LOCAL EXISTENCE AND UNIQUENESS FOR THE HYDROSTATIC EULER EQUATIONS ON A BOUNDED DOMAIN

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ABSTRACT. We address the question of well-posedness in spaces of analytic functions for the Cauchy problem for the hydrostatic incompressible Euler equations (inviscid primitive equations) on domains with boundary. By a suitable extension of the Cauchy-Kowalewski theorem we construct a locally in time, unique, real-analytic solution and give an explicit rate of decay of the radius of real-analyticity.

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1. INTRODUCTION

The literature on incompressible homogeneous geophysical flows in the hydrostatic limit is vast, and several models have been proposed both in the viscous and the inviscid case (cf. J.-L. Lions, Temam, and Wang [13, 14, 15], P.-L. Lions [11], Pedloski [19], Temam and Ziane [27], and references therein). In the present paper we consider the inviscid hydrostatic model which is classical in geophysical fluid mechanics; see [19] and [11, Section 4.6], where the author raises the question of existence and uniqueness of solutions. These equations are formally derived from the three-dimensional incompressible Euler equations for a fluid between two horizontal plates, in the limit of vanishing distance between the plates [4, 5, 11]. The problem is to find a velocity field $\boldsymbol{u} = (v_1, v_2, w) = (\boldsymbol{v}, w)$, a scalar pressure P, and a scalar density ρ solving

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + w\,\partial_z \boldsymbol{v} + \nabla P + f\boldsymbol{v}^\perp = 0, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{v} + \partial_z w = 0, \tag{1.2}$$

$$\partial_z P = -\rho g, \tag{1.3}$$

$$\partial_t \rho + (\boldsymbol{v} \cdot \nabla)\rho + w\,\partial_z \rho = 0, \tag{1.4}$$

in $\mathcal{D} \times (0, T)$, for some $T > 0$. Here

$$\mathcal{D} = \mathcal{M} \times (0, h) = \{(x_1, x_2, z) = (\boldsymbol{x}, z) \in \mathbb{R}^3 : \boldsymbol{x} \in \mathcal{M}, 0 < z < h\}$$

is a 3-dimensional cylinder of height h, where $\mathcal{M} \subset \mathbb{R}^2$ is a smooth domain with real-analytic boundary. We denote by $\operatorname{div}, \nabla$, and Δ the corresponding 2-dimensional operators acting on $\boldsymbol{x} = (x_1, x_2)$, while $\partial_z = \partial/\partial z$. Also, we let $\boldsymbol{v}^\perp = (v_2, -v_1)$ be the first two components of $\boldsymbol{u} \times e_3$, f is the strength of the rotation, and g is the gravitational constant. It follows from (1.9) that the pressure P may be written in terms of the density ρ and the horizontal pressure $p(\boldsymbol{x}, t) = P(\boldsymbol{x}, 0, t)$

$$P(\boldsymbol{x}, z, t) = p(\boldsymbol{x}, t) - g\psi(\boldsymbol{x}, z, t), \tag{1.5}$$

where we have denoted

$$\psi(\boldsymbol{x}, z, t) = \int_0^z \rho(\boldsymbol{x}, \zeta, t)\, d\zeta, \tag{1.6}$$

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for all $(\boldsymbol{x}, z) \in \mathcal{D}$ and $t \geq 0$. The hydrostatic Euler equations (1.1)–(1.4) may then be rewritten as

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + w\, \partial_z \boldsymbol{v} + \nabla p + f\boldsymbol{v}^{\perp} = g\nabla\psi, \tag{1.7}$$

$$\operatorname{div} \boldsymbol{v} + \partial_z w = 0, \tag{1.8}$$

$$\partial_z p = 0, \tag{1.9}$$

$$\partial_t \rho + (\boldsymbol{v} \cdot \nabla)\rho + w\, \partial_z \rho = 0, \tag{1.10}$$

where ψ is obtained from ρ via (1.6). The boundary conditions for the top and bottom $\Gamma_z = \mathcal{M} \times \{0, h\}$ and the side $\Gamma_x = \partial\mathcal{M} \times (0, h)$ of the cylinder $\mathcal{D}$ are

$$w(\boldsymbol{x}, z, t) = 0, \text{ on } \Gamma_z \times (0, T), \tag{1.11}$$

$$\int_0^h \boldsymbol{v}(\boldsymbol{x}, z, t)\, dz \cdot n = 0, \text{ on } \Gamma_x \times (0, T), \tag{1.12}$$

where n is the outward unit normal to $\mathcal{M}$. Note that there is no evolution equation for w. Instead, the incompressibility condition and (1.11) imply that

$$w(\boldsymbol{x}, z, t) = -\int_0^z \operatorname{div} \boldsymbol{v}(\boldsymbol{x}, \zeta, t)\, d\zeta, \tag{1.13}$$

for all $0 < z < h$, and $0 < t < T$, which combined again with (1.11) shows that the vertical average of $\operatorname{div} \boldsymbol{v}$ is zero, i.e.,

$$\int_0^h \operatorname{div} \boldsymbol{v}(\boldsymbol{x}, z, t)\, dz = 0, \tag{1.14}$$

for all $x \in \mathcal{M}$ and $0 < t < T$. We consider a real-analytic initial datum

$$\boldsymbol{v}(\boldsymbol{x}, z, 0) = \boldsymbol{v}_0(\boldsymbol{x}, z), \tag{1.15}$$

$$\rho(\boldsymbol{x}, z, 0) = \rho_0(\boldsymbol{x}, z) \tag{1.16}$$

in $\mathcal{D}$, which satisfies the compatibility conditions arising from (1.12) and (1.14), namely

$$\int_0^h \boldsymbol{v}_0(\boldsymbol{x}, z)\, dz \cdot n = 0, \tag{1.17}$$

for all $\boldsymbol{x} \in \partial\mathcal{M}$, and

$$\int_0^h \operatorname{div} \boldsymbol{v}_0(\boldsymbol{x}, z)\, dz = 0, \tag{1.18}$$

for all $\boldsymbol{x} \in \mathcal{M}$.

The existence and uniqueness of solutions to the hydrostatic Euler equations is an outstanding open problem (cf. P.-L. Lions [11]). The methods and results for hyperbolic systems cannot be applied to (1.7)–(1.12) in order to find a well-posed set of boundary conditions (cf. Oliger and Sündstrom [23]). The instability results of Grenier [6, 5] and Brenier [4] suggest that the problem may be ill-posed in Sobolev spaces, in analogy to the Prandtl equations. Recently, Renardy [20] proved that the linearization of the hydrostatic Euler equations at specific parallel shear flows is ill-posed in the sense of Hadamard. The only local existence result available for the nonlinear problem was obtained in two-dimensions by Brenier [3] under the assumptions of convexity of $\boldsymbol{v}$ in the z-variable, constant normal derivative of $\boldsymbol{v}$ on Γ_z, and of periodicity of $(\boldsymbol{v}, w, p)$ in the x-variable.

The study of this system in spaces of analytic functions is the point of convergence of a number of essential difficulties which we need to recall before we can properly describe the results in detail. For a given system of partial differential equations, there is the issue of well-posedness in the sense of Hadamard [7]: The objective here is to show that the system of partial differential equations supplemented with suitable boundary (and possibly initial) conditions, possesses a unique solution in certain spaces, and that the solution depends continuously on the data.

When we are interested in spaces of analytic functions (in the space variables), two different issues can be considered. The first one is when a well-posedness result has been proven in a suitable function space (e.g. a Sobolev space) for a suitable initial-boundary value problem, and we wish to show that the solution is analytic in space within the domain $\mathcal{D}$, or possibly up to the boundary of $\mathcal{D}$ (in which case the solution extends beyond $\overline{\mathcal{D}}$). Pertaining to this type of approach are the results of Bardos and Benachour [2] for the Euler equations (see also [9, 10, 16]), or Masuda [18] for the incompressible (viscous) Navier-Stokes equations. The second issue would be to look for a Cauchy-Kowalewski-type of result. In such a case a system of partial differential equations and a number of data are given, analytic, on a manifold $\mathcal{V}$. The equations may permit to compute all the derivatives of the unknown functions, and hence the uniqueness of an analytic solution in the neighborhood of $\mathcal{V}$ follows. For the existence, the problem is to derive enough a priori estimates showing that the Taylor series is convergent in a neighborhood of $\mathcal{V}$. Note that for some equations no additional boundary condition, except the data given of $\mathcal{V}$, are necessary. For example one can solve in this way the wave equation $u_{tt} - u_{xx} = 0$, $0 < x < 1$, for a short interval of time, without any boundary condition for u at $x = 0, 1$. Much more elaborate Cauchy-Kowalewski-type results include the results of Sammartino and Cafflish (cf. [24], see also [17] and references therein) for the dissipative Prandtl boundary layer equations. In [24, 17], using the abstract Cauchy-Kowalewski theorem (cf. Asano [1]), the authors prove the existence of a real-analytic solution. We note that due to the non-locality created by the pressure it is challenging to verify that the assumptions of the abstract Cauchy-Kowalewski theorem hold for the system (1.1)–(1.4).

In the case of the hydrostatic Euler equations (1.1)–(1.4) an additional fundamental set of difficulties arises when we raise the question of well-posedness. Indeed, according to an old result of Oliger and Sündstrom [23], revisited in Temam and Tribbia [26], the boundary value problem for these equations, as well as for a number of other equations from geophysical fluid mechanics, cannot be well-posed in $\mathcal{D}\times[0,T]$ for *any set of local boundary conditions*. The approach in [23] and [26] is the following: $\mathcal{D} = \mathcal{M} \times (0,h)$ with $\mathcal{M}$ of the form $(0,L_1)\times(0,L_2)$. An expansion of this system is made in a suitable set of cosine and sine functions in the vertical (z) direction. The boundary conditions which are needed for each mode n depends on n and they are thus of a nonlocal type. In that direction, a linearized system related to (1.1)–(1.4) has been studied by Rousseau, Temam, and Tribbia [21, 22], and a result of well-posedness for this system has been achieved using the linear semigroup theory. Note that in [21, 22] the boundary of $\mathcal{M} = (0,L_1)\times(0,L_2)$ is not analytic.

For the hydrostatic Euler equations (1.1)–(1.4) initial conditions need to be specified for the variables $\boldsymbol{v} = (v_1, v_2)$ and ρ, which are called *prognostic* variables in the language of geophysical fluid mechanics, but P and w are *diagnostic* variables, giving rise to another set of difficulties. At each instant of time, P and w can be expressed as functions (nonlocal functionals) of $\boldsymbol{v}$ and ρ. Furthermore, as it appears below, P is determined by the solution of an elliptic Neumann problem (cf. (7.24)–(7.25)) which introduces some form of ellipticity in the system (1.1)–(1.4), which is otherwise essentially hyperbolic. The requirements for solvability for this Neumann problem, lead us to introduce a novel side-boundary condition (cf. (1.12)). Since z-independent solutions to the 3D hydrostatic Euler equations (1.1)–(1.4) are solutions to the 2D incompressible Euler equations, the natural boundary condition (1.12) is necessary.

