{u}^{(2)}\|_{\mathcal{E}_\tau}\right) \le 1/2$. Then (3.28) shows that $\widetilde{u}^{(1)}$ and $\widetilde{u}^{(2)}$ agree on $[0,\tau)$. For $t \ge \tau$ we have $\widetilde{u}^{(i)}(\cdot,t) \in L^\infty$ for $i = 1,2$, and hence $\widetilde{u}^{(1)}(\cdot,t) = \widetilde{u}^{(2)}(\cdot,t)$ on $[\tau,T)$ (cf. [FJR]).

Thus letting $\phi(t) = \phi^{(1)}(t) - \phi^{(2)}(t)$, we obtain that $\widetilde{u}^{(1)}(x,t) = \widetilde{u}^{(2)}(x-\Phi(t),t) + \phi(t)$, concluding the proof of the Theorem. $\square$

Proof of Theorem 2.2. By Theorem 2.1, in order to show that a weak solution $u \in \mathcal{E}_T$ of $(NSE)_{u_0}$ on $[0,T)$ is unique, it suffices to prove that the function $\phi(t)$ constructed in Lemma 3.3 is identically 0 on $[0,T)$.

We write $\pi = R_{ij}(u_iu_j)$ and $p_h = p - \pi$. It was shown earlier that $\Delta p_h = 0$ in $\mathcal{D}'(\mathbb{R}^n)$ and that $\pi(\cdot,t) \in BMO$ for a.e. $t > 0$. We denote $x + Q_1$ the unit cube centered at x in $\mathbb{R}^n$. It then follows that for a.e. $t > 0$ and $|x| > 1$,

$$\left|\int_{x+Q_1}|\pi(\cdot,t)| - \int_{Q_1}|\pi(\cdot,t)|\right| \le C_t\log|x|.$$

This shows that $\|\pi(\cdot,t)\|_{L^1(x+Q_1)} = o(|x|)$, as $|x| \to \infty$, for a.e. $t \in (0,T)$. Moreover, $p = o(|x|)$, as $|x| \to \infty$ implies $\|p(\cdot,t)\|_{L^1(x+Q_1)} = o(|x|)$, as $|x| \to \infty$, and hence for a.e. $t \in (0,T)$ we have

$$\|p_h(\cdot,t)\|_{L^1(x+Q_1)} = \|p(\cdot,t) - \pi(\cdot,t)\|_{L^1(x+Q_1)} = o(|x|), \text{ as } |x| \to \infty. \tag{3.29}$$

The proof of Theorem 2.1 implies that $\partial_k p_h = \phi_k'(t)$ in $\mathcal{D}'(\mathbb{R}^n \times (0,T))$, which implies

$$p_h = x_k\phi_k'(t) + f, \tag{3.30}$$

where $f \in \mathcal{D}'(\mathbb{R}^n \times (0,T))$ is a distribution of time. Based on the fact that $p_h = p - R_{ij}(u_iu_j) \in L^1_{loc}(\mathbb{R}^n \times (0,T))$ we obtain $\phi_k'(t), f \in L^1_{loc}(\mathbb{R}^n \times (0,T))$ for $k = 1,2,\dots,n$. From (3.29) and (3.30) it follows that $\phi_k'(t) = 0$ for a.e. $t \in (0,T)$, and since $\lim_{t\to 0}\phi_k(t) = 0$, we obtain $\phi_k(t) = 0$ for $t \in (0,T)$, concluding the proof of the theorem. $\square$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CA 90089.
E-mail address: kukavica@usc.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CA 90089.
E-mail address: vicol@usc.edu