After all the preliminaries, we can describe our result as a nonlocal Cauchy-Kowalewski-type of result for the hydrostatic Euler equations (1.1)–(1.4). In the present paper we prove the existence and uniqueness of solutions to the Cauchy problem (1.7)–(1.12) in the two-dimensional case (that is with $\mathcal{M}$ and $(\boldsymbol{u}, \rho, p)$ independent of x_2), and the three-dimensional cases where $\mathcal{M}$ is a half-plane or a periodic box. These results were announced in the note [8]. The three dimensional case when $\mathcal{M}$ is a generic bounded domain with analytic boundary will be treated in a forthcoming paper. Note that this is not a boundary value problem. Indeed we do not require an infinite set of boundary conditions on $\partial\mathcal{D}$, as seems to be required by the results of [21, 22, 23, 26]. However, the Neumann problem needed for the determination of P destroys the hyperbolic nature of the equations (and thus also the finite speed of propagation), and for the uniqueness of solutions we cannot argue by unique continuation, but we rather employ methods which pertain to the evolution problem. Furthermore, we obtain an explicit rate of decay in time of the radius of analyticity of the solution. To the best of our knowledge this is the first local well-posedness result for the hydrostatic Euler equations in three dimensions, and in the absence of convexity, even in two dimensions.

Organization of the paper is as follows. Section 2 contains the functional setting and the statements of the main theorems. In Section 3 we give the *a priori* estimates for the two-dimensional case. In Section 4 we prove the derivation of the estimates for the velocity. The construction of solutions is given in Section 5, and the uniqueness is proven in Section 6. Section 7 contains the proof for the three-dimensional periodic domain and for the half-space.

2. Main Theorems

In the following, $\alpha = (\alpha_1, \alpha_2, \alpha_z) \in \mathbb{N}^3$ and $\beta = (\beta_1, \beta_2, \beta_z) \in \mathbb{N}^3$ denote multi-indices, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of all non-negative integers. The notation $\partial^\alpha = \partial_x^{\alpha'} \partial_z^{\alpha_z} = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_z^{\alpha_z}$, where $\alpha' = (\alpha_1, \alpha_2)$ will be used throughout. We denote the homogeneous Sobolev semi-norms $|\cdot|_m$, for $m \in \mathbb{N}$, by

$$|\boldsymbol{v}|_m = \sum_{|\alpha|=m} \|\partial^\alpha \boldsymbol{v}\|_{L^2(\mathcal{D})}, \tag{2.1}$$

where $\|\partial^\alpha \boldsymbol{v}\|_{L^2}^2 = \|\partial^\alpha v_1\|_{L^2}^2 + \|\partial^\alpha v_2\|_{L^2}^2$.

Remark 2.1. In the two-dimensional case, when $\mathcal{M} = (0,1) \times \mathbb{R}$ is independent of x_2, the L^2 norms in the definition of $|\cdot|_m$ are considered only with respect to x_1 and z, that is

$$\|\partial^\alpha v_i\|_{L^2(\mathcal{D})}^2 = \int_0^1 \int_0^h |\partial^\alpha v_i(\boldsymbol{x}, z)|^2 \, dz \, dx_1,$$

for $i \in \{1, 2\}$.

We recall that a function $\boldsymbol{v}(\boldsymbol{x}, z)$ is real-analytic in $\boldsymbol{x}$ and z, with radius of analyticity τ if there exists $M > 0$ such that

$$|\partial^\alpha \boldsymbol{v}(\boldsymbol{x}, z)| \leq M \frac{|\alpha|!}{\tau^{|\alpha|}},$$

for all $(\boldsymbol{x}, z) \in \mathcal{D}$ and $\alpha \in \mathbb{N}^3$, where $|\alpha| = \alpha_1 + \alpha_2 + \alpha_z$. For $r \geq 0$ and $\tau > 0$ fixed, we define the spaces of real-analytic functions

$$X_\tau = \left\{ \boldsymbol{v} \in C^\infty(\mathcal{D}) : \int_0^h \boldsymbol{v}|_{\Gamma_x} \, dz \cdot n = 0, \int_0^h \operatorname{div} \boldsymbol{v} \, dz = 0, \|\boldsymbol{v}\|_{X_\tau} < \infty \right\}, \tag{2.2}$$

where

$$\|\boldsymbol{v}\|_{X_\tau} = \sum_{m=0}^\infty |\boldsymbol{v}|_m \frac{(m+1)^r \tau^m}{m!}. \tag{2.3}$$

Similarly, denote

$$Y_\tau = \left\{ \boldsymbol{v} \in X_\tau, \|\boldsymbol{v}\|_{Y_\tau} < \infty \right\}, \tag{2.4}$$

where the semi-norm $\|\cdot\|_{Y_\tau}$ is given by

$$\|\boldsymbol{v}\|_{Y_\tau} = \sum_{m=1}^\infty |\boldsymbol{v}|_m \frac{(m+1)^r \tau^{m-1}}{(m-1)!}. \tag{2.5}$$

We write $\rho \in X_\tau$ if $\rho \in C^\infty(\mathcal{D})$ and $\|\rho\|_{X_\tau} < \infty$, and $\rho \in Y_\tau$ if $\rho \in X_\tau$ and $\|\rho\|_{Y_\tau} < \infty$. For ease of notation, let

$$\|(\boldsymbol{v}, \rho)\|_{X_\tau} = \|\boldsymbol{v}\|_{X_\tau} + \|\rho\|_{X_\tau},$$

and similarly

$$\|(\boldsymbol{v},\rho)\|_{Y_\tau} = \|\boldsymbol{v}\|_{Y_\tau} + \|\rho\|_{Y_\tau}.$$

Using the Sobolev embedding theorem it is clear from (2.3) that if $\boldsymbol{v} \in X_\tau$ then $\boldsymbol{v}$ is real-analytic with radius of analyticity τ. Conversely, if $\boldsymbol{v}$ is real-analytic with radius of analyticity τ (and satisfies the boundary conditions), then $\boldsymbol{v} \in X_{\tau'}$ for any $\tau' < \tau$ and $r \geq 0$, since $\sum_{m\geq 0} m^{r+2}(\tau'/\tau)^m < \infty$. Moreover, we have the estimate $\|\boldsymbol{v}\|_{X_\tau} \leq \|\boldsymbol{v}\|_{L^2(\mathcal{D})} + \tau\|\boldsymbol{v}\|_{Y_\tau}$, and for any $\varepsilon > 0$ we have $X_{\tau+\varepsilon} \subset Y_\tau$ since $\|\boldsymbol{v}\|_{Y_\tau} \leq (e\tau \ln(1+\varepsilon/\tau))^{-1}\|\boldsymbol{v}\|_{X_{\tau+\varepsilon}}$.

The following theorem is our main result for dimension two.

Theorem 2.2. *Let the functions $\boldsymbol{u}, p, \rho$ be independent of x_2, and let $r \geq 2$. Assume that $\boldsymbol{v}_0$ and ρ_0 are real-analytic with radius of analyticity strictly larger than τ_0, and suppose that $\boldsymbol{v}_0$ satisfies the compatibility conditions (1.17)–(1.18). Then there exists $T_* = T_*(r, g, \tau_0, \|(\boldsymbol{v}_0,\rho_0)\|_{X_{\tau_0}}) > 0$, and a unique real-analytic solution $(\boldsymbol{v}(t),\rho(t))$ of the initial value problem associated with (1.7)–(1.12) with radius of analyticity $\tau(t)$, such that*

$$\begin{aligned}\|(\boldsymbol{v}(t),\rho(t))\|_{X_{\tau(t)}} &+ Cg\int_0^t e^{g(t-s)}\,\|(\boldsymbol{v}(s),\rho(s))\|_{Y_{\tau(s)}}\,ds \\ &+ C\|(\boldsymbol{v}_0,\rho_0)\|_{X_{\tau_0}} e^{gt}\int_0^t \left(1+\tau^{-2}(s)\right)\|(\boldsymbol{v}(s),\rho(s))\|_{Y_{\tau(s)}}\,ds \leq \|(\boldsymbol{v}_0,\rho_0)\|_{X_{\tau_0}} e^{gt},\end{aligned} \tag{2.6}$$

for all $t \in [0, T_*)$, where $C = C(\mathcal{D})$ is a fixed positive constant. Moreover, the radius of analyticity of the solution, $\tau : [0,T_*) \mapsto \mathbb{R}_+$, may be computed explicitly from (3.26) below.

In the three-dimensional case, the boundary condition (1.12) allows us to find the pressure implicitly as the solution of an elliptic Neumann problem (cf. (7.24)–(7.25)). The classical higher-regularity estimates (cf. J.-L. Lions and Magenes [12], Temam [25]) may not be used to prove that the pressure has the same radius of analyticity as the velocity, preventing the estimates from closing. To overcome this obstacle we introduce a new analytic norm which combinatorially encodes the transfer of normal to tangential derivatives in the pressure estimate. The following theorem treats the case when $\mathcal{M}$ is a half-plane, or the periodic domain. The case when $\mathcal{M}$ is a generic real-analytic bounded domain in $\mathbb{R}^2$, and (1.12) holds on $\partial\mathcal{M}$, requires new ideas and will be treated in a forthcoming paper.

Theorem 2.3. *Let $r \geq 5/2$, and let $\mathcal{M}$ be either the upper half-plane $\{x_1 > 0\}$ or the periodic box $[0,2\pi]^2$. Assume that $\boldsymbol{v}_0$ and ρ_0 are real-analytic with radius of analyticity strictly larger than τ_0, and suppose that $\boldsymbol{v}_0$ satisfies the compatibility conditions (1.17)–(1.18). Then there exists $T_* = T_*(r, f, g, \tau_0, \|(\boldsymbol{v}_0,\rho_0)\|_{X_{\tau_0}}) > 0$, and a unique real-analytic solution $(\boldsymbol{v}(t),\rho(t))$ of the initial value problem associated with (1.7)–(1.12) with radius of analyticity $\tau(t)$, such that*

$$\begin{aligned}\|(\boldsymbol{v}(t),\rho(t))\|_{X_{\tau(t)}} &+ Cg\int_0^t e^{C_1(t-s)}\,\|(\boldsymbol{v}(s),\rho(s))\|_{Y_{\tau(s)}}\,ds \\ &+ C\|(\boldsymbol{v}_0,\rho_0)\|_{X_{\tau_0}} e^{C_1 t}\int_0^t \left(1+\tau^{-5/2}(s)\right)\|(\boldsymbol{v}(s),\rho(s))\|_{Y_{\tau(s)}}\,ds \leq \|(\boldsymbol{v}_0,\rho_0)\|_{X_{\tau_0}} e^{C_1 t},\end{aligned} \tag{2.7}$$

for all $t \in [0,T_*)$, where $C = C(\mathcal{D})$ and $C_1 = C_1(f,g)$ are fixed positive constants. Moreover, the radius of analyticity of the solution, $\tau : [0,T_*) \mapsto \mathbb{R}_+$ may be computed explicitly from (7.13) or (7.46) below, for the periodic box or the half-plane respectively.

The different powers of τ in (2.6) and (2.7) are due to the different exponents in the two-dimensional and three-dimensional Agmon's inequalities.

Remark 2.4. We note that the solutions $\boldsymbol{v}(t)$ constructed in Theorem 2.2 and Theorem 2.3 do not depend on the function $\tau(t)$. Moreover, if $\boldsymbol{v}^{(1)}(t)$ and $\boldsymbol{v}^{(2)}(t)$ are two such real-analytic solutions, with radii of analyticity $\tau^{(1)}(t)$ and $\tau^{(2)}(t)$ respectively, then $\boldsymbol{v}^{(1)}(t) = \boldsymbol{v}^{(2)}(t)$ for all t on the common interval of existence.

3. The two-dimensional case

In this section we give the formal *a priori* estimates needed to prove Theorem 2.2. These estimates will be made rigorous in Sections 5 and 6. Let $r \geq 2$ be fixed throughout the rest of the section. Assume that $(\boldsymbol{v}, w)$, p, and ρ are smooth solutions of (1.7)–(1.12), with $\boldsymbol{v}_0, \rho_0 \in X_{\tau_0}$, for some $\tau_0 > 0$, which are independent of x_2. The horizontal domain $\mathcal{M}$ needs to be x_2 independent, translation invariant in x_2, and hence without loss of generality we may consider $\mathcal{M} = (0,1) \times \mathbb{R}$. Since all derivatives with respect to x_2 vanish we denote $\partial_x = \partial_{x_1}$, and similarly $\Delta = \partial_{x_1 x_1}$. It is convenient to write the system of equations (1.7)–(1.12) in component form

$$\partial_t v_1 + v_1 \partial_x v_1 + w \partial_z v_1 + \partial_x p + f v_2 = g \partial_x \psi, \tag{3.1}$$

$$\partial_t v_2 + v_1 \partial_x v_2 + w \partial_z v_2 - f v_1 = 0, \tag{3.2}$$

$$\partial_x v_1 + \partial_z w = 0, \tag{3.3}$$

$$\partial_z p = 0, \tag{3.4}$$

$$\partial_t \rho + v_1 \partial_x \rho + w \partial_z \rho = 0, \tag{3.5}$$

where $\psi(\boldsymbol{x}, z) = \int_0^z \rho(\boldsymbol{x}, \zeta)\, d\zeta$. The boundary conditions for w and v_1 are

$$w = 0, \text{ on } \Gamma_z, \tag{3.6}$$

$$\int_0^h v_1 \, dz = 0, \text{ on } \Gamma_x, \tag{3.7}$$

where $\Gamma_x = \{0,1\} \times \mathbb{R}$. We note that there is no boundary condition for v_2. Integrating the incompressibility condition (3.3) in z, and using (3.6) we get $\partial_x \int_0^h v_1 \, dz = 0$. Combined with (3.7) this implies that for all $\boldsymbol{x} \in \mathcal{M}$ we have

$$\int_0^h v_1 \, dz = 0. \tag{3.8}$$

In the two-dimensional case, the boundary conditions (3.6) and (3.8) give the pressure explicitly as a function of $\boldsymbol{v}$ and ρ, and imply the cancelation property (3.10) below, which turns out to be convenient in the *a priori* estimates below.

Lemma 3.1. *Let $(\boldsymbol{v}, w, p, \rho)$ be a smooth solution of (3.1)–(3.8). Then, after subtracting from p a function of time, the pressure is given at each instant of time t by*

$$p(\boldsymbol{x}) = -\fint_0^h v_1^2(\boldsymbol{x}, z)\, dz - f \int_0^{x_1} \fint_0^h v_2(x_1', x_2, z)\, dz\, dx_1' + g \fint_0^h \psi(\boldsymbol{x}, z)\, dz, \tag{3.9}$$

where ψ is obtained from ρ via (1.6). In (3.9) we have suppressed the dependence in t for convenience. Also, we have the cancelation property

$$\langle \partial_x \partial^\alpha p, \partial^\alpha v_1 \rangle = 0, \tag{3.10}$$

for any multi-index $\alpha \in \mathbb{N}^3$.

In (3.9) and in the following we use the notation $\fint_0^h \phi(\boldsymbol{x}, z) dz = (1/h) \int_0^h \phi(\boldsymbol{x}, z) dz$, for any function ϕ. In the two dimensional case we do not integrate in x_2, so that in (3.10) we denoted $\langle \phi_1, \phi_2 \rangle = \int_0^1 \int_0^h \phi_1(\boldsymbol{x}, z) \phi_2(\boldsymbol{x}, z)\, dz\, dx_1$, for any pair of smooth real functions ϕ_1 and ϕ_2.

Proof of Lemma 3.1. Integrating (3.1) in z and using the boundary condition (3.8), we obtain

$$\partial_x \left(\int_0^h v_1^2(\boldsymbol{x}, z)\, dz + h p(\boldsymbol{x}) - g \int_0^h \psi(\boldsymbol{x}, z)\, dz \right) = -f \int_0^h v_2(\boldsymbol{x}, z)\, dz. \tag{3.11}$$

Here we used

$$\int_0^h w\partial_z v_1\,dz = -\int_0^h v_1\partial_z w\,dz = \int_0^h v_1\partial_x v_1\,dz, \tag{3.12}$$

which holds by $w|_{\Gamma_z} = 0$, and the incompressibility condition (3.3). The identity (3.9) then follows from (3.11) by integrating in x_1 and subtracting a function of time.

In order to prove (3.10), note that if $\alpha_z \neq 0$ or $\alpha_2 \neq 0$, then $\partial^\alpha p = 0$. On the other hand, if $\alpha_2 = \alpha_z = 0$ and $\alpha_1 \geq 1$, then by (3.3) we have $\partial^\alpha v_1 = \partial_x^{\alpha_1} v_1 = -\partial_x^{\alpha_1 - 1}\partial_z w$. Moreover, the boundary condition (3.6) for w implies $\partial_x^{\alpha_1-1} w = 0$ on Γ_z; therefore integrating by parts in z gives

$$\langle \partial_x\partial^\alpha p, \partial^\alpha v_1\rangle = -\langle \partial_x\partial^\alpha p, \partial_x^{\alpha_1-1}\partial_z w\rangle = \langle \partial_z\partial_x^{\alpha_1+1} p, \partial_x^{\alpha_1-1} w\rangle = 0.$$

Lastly, we need to prove (3.10) in the case $\alpha = (0,0,0)$. Integrating by parts in x_1 and using (3.3) we have $\langle \partial_x p, v_1\rangle = \langle p, \partial_z w\rangle$ since on $\Gamma_x = \{0,1\}\times\mathbb{R}$ by (3.7) we have

$$\int_0^h p|_{\Gamma_x} v_1|_{\Gamma_x} dz = p|_{\Gamma_x}\int_0^h v_1|_{\Gamma_x}\,dz = 0. \tag{3.13}$$

Integrating by parts in z we obtain $\langle p, \partial_z w\rangle = 0$ since $w|_{\Gamma_z} = 0$. Therefore, (3.10) is proven. $\square$

We now turn to *a priori* estimates needed to prove Theorem 2.2. From (2.1), (2.3), and (2.5) it follows that

$$\frac{d}{dt}\|\boldsymbol{v}(t)\|_{X_{\tau(t)}} = \dot\tau(t)\|\boldsymbol{v}(t)\|_{Y_{\tau(t)}} + \sum_{m=0}^\infty\sum_{|\alpha|=m}\frac{d}{dt}\|\partial^\alpha \boldsymbol{v}(t)\|_{L^2}\frac{(m+1)^r\tau(t)^m}{m!}. \tag{3.14}$$

Given a multi-index $\alpha\in\mathbb{N}^3$, we estimate $(d/dt)\|\partial^\alpha\boldsymbol{v}(t)\|_{L^2}$ by applying ∂^α to (3.1)–(3.2) and taking the $L^2(\mathcal{D})$-inner product $\langle\cdot,\cdot\rangle$ with $\partial^\alpha\boldsymbol{v}$. Recall that in $\langle\cdot,\cdot\rangle$ we integrate only with respect to x_1 and z. Since $\langle\partial^\alpha v^\perp, \partial^\alpha v\rangle = 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial^\alpha\boldsymbol{v}(t)\|_{L^2}^2 + \langle\partial^\alpha(\boldsymbol{v}\cdot\nabla\boldsymbol{v} + w\,\partial_z\boldsymbol{v}), \partial^\alpha\boldsymbol{v}\rangle + \langle\partial^\alpha\nabla p, \partial^\alpha\boldsymbol{v}\rangle = g\langle\partial^\alpha\nabla\psi, \partial^\alpha v\rangle. \tag{3.15}$$

To treat the second term on the left of (3.15), we use the Leibniz rule to write

$$\langle\partial^\alpha(\boldsymbol{v}\cdot\nabla\boldsymbol{v} + w\,\partial_z\boldsymbol{v}), \partial^\alpha\boldsymbol{v}\rangle = \sum_{0\leq\beta\leq\alpha}\binom{\alpha}{\beta}\langle\partial^\beta\boldsymbol{v}\cdot\nabla\partial^{\alpha-\beta}\boldsymbol{v}, \partial^\alpha\boldsymbol{v}\rangle + \sum_{0\leq\beta\leq\alpha}\binom{\alpha}{\beta}\langle\partial^\beta w\,\partial_z\partial^{\alpha-\beta}\boldsymbol{v}, \partial^\alpha\boldsymbol{v}\rangle.$$

The third term on the left of (3.15) vanishes by Lemma 3.1, and therefore by (3.15) and the Schwarz inequality we have

$$\frac{d}{dt}\|\partial^\alpha\boldsymbol{v}\|_{L^2} \leq \sum_{0\leq\beta\leq\alpha}\binom{\alpha}{\beta}\|\partial^\beta\boldsymbol{v}\cdot\nabla\partial^{\alpha-\beta}\boldsymbol{v}\|_{L^2} + \sum_{0\leq\beta\leq\alpha}\binom{\alpha}{\beta}\|\partial^\beta w\,\partial_z\partial^{\alpha-\beta}\boldsymbol{v}\|_{L^2} + g\|\nabla\partial^\alpha\psi\|_{L^2}. \tag{3.16}$$

Substituting estimate (3.16) above into (3.14), and using the independence on the x_2 variable, we have the *a priori* estimate

$$\frac{d}{dt}\|\boldsymbol{v}\|_{X_\tau} \leq \dot\tau\|\boldsymbol{v}\|_{Y_\tau} + \mathcal{U}(v_1,v_1) + \mathcal{U}(v_1,v_2) + \mathcal{V}(w,v_1) + \mathcal{V}(w,v_2) + g\|\partial_x\psi\|_{X_\tau}, \tag{3.17}$$

where for $i\in\{1,2\}$, a vector function $\boldsymbol{v}\in X_\tau$, and a scalar function $\tilde v\in C^\infty(\mathcal{D})$, we denoted

$$\mathcal{U}(v_i,\tilde v) = \sum_{m=0}^\infty\sum_{j=0}^m\sum_{|\alpha|=m}\sum_{|\beta|=j,\beta\leq\alpha}\binom{\alpha}{\beta}\|\partial^\beta v_i\,\partial_{x_i}\partial^{\alpha-\beta}\tilde v\|_{L^2}\frac{(m+1)^r\tau^m}{m!}, \tag{3.18}$$

and

$$\mathcal{V}(w,\tilde v) = \sum_{m=0}^\infty\sum_{j=0}^m\sum_{|\alpha|=m}\sum_{|\beta|=j,\beta\leq\alpha}\binom{\alpha}{\beta}\|\partial^\beta w\,\partial_z\partial^{\alpha-\beta}\tilde v\|_{L^2}\frac{(m+1)^r\tau^m}{m!}. \tag{3.19}$$

In (3.19) we denoted as usual $w(\boldsymbol{x},z) = -\int_0^z \operatorname{div} \boldsymbol{v}(\boldsymbol{x},\zeta)\, d\zeta$. The following lemma summarizes the estimates on $\mathcal{U}(v_1, v_i)$ and $\mathcal{V}(w, v_i)$, for $i \in \{1,2\}$. It is a corollary of Lemma 4.1, which is proven in Section 4.

Lemma 3.2. *Let $\boldsymbol{v} \in Y_\tau$, for some $\tau > 0$ and $r \geq 2$, and let w be given by (1.13). Then we have*

$$\mathcal{U}(v_1, v_1) + \mathcal{U}(v_1, v_2) \leq C_0(1+\tau^{-1})\|\boldsymbol{v}\|_{X_\tau}\|\boldsymbol{v}\|_{Y_\tau}, \tag{3.20}$$

and

$$\mathcal{V}(w, v_1) + \mathcal{V}(w, v_2) \leq C_0(1+\tau^{-2})\|\boldsymbol{v}\|_{X_\tau}\|\boldsymbol{v}\|_{Y_\tau}, \tag{3.21}$$

for some sufficiently large positive constant $C_0 = C(\mathcal{D})$.

To bound the last term on the right of (3.17) we note that

$$\begin{aligned}|\partial_x \psi|_m &= \sum_{|\alpha|=m} \|\partial^\alpha \partial_x \psi\|_{L^2} = \sum_{|\alpha|=m,\alpha_z=0} \left\| \int_0^z \partial^\alpha \partial_x \rho(\boldsymbol{x},\zeta)\, d\zeta \right\|_{L^2} + \sum_{|\alpha|=m,\alpha_z\geq 1} \|\partial_x^{\alpha_1+1}\partial_z^{\alpha_z-1}\rho(\boldsymbol{x},z)\|_{L^2} \\ &\leq C(h)|\rho|_{m+1} + |\rho|_m,\end{aligned}$$

so that by possibly enlarging the constant C_0 from (3.20) and (3.21), we have

$$g\|\partial_x\psi\|_{X_\tau} \leq C_0 g\|\rho\|_{Y_\tau} + g\|\rho\|_{X_\tau}. \tag{3.22}$$

We fix $C_0 = C_0(\mathcal{D})$ as in (3.20)–(3.22) throughout the rest of this section. By (3.17), (3.20), (3.21), and (3.22) we have

$$\frac{d}{dt}\|\boldsymbol{v}(t)\|_{X_{\tau(t)}} \leq \left(\dot{\tau}(t) + 3C_0(1+\tau(t)^{-2})\|\boldsymbol{v}(t)\|_{X_{\tau(t)}}\right)\|\boldsymbol{v}(t)\|_{Y_{\tau(t)}} + C_0 g\|\rho(t)\|_{Y_{\tau(t)}} + g\|\rho(t)\|_{X_{\tau(t)}}. \tag{3.23}$$

Similarly, by Lemma 4.1 we obtain from (3.5) an estimate for the growth of $\|\rho(t)\|_{X_{\tau(t)}}$, namely

$$\begin{aligned}\frac{d}{dt}\|\rho(t)\|_{X_{\tau(t)}} &\leq \dot{\tau}(t)\|\rho(t)\|_{Y_{\tau(t)}} + \mathcal{U}(v_1(t),\rho(t)) + \mathcal{V}(w(t),\rho(t)) \\ &\leq \dot{\tau}(t)\|\rho\|_{Y_{\tau(t)}} + C_0(1+\tau(t)^{-2})\|\boldsymbol{v}(t)\|_{X_{\tau(t)}}\|\rho(t)\|_{Y_{\tau(t)}} + 2C_0(1+\tau(t)^{-2})\|\boldsymbol{v}(t)\|_{Y_{\tau(t)}}\|\rho(t)\|_{X\tau(t)}.\end{aligned} \tag{3.24}$$

By summing the estimates (3.23) and (3.24) we obtain

$$\frac{d}{dt}\|(\boldsymbol{v},\rho)\|_{X_\tau} \leq \left(\dot{\tau} + C_0 g + 3C_0(1+\tau^{-2})\|(\boldsymbol{v},\rho)\|_{X_\tau}\right)\|(\boldsymbol{v},\rho)\|_{Y_\tau} + g\|(\boldsymbol{v},\rho)\|_{X_\tau}. \tag{3.25}$$

Define the decreasing function $\tau(t)$ by

$$\dot{\tau} + 20\, C_0 g + 20\, C_0(1+\tau^{-2})\|(\boldsymbol{v}_0,\rho_0)\|_{X_{\tau_0}} e^{gt} = 0, \tag{3.26}$$

and $\tau(0) = \tau_0$; this uniquely determines τ in terms of the initial data. Let T_* be the maximal time such that $\tau(t) \geq 0$. It is clear that $\tau(t)$ and T_* may be computed from (3.26) in terms of the initial data. By construction, we have at $t = 0$

$$\dot{\tau}(t) + C_0 g + 3C_0(1+\tau(t)^{-2})\|(\boldsymbol{v}(t),\rho(t))\|_{X_{\tau(t)}} < 0, \tag{3.27}$$

and by (3.25) we then have for a short time

$$\|(\boldsymbol{v}(t),\rho(t))\|_{X_{\tau(t)}} \leq \|(\boldsymbol{v}_0,\rho_0)\|_{X_{\tau_0}} e^{gt}. \tag{3.28}$$

It follows that (3.27), and hence (3.28), holds for all $t < T_*$. Moreover, by (3.25), we obtain that the solution is *a priori* bounded in $L^\infty(0,T_*;X_\tau) \cap L^1(0,T_*;(1+\tau^{-2})\,Y_\tau)$ in the sense

$$\begin{aligned}&\|(\boldsymbol{v}(t),\rho(t))\|_{X_{\tau(t)}} + 10\, C_0 g \int_0^t e^{g(t-s)}\|(\boldsymbol{v}(s),\rho(s))\|_{Y_{\tau(s)}}\, ds \\ &\qquad + 10\, C_0\|(\boldsymbol{v}_0,\rho_0)\|_{X_{\tau_0}} e^{gt}\int_0^t \left(1+\tau(s)^{-2}\right)\|(\boldsymbol{v}(s),\rho(s))\|_{Y_{\tau(s)}}\, ds \leq \|\boldsymbol{v}_0\|_{X_{\tau_0}} e^{gt},\end{aligned} \tag{3.29}$$

for all $t < T_*$. The formal construction of the analytic solution $(\boldsymbol{v}, \rho)$ satisfying (3.29), with τ given by (3.26) is given in Section 5, which combined with the uniqueness result given in Section 6 completes the proof of Theorem 2.2.

4. The velocity estimate

The goal of this section is to prove the next lemma. For convenience of notation, we suppress the time dependence of $\boldsymbol{v}, w, \tilde{v}$, and τ. By the two-dimensional case $d = 2$ we mean that all functions and the domain are independent of the variable x_2. In that case we recall that by $\|\phi\|^2_{L^2(\mathcal{D})}$ we mean $\int_0^1 \int_0^h |\phi(\boldsymbol{x}, z)|^2 \, dz \, dx_1$, for any smooth function ϕ independent of x_2.

Lemma 4.1. *Let $\boldsymbol{v} \in Y_\tau$ for some $\tau > 0$, and $\tilde{v} \in C^\infty(\mathcal{D}) \cap L^2(\mathcal{D})$ with $\|\tilde{v}\|_{Y_\tau} < \infty$. Fix $r \geq d/2 + 1$ and $d \in \{2, 3\}$. Let w be determined from $\boldsymbol{v}$ via (1.13) i.e., $w(x, z) = -\int_0^z \operatorname{div} \boldsymbol{v}(x, \zeta) \, d\zeta$. If $\mathcal{U}(v_i, \tilde{v})$, for $i \in \{1, 2\}$, and $\mathcal{V}(w, \tilde{v})$ are as in (3.18) and (3.19) respectively, then we have the estimates*

$$\mathcal{U}(v_i, \tilde{v}) \leq C_0(1 + \tau^{-\theta(d)}) \|\boldsymbol{v}\|_{X_\tau} \|\tilde{v}\|_{Y_\tau}, \tag{4.1}$$

and

$$\mathcal{V}(w, \tilde{v}) \leq C_0(1 + \tau^{-1-\theta(d)}) \|\boldsymbol{v}\|_{X_\tau} \|\tilde{v}\|_{Y_\tau} + C_0(\tau^{-1} + \tau^{-1-\theta(d)}) \|\boldsymbol{v}\|_{Y_\tau} \|\tilde{v}\|_{X_\tau}, \tag{4.2}$$

for some positive constant $C_0 = C(r, \mathcal{D})$, where $\theta(d) = 1$ if $d = 2$, and $\theta(d) = 3/2$ if $d = 3$.

By setting $\tilde{v} = v_1$, and then $\tilde{v} = v_2$, Lemma 3.2 is a corollary of Lemma 4.1 for the case $d = 2$. For the rest of this section we fix $d = 2$. In the three-dimensional case the proof is identical except for two modifications. The integration is done also in x_2, and hence the exponents in Agmon's inequality are different. We omit further details. Here, $\|\cdot\|_{L^p} = \|\cdot\|_{L^p(\mathcal{D})}$, for all $1 \leq p \leq \infty$.

Proof of Lemma 4.1. In order to prove the lemma, we need to estimate the terms $\|\partial^\beta v_i \, \partial_{x_i} \partial^{\alpha-\beta} \tilde{v}\|_{L^2}$ and $\|\partial^\beta w \, \partial_z \partial^{\alpha-\beta} \tilde{v}\|_{L^2}$, for all $\alpha, \beta \in \mathbb{N}^3$. For this purpose it is convenient to distinguish between two cases: $0 \leq |\beta| \leq |\alpha - \beta|$ and $|\alpha - \beta| < |\beta| \leq |\alpha|$. When $0 \leq |\beta| \leq |\alpha - \beta|$, by the Hölder inequality, and by the two-dimensional Agmon inequality we have

$$\begin{aligned} \|\partial^\beta v_i \, \partial_{x_i} \partial^{\alpha-\beta} \tilde{v}\|_{L^2} &\leq C \|\partial^\beta v_i\|_{L^\infty} \|\partial_{x_i} \partial^{\alpha-\beta} \tilde{v}\|_{L^2} \\ &\leq C \|\partial^\beta v_i\|^{1/2}_{L^2} \|(\Delta + \partial_{zz}) \partial^\beta v_i\|^{1/2}_{L^2} \|\partial_{x_i} \partial^{\alpha-\beta} \tilde{v}\|_{L^2} + C \|\partial^\beta v_i\|_{L^2} \|\partial_{x_i} \partial^{\alpha-\beta} \tilde{v}\|_{L^2}, \end{aligned} \tag{4.3}$$

for some constant $C = C(\mathcal{D}) > 0$. Recall that Δ is the horizontal Laplacian $\partial_{x_1 x_1}$ when $d = 2$, and $\partial_{x_1 x_1} + \partial_{x_2 x_2}$ when $d = 3$. Similarly, we estimate

$$\begin{aligned} \|\partial^\beta w \partial_z \partial^{\alpha-\beta} \tilde{v}\|_{L^2} &\leq C \|\partial^\beta w\|_{L^\infty} \|\partial_z \partial^{\alpha-\beta} \tilde{v}\|_{L^2} \\ &\leq C \|\partial^\beta w\|^{1/2}_{L^2} \|(\Delta + \partial_{zz}) \partial^\beta w\|^{1/2}_{L^2} \|\partial_z \partial^{\alpha-\beta} \tilde{v}\|_{L^2} + C \|\partial^\beta w\|_{L^2} \|\partial_z \partial^{\alpha-\beta} \tilde{v}\|_{L^2}. \end{aligned} \tag{4.4}$$

As in the above estimates, for multi-indices such that $|\alpha - \beta| < |\beta| \leq |\alpha|$, we have

$$\begin{aligned} \|\partial^\beta v_i \, \partial_{x_i} \partial^{\alpha-\beta} \tilde{v}\|_{L^2} &\leq C \|\partial^\beta v_i\|_{L^2} \|\partial_{x_i} \partial^{\alpha-\beta} \tilde{v}\|_{L^\infty} \\ &\leq C \|\partial^\beta v_i\|_{L^2} \|\partial_{x_i} \partial^{\alpha-\beta} \tilde{v}\|^{1/2}_{L^2} \|\partial_{x_i} (\Delta + \partial_{zz}) \partial^{\alpha-\beta} \tilde{v}\|^{1/2}_{L^2} + C \|\partial^\beta v_i\|_{L^2} \|\partial_{x_i} \partial^{\alpha-\beta} \tilde{v}\|_{L^2}, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} \|\partial^\beta w \, \partial_z \partial^{\alpha-\beta} \tilde{v}\|_{L^2} &\leq C \|\partial^\beta w\|_{L^2} \|\partial_z \partial^{\alpha-\beta} \tilde{v}\|_{L^\infty} \\ &\leq C \|\partial^\beta w\|_{L^2} \|\partial_z \partial^{\alpha-\beta} \tilde{v}\|^{1/2}_{L^2} \|\partial_z (\Delta + \partial_{zz}) \partial^{\alpha-\beta} \tilde{v}\|^{1/2}_{L^2} + C \|\partial^\beta w\|_{L^2} \|\partial_z \partial^{\alpha-\beta} \tilde{v}\|_{L^2}. \end{aligned} \tag{4.6}$$

Throughout this section we use the inequality $\binom{\alpha}{\beta} \leq \binom{|\alpha|}{|\beta|}$, which holds for all $\alpha, \beta \in \mathbb{N}^3$ with $\beta \leq \alpha$. Moreover, since $|\alpha - \beta| = |\alpha| - |\beta|$ for $\beta \leq \alpha$, the identity

$$\sum_{|\alpha|=m} \sum_{|\beta|=j, \beta\leq\alpha} a_\beta b_{\alpha-\beta} = \left(\sum_{|\beta|=j} a_\beta \right) \left(\sum_{|\gamma|=m-j} b_\gamma \right), \tag{4.7}$$

holds for all sequences $\{a_\beta\}$ and $\{b_\gamma\}$, and for $j \leq m$; this identity is useful when estimating $\mathcal{U} = \mathcal{U}(v_i, \tilde{v})$ and $\mathcal{V} = \mathcal{V}(w, \tilde{v})$.

With (4.3) and (4.5) in mind, we split $\mathcal{U} = \mathcal{U}_{low} + \mathcal{U}_{high}$, according to $0 \leq j \leq [m/2]$ and $[m/2] + 1 \leq j \leq m$ respectively. Using (4.3), we have

$$\begin{aligned} \mathcal{U}_{low} \leq C \sum_{m=0}^{\infty} \sum_{j=0}^{[m/2]} \sum_{|\alpha|=m} \sum_{|\beta|=j,\beta\leq\alpha} \binom{\alpha}{\beta} \|\partial^\beta v_i\|_{L^2}^{1/2} \|(\Delta + \partial_{zz})\partial^\beta v_i\|_{L^2}^{1/2} \|\partial_{x_i} \partial^{\alpha-\beta} \tilde{v}\|_{L^2} \frac{(m+1)^r \tau^m}{m!} \\ + C \sum_{m=0}^{\infty} \sum_{j=0}^{[m/2]} \sum_{|\alpha|=m} \sum_{|\beta|=j,\beta\leq\alpha} \binom{\alpha}{\beta} \|\partial^\beta v_i\|_{L^2} \|\partial_{x_i} \partial^{\alpha-\beta} \tilde{v}\|_{L^2} \frac{(m+1)^r \tau^m}{m!}. \end{aligned} \tag{4.8}$$

By (4.7) it follows that

$$\begin{aligned} &\sum_{|\alpha|=m} \sum_{|\beta|=j,\beta\leq\alpha} \binom{\alpha}{\beta} \|\partial^\beta v_i\|_{L^2}^{1/2} \|(\Delta + \partial_{zz})\partial^\beta v_i\|_{L^2}^{1/2} \|\partial_{x_i} \partial^{\alpha-\beta} \tilde{v}\|_{L^2} \\ &\quad \leq C \binom{m}{j} \left(\sum_{|\beta|=j} \|\partial^\beta v_i\|_{L^2} \right)^{1/2} \left(\sum_{|\beta|=j} \|(\Delta + \partial_{zz})\partial^\beta v_i\|_{L^2} \right)^{1/2} \left(\sum_{|\gamma|=m-j} \|\partial^\gamma \partial_{x_i} \tilde{v}\|_{L^2} \right). \end{aligned} \tag{4.9}$$

We observe that $\sum_{|\gamma|=m-j} \|\partial_{x_i} \partial^\gamma \tilde{v}\|_{L^2} \leq C|\tilde{v}|_{m-j+1}$ and $\sum_{|\beta|=j} \|(\Delta + \partial_{zz})\partial^\beta v_i\|_{L^2} \leq C|v|_{j+2}$. Hence, from (4.8) it follows by the discrete Hölder inequality and (4.9) that $\mathcal{U}_{low}$ is bounded from above by

$$\begin{aligned} &C \sum_{m=0}^{\infty} \sum_{j=0}^{[m/2]} |v_i|_j^{1/2} |v_i|_{j+2}^{1/2} |\tilde{v}|_{m-j+1} \binom{m}{j} \frac{(m+1)^r \tau^m}{m!} + C \sum_{m=0}^{\infty} \sum_{j=0}^{[m/2]} |v_i|_j |\tilde{v}|_{m-j+1} \binom{m}{j} \frac{(m+1)^r \tau^m}{m!} \\ &\quad \leq C \sum_{m=0}^{\infty} \sum_{j=0}^{[m/2]} \left(|v_i|_j \frac{(j+1)^r \tau^j}{j!} \right)^{\frac{1}{2}} \left(|v_i|_{j+2} \frac{(j+3)^r \tau^{j+2}}{(j+2)!} \right)^{\frac{1}{2}} \left(|\tilde{v}|_{m-j+1} \frac{(m-j+2)^r \tau^{m-j}}{(m-j)!} \right) \tau^{-1} \\ &\quad\quad + C \sum_{m=0}^{\infty} \sum_{j=0}^{[m/2]} \left(|v_i|_j \frac{(j+1)^r \tau^j}{j!} \right) \left(|\tilde{v}|_{m-j+1} \frac{(m-j+2)^r \tau^{m-j}}{(m-j)!} \right), \end{aligned} \tag{4.10}$$

where we have used the inequality

$$\binom{m}{j} \frac{(m+1)^r}{m!} \frac{(m-j)!}{(m-j+2)^r} \frac{j!^{1/2}(j+2)!^{1/2}}{(j+1)^{r/2}(j+3)^{r/2}} = \frac{(m+1)^r}{(m-j+2)^r} \frac{(j+1)^{1/2}(j+2)^{1/2}}{(j+1)^{r/2}(j+3)^{r/2}} \leq C, \tag{4.11}$$

which holds for all $m \geq 0$, $0 \leq j \leq [m/2]$, $r \geq 1$, and a sufficiently large constant C, depending only on r. By (4.10), the discrete Hölder and Young inequalities imply

$$\mathcal{U}_{low} \leq C(1 + \tau^{-1}) \|\boldsymbol{v}\|_{X_\tau} \|\tilde{v}\|_{Y_\tau}.$$

By symmetry, from (4.5) and (4.7), we obtain an estimate for the high values of j. Namely

$$\begin{aligned}\mathcal{U}_{high} &\leq C\sum_{m=1}^{\infty}\sum_{j=[m/2]+1}^{m}|v_i|_j|\partial_{x_i}\tilde{v}|_{m-j}^{1/2}|\partial_{x_i}(\Delta+\partial_{zz})\tilde{v}|_{m-j}^{1/2}\binom{m}{j}\frac{(m+1)^r\tau^m}{m!}\\&\qquad+C\sum_{m=1}^{\infty}\sum_{j=[m/2]+1}^{m}|v_i|_j|\partial_{x_i}\tilde{v}|_{m-j}\binom{m}{j}\frac{(m+1)^r\tau^m}{m!}\\&\leq C(1+\tau^{-1})\|\boldsymbol{v}\|_{X_\tau}\|\tilde{v}\|_{Y_\tau},\end{aligned}\tag{4.12}$$

for some positive constant C depending only on r and $\mathcal{D}$. This completes the proof of (4.1). The estimate on $\mathcal{V}$ is in the same spirit, but we need to account for the derivative loss that occurs when estimating w in terms of $\boldsymbol{v}$. First, note that the definition (1.13) of w and that of the semi-norms $|\cdot|_j$ imply

$$|w|_j\leq\sum_{|\beta|=j,\beta_3\geq1}\|\partial_x^{\beta'}\partial_z^{\beta_3-1}\operatorname{div}\boldsymbol{v}\|_{L^2}+\sum_{|\beta|=j,\beta_3=0}\left\|\int_0^z\partial^\beta\operatorname{div}\boldsymbol{v}(\boldsymbol{x},\zeta)d\zeta\right\|_{L^2}\leq|\boldsymbol{v}|_j+C|\boldsymbol{v}|_{j+1},$$

for some constant $C=C(h)>0$. We recall that div is the divergence operator acting on $\boldsymbol{x}$. Similarly we have that $|\Delta w|_j\leq|\boldsymbol{v}|_{j+2}+C|\boldsymbol{v}|_{j+3}$, and also $|\partial_{zz}w|_j\leq|w|_{j+2}$. Next, we split $\mathcal{V}=\mathcal{V}_{low}+\mathcal{V}_{high}$, according to $0\leq j\leq[m/2]$ and $[m/2]<j\leq m$. By (4.4) and (4.7) we have

$$\begin{aligned}\mathcal{V}_{low}&\leq C\sum_{m=0}^{\infty}\sum_{j=0}^{[m/2]}\left(|\boldsymbol{v}|_j^{1/2}+|\boldsymbol{v}|_{j+1}^{1/2}\right)\left(|\boldsymbol{v}|_{j+2}^{1/2}+|\boldsymbol{v}|_{j+3}^{1/2}\right)|\tilde{v}|_{m-j+1}\binom{m}{j}\frac{(m+1)^r\tau^m}{m!}\\&\qquad+C\sum_{m=0}^{\infty}\sum_{j=0}^{[m/2]}\left(|\boldsymbol{v}|_j+|\boldsymbol{v}|_{j+1}\right)|\tilde{v}|_{m-j+1}\binom{m}{j}\frac{(m+1)^r\tau^m}{m!},\end{aligned}\tag{4.13}$$

for some constant $C=C(r,\mathcal{D})>0$. Using the fact that

$$\binom{m}{j}\frac{(m+1)^r}{m!}\frac{(m-j)!}{(m-j+2)^r}\frac{(j+1)!^{1/2}(j+3)!^{1/2}}{(j+2)^{r/2}(j+4)^{r/2}}\leq C,$$

holds for all $m\geq0$ and $0\leq j\leq[m/2]$, for some sufficiently large positive constant C depending only on $r\geq2$, we obtain

$$\mathcal{V}_{low}\leq C(1+\tau^{-2})\|\boldsymbol{v}\|_{X_\tau}\|\tilde{v}\|_{Y_\tau}.$$

Lastly, to estimate $\mathcal{V}_{high}$, we note that by (4.6) and (4.7), we have

$$\begin{aligned}\mathcal{V}_{high}&\leq C\sum_{m=1}^{\infty}\sum_{j=[m/2]+1}^{m}|w|_j|\partial_z\tilde{v}|_{m-j}^{1/2}|\partial_z(\Delta+\partial_{zz})\tilde{v}|_{m-j}^{1/2}\binom{m}{j}\frac{(m+1)^r\tau^m}{m!}\\&\qquad+C\sum_{m=1}^{\infty}\sum_{j=[m/2]+1}^{m}|w|_j|\partial_z\tilde{v}|_{m-j}\binom{m}{j}\frac{(m+1)^r\tau^m}{m!}\\&\leq C\sum_{m=1}^{\infty}\sum_{j=[m/2]+1}^{m}\left(|\boldsymbol{v}|_j+|\boldsymbol{v}|_{j+1}\right)|\tilde{v}|_{m-j+1}^{1/2}|\tilde{v}|_{m-j+3}^{1/2}\binom{m}{j}\frac{(m+1)^r\tau^m}{m!}\\&\qquad+C\sum_{m=1}^{\infty}\sum_{j=[m/2]+1}^{m}\left(|\boldsymbol{v}|_j+|\boldsymbol{v}|_{j+1}\right)|\tilde{v}|_{m-j+1}\binom{m}{j}\frac{(m+1)^r\tau^m}{m!}.\end{aligned}\tag{4.14}$$

By symmetry with the $\mathcal{V}_{low}$ estimate, the lower order terms (the ones containing $|\boldsymbol{v}|_j$) on the right side of (4.14) are estimated by $C(1+\tau^{-1})\|\boldsymbol{v}\|_{X_\tau}\|\tilde{v}\|_{Y_\tau}$. On the other hand, the terms containing $|\boldsymbol{v}|_{j+1}$ are similarly bounded by $C(\tau^{-1}+\tau^{-2})\|\boldsymbol{v}\|_{Y_\tau}\|\tilde{v}\|_{X_\tau}$ concluding the proof of (4.2), and hence of the lemma. $\square$

5. Construction of the solution

The formal construction of the solutions is via the Picard iteration. Let $\boldsymbol{v}^{(0)} = \boldsymbol{v}_0$, $\rho^{(0)} = \rho_0$ be given, with $\boldsymbol{v}_0$ satisfying the compatibility conditions (1.17) and (1.18). For $n \in \mathbb{N}$, let

$$w^{(n)}(\boldsymbol{x}, z, t) = -\int_0^z \operatorname{div} \boldsymbol{v}^{(n)}(\boldsymbol{x}, \zeta, t)\, d\zeta, \tag{5.1}$$

and

$$\psi^{(n)}(\boldsymbol{x}, z, t) = \int_0^z \rho^{(n)}(\boldsymbol{x}, \zeta, t)\, d\zeta. \tag{5.2}$$

The density iterate is given by

$$\rho^{(n+1)}(t) = \rho_0 - \int_0^t \left(\boldsymbol{v}^{(n)}(s) \cdot \nabla + w^{(n)}(s)\partial_z \right) \rho^{(n)}(s)\, ds, \tag{5.3}$$

and motivated by Lemma 3.1, we define the pressure

$$\begin{aligned} p^{(n+1)}(\boldsymbol{x}, t) = &-\fint_0^h \left(v_1^{(n)} \right)^2 (\boldsymbol{x}, z, t)\, dz - f \int_0^{x_1} \fint_0^h v_2^{(n)}(x_1', x_2, z, t)\, dz\, dx_1' \\ &+ g \fint_0^h \psi^{(n+1)}(\boldsymbol{x}, z, t)\, dz. \end{aligned} \tag{5.4}$$

Lastly, the velocity iterate is constructed as

$$\begin{aligned} \boldsymbol{v}^{(n+1)}(t) = \boldsymbol{v}_0 &- \int_0^t \left(\boldsymbol{v}^{(n)}(s) \cdot \nabla + w^{(n)}(s)\partial_z \right) \boldsymbol{v}^{(n)}(s)\, ds \\ &- \int_0^t \left(\nabla p^{(n+1)}(s) - g\nabla\psi^{(n+1)}(s) + f\boldsymbol{v}^{(n)\perp}(s) \right) ds, \end{aligned} \tag{5.5}$$

for all $n \in \mathbb{N}$. Taking the time derivative of the first component of (5.5), integrating in z, and using the fact that $\partial_{x_1} p^{(n)}$ is obtained from (5.4), we obtain that $\partial_t \int_0^h v_1^{(n+1)}\, dz = 0$. Since the first component of the initial data, v_{01}, has zero vertical average, we obtain $\int_0^h v_1^{(n)}(\boldsymbol{x}, z)\, dz = 0$ for all $\boldsymbol{x} \in \mathcal{M}$ and $n \geq 0$. Therefore the compatibility conditions $\int_0^h \operatorname{div} v_1^{(n)} dz = 0$ and the boundary condition $\int_0^h v_1^{(n)}|_{\Gamma_x} dz = 0$ are conserved for all $n \in \mathbb{N}$. Recall that we denote by div the differential operator acting only on x.

Assume that $(\boldsymbol{v}_0, \rho_0) \in X_{\tau_0+\varepsilon}$, for some $0 < \epsilon < \tau_0$. In particular, we have $(\boldsymbol{v}_0, \rho_0) \in Y_{\tau_0}$. We define $\tau(t)$ by $\tau(0) = \tau_0$ and

$$\dot{\tau}(t) + 20\, C_0 g + 20\, C_0 (1 + \tau^{-2}(t)) \|(\boldsymbol{v}_0, \rho_0)\|_{X_{\tau_0}} e^{gt} = 0, \tag{5.6}$$

where the constant $C_0 = C_0(r, \mathcal{D})$ is fixed in Lemma 4.1. First we show that the sequence of approximations $\boldsymbol{v}^{(n)}$ is bounded in $L^\infty(0, T; X_\tau) \cap L^1(0, T; (1 + \tau^{-2}) Y_\tau)$ for some sufficiently small $T > 0$, depending solely on the initial data.

Lemma 5.1. *Let $(\boldsymbol{v}_0, \rho_0) \in X_{\tau_0+\varepsilon}$ and $\tau(t)$ be defined by (5.6). The approximating sequence $\{(\boldsymbol{v}^{(n)}, \rho^{(n)})\}_{n\geq 0}$, constructed via (5.1)–(5.5), satisfies*

$$\begin{aligned} &\sup_{t\in[0,T]} \|(\boldsymbol{v}^{(n)}(t), \rho^{(n)}(t))\|_{X_{\tau(t)}} + 10\, C_0 g \int_0^T e^{g(T-t)} \|(\boldsymbol{v}^{(n)}(t), \rho^{(n)}(t))\|_{Y_{\tau(t)}}\, dt \\ &\quad + 20\, C_0 \|(\boldsymbol{v}_0, \rho_0)\|_{X_{\tau_0}} e^{gT} \int_0^T \left(1 + \tau^{-2}(t) \right) \|(\boldsymbol{v}^{(n)}(t), \rho^{(n)}(t))\|_{Y_{\tau(t)}}\, dt \leq 3 e^{gT} \|(\boldsymbol{v}_0, \rho_0)\|_{X_{\tau_0}}, \end{aligned} \tag{5.7}$$

for all $n \geq 0$, where $T = T(\boldsymbol{v}_0, \rho_0) > 0$ is sufficiently small.

Proof of Lemma 5.1. When $n = 0$, since τ is decreasing and since $\epsilon < \tau_0$, for all $t \geq 0$ we have $\|(\boldsymbol{v}_0, \rho_0)\|_{Y_{\tau_0}} \leq C\|(\boldsymbol{v}_0, \rho_0)\|_{X_{\tau_0+\varepsilon}}/\varepsilon$, for some constant C. Sufficient conditions for the bound (5.7) to hold in the case $n = 0$ are that T is chosen such that

$$10\, C_0\, C(e^{gT} - 1)\|(\boldsymbol{v}_0, \rho_0)\|_{X_{\tau_0+\varepsilon}} \leq \varepsilon\|(\boldsymbol{v}_0, \rho_0)\|_{X_{\tau_0}} e^{gT} \tag{5.8}$$

and

$$20\, C_0\, C\|(\boldsymbol{v}_0, \rho_0)\|_{X_{\tau_0+\varepsilon}} \int_0^T \Big(1 + \tau^{-2}(t)\Big)\, dt \leq \varepsilon. \tag{5.9}$$

The condition (5.8) holds if $T \leq T_1$, where $T_1(\epsilon, C_0, C, g, \|(\boldsymbol{v}_0, \rho_0)\|_{X_{\tau_0}}, \|(\boldsymbol{v}_0, \rho_0)\|_{X_{\tau_0+\varepsilon}}) > 0$ is determined explicitly. Also, by the construction of τ (cf. (5.6)) we have $20\, C_0(1 + \tau^{-2}) < -\dot{\tau}/\|(\boldsymbol{v}_0, \rho_0)\|_{X_{\tau_0}}$, so that the condition (5.9) is satisfied if we choose T so that

$$\tau_0 - \tau(T) \leq \frac{\varepsilon\|(\boldsymbol{v}_0, \rho_0)\|_{X_{\tau_0}}}{C\|(\boldsymbol{v}_0, \rho_0)\|_{X_{\tau_0+\varepsilon}}}, \tag{5.10}$$

which is satisfied if $T \leq T_2$, where $T_2(\varepsilon, \tau_0, C_0, C, g, \|(\boldsymbol{v}_0, \rho_0)\|_{X_{\tau_0}}, \|(\boldsymbol{v}_0, \rho_0)\|_{X_{\tau_0+\varepsilon}}) > 0$ may be computed explicitly from (5.6) and (5.10). Thus (5.7) holds for $n = 0$ if $T \leq \min\{T_1, T_2\}$.

We proceed by induction. By (5.3), (5.5), and Lemma 4.1, similarly to estimate (3.25), we obtain

$$\begin{aligned}
\frac{d}{dt}\|(\boldsymbol{v}^{(n+1)}, \rho^{(n+1)})\|_{X_\tau} &\leq (\dot{\tau} + C_0 g)\|(\boldsymbol{v}^{(n+1)}, \rho^{(n+1)})\|_{Y_\tau} + g\|(\boldsymbol{v}^{(n+1)}, \rho^{(n+1)})\|_{X_\tau} \\
&\qquad\qquad + 3C_0(1 + \tau^{-2})\|(\boldsymbol{v}^{(n)}, \rho^{(n)})\|_{X_\tau}\|(\boldsymbol{v}^{(n)}, \rho^{(n)})\|_{Y_\tau} \\
&\leq (\dot{\tau} + C_0 g)\|(\boldsymbol{v}^{(n+1)}, \rho^{(n+1)})\|_{Y_\tau} + g\|(\boldsymbol{v}^{(n+1)}, \rho^{(n+1)})\|_{X_\tau} \\
&\qquad\qquad + 9C_0(1 + \tau^{-2})e^{gT}\|(\boldsymbol{v}_0, \rho_0)\|_{X_{\tau_0}}\|(\boldsymbol{v}^{(n)}, \rho^{(n)})\|_{Y_\tau},
\end{aligned} \tag{5.11}$$

by the induction assumption. In the above we also used the fact that by Lemma 3.1 we have $\langle \partial^\alpha \nabla p^{(n)}, \partial^\alpha \boldsymbol{v}^{(n+1)} \rangle = 0$. Using that τ was chosen to satisfy (5.6), the above estimate and Grönwall's inequality give

$$\begin{aligned}
&\|(\boldsymbol{v}^{(n+1)}(t), \rho^{(n+1)}(t))\|_{X_{\tau(t)}} + 10\, C_0 g \int_0^t e^{g(t-s)}\|\boldsymbol{v}^{(n+1)}(s), \rho^{(n+1)}(s))\|_{Y_{\tau(s)}}\, ds \\
&\qquad + 20\, C_0\|(\boldsymbol{v}_0, \rho_0)\|_{X_{\tau_0}} e^{gt} \int_0^t \Big(1 + \tau^{-2}(s)\Big)\|(\boldsymbol{v}^{(n+1)}(s), \rho^{(n+1)}(s))\|_{Y_{\tau(s)}}\, ds \\
&\leq \|(\boldsymbol{v}_0, \rho_0)\|_{X_{\tau_0}} e^{gt} + 9e^{gT} C_0\|(\boldsymbol{v}_0, \rho_0)\|_{X_{\tau_0}} e^{gt} \int_0^t \Big(1 + \tau^{-2}(s)\Big)\|(\boldsymbol{v}^{(n)}(s), \rho^{(n)}(s))\|_{Y_{\tau(s)}}\, ds.
\end{aligned} \tag{5.12}$$

The proof of Lemma 5.1 is completed by taking the supremum over $t \in [0, T]$ of the above inequality, by the induction assumption, and by additionally letting T be small enough so that $27e^{gT} \leq 40$. $\square$

We conclude the construction of the solution by showing that the map $\boldsymbol{v}^{(n)} \mapsto \boldsymbol{v}^{(n+1)}$ is a contraction in $L^\infty(0, T; X_\tau) \cap L^1(0, T; (1 + \tau^{-2})Y_\tau)$, for some sufficiently small T depending on the initial data.

Lemma 5.2. *Let $\tilde{\boldsymbol{v}}^{(n)} = \boldsymbol{v}^{(n+1)} - \boldsymbol{v}^{(n)}$, and $\tilde{\rho}^{(n)} = \rho^{(n+1)} - \rho^{(n)}$ for all $n \geq 0$. Let $(\boldsymbol{v}_0, \rho_0) \in X_{\tau_0+\varepsilon}$, $\tau(t)$ be defined by (5.6), and T be as in Lemma 5.1. If for all $n \geq 0$ we let*

$$\begin{aligned}
a_n = \sup_{t \in [0,T]} &\|(\tilde{\boldsymbol{v}}^{(n)}(t), \tilde{\rho}^{(n)}(t))\|_{X_{\tau(t)}} + 10\, C_0 g \int_0^T e^{g(T-t)}\|(\tilde{\boldsymbol{v}}^{(n)}(t), \tilde{\rho}^{(n)}(t))\|_{Y_{\tau(t)}}\, dt \\
&+ 20\, C_0\|(\boldsymbol{v}_0, \rho_0)\|_{X_{\tau_0}} e^{gT} \int_0^T \Big(1 + \tau^{-2}(t)\Big)\|(\tilde{\boldsymbol{v}}^{(n)}(t), \tilde{\rho}^{(n)}(t))\|_{Y_{\tau(t)}} dt,
\end{aligned} \tag{5.13}$$

then we have $20\, a_n \leq 19\, a_{n-1}$ for all $n \geq 1$.

Proof of Lemma 5.2. Denote $\tilde{w}^{(n)} = w^{(n+1)} - w^{(n)}$, $\tilde{\psi}^{(n)} = \psi^{(n+1)} - \psi^{(n)}$, and $\tilde{p}^{(n)} = p^{(n+1)} - p^{(n)}$. Then we have that the difference of two iterations satisfies the equations

$$\partial_t \tilde{\boldsymbol{v}}^{(n)} + \big(\boldsymbol{v}^{(n)} \cdot \nabla + w^{(n)} \partial_z\big) \tilde{\boldsymbol{v}}^{(n-1)} + \big(\tilde{\boldsymbol{v}}^{(n-1)} \cdot \nabla + \tilde{w}^{(n-1)} \partial_z\big) \boldsymbol{v}^{(n-1)} \\ + \nabla \tilde{p}^{(n)} - g \nabla \tilde{\psi}^{(n)} + f \tilde{\boldsymbol{v}}^{(n-1)\perp} = 0, \tag{5.14}$$

and

$$\partial_t \tilde{\rho}^{(n)} + \big(\boldsymbol{v}^{(n)} \cdot \nabla + w^{(n)}\, \partial_z\big) \tilde{\rho}^{(n-1)} + \big(\tilde{\boldsymbol{v}}^{(n-1)} \cdot \nabla + \tilde{w}^{(n-1)} \partial_z\big) \rho^{(n-1)} = 0, \tag{5.15}$$

with initial conditions $\tilde{\boldsymbol{v}}^{(n)}(0) = 0$ and $\tilde{\rho}^{(n)}(0) = 0$, for all $n \geq 0$. Since the approximate solutions $\boldsymbol{u}^{(n)}$ satisfy the boundary conditions (1.11)–(1.12), similarly to the proof of (3.10), it can be shown that $\langle \partial^\alpha \nabla \tilde{p}^{(n-1)}, \partial^\alpha \tilde{\boldsymbol{v}}^{(n)} \rangle = 0$ for all $\alpha \in \mathbb{N}^3$. Hence, from (5.14), (5.15), and Lemma 4.1, we obtain

$$\frac{d}{dt} \|(\tilde{\boldsymbol{v}}^{(n)}, \tilde{\rho}^{(n)})\|_{X_\tau} \leq (\dot{\tau} + C_0 g) \|(\tilde{\boldsymbol{v}}^{(n)}, \tilde{\rho}^{(n)})\|_{Y_\tau} + g \|(\tilde{\boldsymbol{v}}^{(n)}, \tilde{\rho}^{(n)})\|_{X_\tau} \\ + 3\, C_0 (1 + \tau^{-2}) \Big(\|(\boldsymbol{v}^{(n)}, \rho^{(n)})\|_{X_\tau} + \|(\boldsymbol{v}^{(n-1)}, \rho^{(n-1)})\|_{X_\tau} \Big) \|(\tilde{\boldsymbol{v}}^{(n-1)}, \tilde{\rho}^{(n-1)})\|_{Y_\tau} \\ + 3\, C_0 (1 + \tau^{-2}) \Big(\|(\boldsymbol{v}^{(n)}, \rho^{(n)})\|_{Y_\tau} + \|(\boldsymbol{v}^{(n-1)}, \rho^{(n-1)})\|_{Y_\tau} \Big) \|(\tilde{\boldsymbol{v}}^{(n-1)}, \tilde{\rho}^{(n-1)})\|_{X_\tau}. \tag{5.16}$$

Using the definition of $\dot{\tau}$ (cf. (5.6)), the estimate in Lemma 5.1, Grönwall's inequality, and taking the supremum for $t \in [0, T]$, we obtain

$$a_n \leq 18 C_0 e^{gT} \|(\boldsymbol{v}_0, \rho_0)\|_{X_{\tau_0}} \int_0^T (1 + \tau^{-2}) e^{g(T-t)} \|(\tilde{\boldsymbol{v}}^{(n-1)}, \tilde{\rho}^{(n-1)})\|_{Y_\tau}\, dt \\ + \left(\sup_{t \in [0,T]} \|(\tilde{\boldsymbol{v}}^{(n-1)}, \tilde{\rho}^{(n-1)})\|_{X_\tau} \right) \int_0^T 3 C_0 (1 + \tau^{-2}) e^{g(T-t)} \Big(\|(\boldsymbol{v}^{(n)}, \rho^{(n)})\|_{Y_\tau} + \|(\boldsymbol{v}^{(n-1)}, \rho^{(n-1)})\|_{Y_\tau} \Big) dt. \tag{5.17}$$

If T is taken such that $18\, e^{gT} \leq 19$, then the above estimate and the definition of a_n (cf. 5.13) imply that

$$a_n \leq \frac{19}{20}\, a_{n-1}. \tag{5.18}$$

This concludes the proof of the lemma, showing that the map $(\boldsymbol{v}^{(n)}, \rho^{(n)}) \mapsto (\boldsymbol{v}^{(n-1)}, \rho^{(n-1)})$ is a strict contraction. The existence of a solution to (1.7)–(1.12) in the class $L^\infty(0, T; X_\tau) \cap L^1(0, T; (1 + \tau^{-2}) Y_\tau)$, with $\tau(t)$ given by (5.6) follows from the classical fixed point theorem. □

6. Uniqueness

Fix $(\boldsymbol{v}_0, \rho_0) \in X_{\tau_0 + \varepsilon}$, a real-analytic function on $\mathcal{D}$ with radius of analyticity strictly larger than τ_0, for some positive $\varepsilon < \tau_0$. Let $\tau(t)$ be defined by $\tau(0) = \tau_0$ and $\dot{\tau} + 20 C_0 g + 20 C_0 (1 + \tau^{-2}) \|(\boldsymbol{v}_0, \rho_0)\|_{X_{\tau_0}} e^{gt} = 0$, where $C_0 = C(\mathcal{D}, r) > 0$ is the fixed constant defined in Lemma 3.2. Let T_* be the maximal time such that $\tau(t) \geq 0$.

Assume that there exist two solutions $(\boldsymbol{v}^{(1)}, \rho^{(1)})$ and $(\boldsymbol{v}^{(2)}, \rho^{(2)})$ to (1.7)–(1.12) evolving from initial data $(\boldsymbol{v}_0, \rho_0)$, such that for $i = 1, 2$, we have

$$\|(\boldsymbol{v}^{(i)}(t), \rho^{(i)})\|_{X_{\tau(t)}} + 10 C_0 g \int_0^t e^{g(t-s)} \|(\boldsymbol{v}^{(i)}(s), \rho^{(i)}(s))\|_{Y_{\tau(s)}}\, ds \\ + 10 C_0 \|(\boldsymbol{v}_0, \rho_0)\|_{X_{\tau_0}} e^{gT} \int_0^t \Big(1 + \tau^{-2}(s)\Big) \|(\boldsymbol{v}^{(i)}(s), \rho^{(i)}(s))\|_{Y_{\tau(s)}} ds \; < \infty, \tag{6.1}$$

for all $0 \leq t < T_*$. Similarly to (3.28) we have that $\|\boldsymbol{v}^{(i)}(t)\|_{X_{\tau(t)}} \leq \|(\boldsymbol{v}_0, \rho_0)\|_{X_{\tau_0}} e^{gt}$ for $i \in \{1,2\}$ and for all $0 \leq t < T_*$. Let $w^{(i)}$ and $p^{(i)}$ be the vertical velocity and the pressure associated to $\boldsymbol{v}^{(i)}$, and let $\psi^{(i)}(\boldsymbol{x}, z) = \int_0^z \rho^{(i)}(\boldsymbol{x}, \zeta)\, d\zeta$, for $i \in \{1,2\}$. We denote the difference of the solutions $\boldsymbol{v}^{(1)} - \boldsymbol{v}^{(2)} = \boldsymbol{v}$, $\rho^{(1)} - \rho^{(2)} = \rho$, and similarly define w, ψ and p. Then $(\boldsymbol{v}, w, p, \rho)$ satisfy the equations

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}^{(1)} \cdot \nabla + w^{(1)}\partial_z)\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla + w\partial_z)\boldsymbol{v}^{(2)} + \nabla p + f\boldsymbol{v}^{\perp} = g\nabla\psi, \tag{6.2}$$

$$\operatorname{div} \boldsymbol{v} + \partial_z w = 0, \tag{6.3}$$

$$\partial_z p = 0, \tag{6.4}$$

$$\partial_t \rho + (\boldsymbol{v}^{(1)} \cdot \nabla + w^{(1)}\partial_z)\rho + (\boldsymbol{v} \cdot \nabla + w\partial_z)\rho^{(2)} = 0, \tag{6.5}$$

in $\mathcal{D} \times (0, T)$, with the corresponding boundary and initial value conditions

$$w(\boldsymbol{x}, z, t) = 0, \text{ on } \Gamma_z \times (0, T), \tag{6.6}$$

$$\int_0^h \boldsymbol{v}(\boldsymbol{x}, z, t) dz \cdot n = 0, \text{ on } \Gamma_x \times (0, T), \tag{6.7}$$

$$\boldsymbol{v}(\boldsymbol{x}, z, 0) = 0, \text{ in } \mathcal{D}, \tag{6.8}$$

$$\rho(\boldsymbol{x}, z, 0) = 0, \text{ in } \mathcal{D}. \tag{6.9}$$

Similarly to the *a priori* estimates of Section 3, by (6.2)–(6.9) and Lemma 4.1, we obtain that

$$\begin{aligned}\frac{d}{dt}\|(\boldsymbol{v}, \rho)\|_{X_\tau} \leq &\Big(\dot\tau + C_0 g + 3C_0(1+\tau^{-2})\big(\|(\boldsymbol{v}^{(1)}, \rho^{(1)})\|_{X_\tau} + \|(\boldsymbol{v}^{(2)}, \rho^{(2)})\|_{X_\tau}\big)\Big)\|(\boldsymbol{v}, \rho)\|_{Y_\tau} \\ &+ g\|(\boldsymbol{v}, \rho)\|_{X_\tau} + 3C_0(1+\tau^{-2})\big(\|(\boldsymbol{v}^{(1)}, \rho^{(1)})\|_{Y_\tau} + \|(\boldsymbol{v}^{(2)}, \rho^{(2)})\|_{Y_\tau}\big)\|(\boldsymbol{v}, \rho)\|_{X_\tau},\end{aligned} \tag{6.10}$$

where $C_0 > 0$ is the constant from Lemma 3.2. But by the construction of τ we have $\dot\tau + 20C_0 g + 20C_0(1 + \tau^{-2})\|(\boldsymbol{v}_0, \rho_0)\|_{X_{\tau_0}} e^{gt} = 0$, and by also using $\|(\boldsymbol{v}^{(1)}, \rho^{(1)})\|_{X_\tau} + \|(\boldsymbol{v}^{(2)}, \rho^{(2)})\|_{X_\tau} \leq 2\|(\boldsymbol{v}_0, \rho_0)\|_{X_{\tau_0}} e^{gt}$, we obtain

$$\begin{aligned}\frac{d}{dt}\|(\boldsymbol{v}, \rho)\|_{X_\tau} + 10C_0 g\|(\boldsymbol{v}, \rho)\|_{Y_\tau} + 10C_0\|(\boldsymbol{v}_0, \rho_0)\|_{X_{\tau_0}} e^{gt}(1+\tau^{-2})\|(\boldsymbol{v}, \rho)\|_{Y_\tau} \\ \leq g\|(\boldsymbol{v}, \rho)\|_{X_\tau} + 3C_0(1+\tau^{-2})(\|\boldsymbol{v}^{(1)}\|_{Y_\tau} + \|\boldsymbol{v}^{(2)}\|_{Y_\tau})\|(\boldsymbol{v}, \rho)\|_{X_\tau}.\end{aligned} \tag{6.11}$$

It is straightforward to check that (6.1), (6.8), (6.9), (6.11), and Grönwall's inequality imply that $\|(\boldsymbol{v}, \rho)\|_{X_\tau} = 0$ for all $t \in [0, T_*)$, and thereby proving the uniqueness of the solutions.

7. The three-dimensional case

In this section we sketch the proof of Theorem 2.3. As opposed to the two-dimensional case, here Lemma 3.1 does not hold, and hence we need to estimate the analytic norm of the pressure. We only emphasize the necessary changes from the two-dimensional case.

In Section 7.1 we give the proof of the pressure estimate in the case of periodic boundary conditions in the x-variable. In this case p may be written explicitly as a function of $\boldsymbol{v}$ and ρ (cf. (7.6) below), thereby simplifying the analysis.

When $\mathcal{M}$ is an analytic domain with boundary, the pressure is given implicitly as a solution of an elliptic Neumann problem (cf. Temam [25] for the Euler equations). We explore the transfer of normal to tangential derivatives in the higher-order estimates for the pressure and introduce a new suitable analytic norm to combinatorially encode this transfer. This gives us the necessary estimate (cf. Lemma 7.1) to prove that the pressure has the same radius of analyticity as the velocity. In Section 7.2 we give the proof of the pressure estimate in the case when $\mathcal{M}$ is a half-space.

7.1. **The periodic case: $\mathcal{M} = [0, 2\pi]^2$.** Here we give the *a priori* estimates for the case when the boundary condition (1.12) is replaced by the periodic boundary condition in the x-variable. Assume that $(\boldsymbol{u}, p, \rho)$ is a smooth solution to (1.7)–(1.11), $\mathcal{M}$-periodic in the x-variable. Since the pressure is defined up to a function of time, we may assume that $\int_{\mathcal{M}} p\, dx = 0$. Let $\boldsymbol{v}_0, \rho_0 \in X_{\tau_0}$ for some $\tau_0 > 0$, and fixed $r \geq 5/2$. Similarly to estimate (3.17) we have

$$\frac{d}{dt}\|\boldsymbol{v}(t)\|_{X_{\tau(t)}} \leq \dot{\tau}(t)\|\boldsymbol{v}(t)\|_{Y_{\tau(t)}} + \mathcal{U}(\boldsymbol{v}, \boldsymbol{v}) + \mathcal{V}(w, \boldsymbol{v}) + \mathcal{P} + g\|\nabla\psi\|_{X_\tau}, \tag{7.1}$$

where $\mathcal{U}(\boldsymbol{v}, \boldsymbol{v}) = \sum_{i,j=1}^{2} \mathcal{U}(v_i, v_j)$ and $\mathcal{V}(w, \boldsymbol{v}) = \sum_{j=1}^{2} \mathcal{V}(w, v_j)$, with $\mathcal{U}(v_i, v_j)$ and $\mathcal{V}(w, v_j)$ being defined by (3.18) and (3.19) respectively. In (7.1) above, we have denoted the upper bound on the pressure term by

$$\mathcal{P} = \sum_{m=1}^{\infty} \sum_{|\alpha|=m} \|\nabla\partial^\alpha p\|_{L^2(\mathcal{D})} \frac{(m+1)^r \tau^m}{m!} = h^{1/2} \sum_{m=1}^{\infty} \sum_{|\alpha|=m, \alpha_3=0} \|\nabla\partial^\alpha p\|_{L^2(\mathcal{M})} \frac{(m+1)^r \tau^m}{m!}. \tag{7.2}$$

Here we used the fact that p is z-independent, and the fact that due to the boundary condition (1.12) we have $\langle \nabla p, \boldsymbol{v}\rangle = \langle p, \operatorname{div} \boldsymbol{v}\rangle = -\langle p, \partial_z w\rangle = 0$. We note that in the three-dimensional case the cancelation property (3.10) does not hold, and therefore the pressure term does not vanish in the estimate (7.1). To estimate $\mathcal{P}$, we use the fact that the pressure may be computed explicitly from the velocity. First, note that $\int_0^h \operatorname{div} \boldsymbol{v}\, dz = 0$, and therefore, by integrating (1.7) in the z-variable, and then applying the divergence operator in the x-variable, we obtain

$$-\Delta p = \partial_k \fint_0^h \Big(v_j \partial_j v_k + w\partial_z v_k\Big)\, dz + f \fint_0^h (\partial_1 v_2 - \partial_2 v_1)\, dz - g\Delta \fint_0^h \psi\, dz. \tag{7.3}$$

In (7.3) we have used the summation convention over repeated indices $1 \leq j, k \leq 2$, and denoted by ∂_j the partial derivative $\partial/\partial x_j$, for all $1 \leq j \leq 2$. Integrating by parts in the z variable, it follows from (1.8) and (1.11) that

$$\int_0^h w\partial_z v_k\, dz = -\int_0^h v_k \partial_z w\, dz = \int_0^h v_k \partial_j v_j\, dz, \tag{7.4}$$

and therefore, by (7.3) we have

$$-\Delta p = \partial_k \partial_j \fint_0^h (v_j v_k)\, dz + f \fint_0^h (\partial_1 v_2 - \partial_2 v_1)\, dz - g\Delta \fint_0^h \psi\, dz. \tag{7.5}$$

The periodic boundary conditions in the x-variable allow for a simple solution to (7.5), namely

$$p = R_j R_k \fint_0^h (v_j v_k)\, dz + f(-\Delta)^{-1/2} \fint_0^h (R_1 v_2 - R_2 v_1)\, dz + g \fint_0^h \psi\, dz, \tag{7.6}$$

where R_j is the j^{th} Riesz transform, classically defined by its Fourier symbol $i\xi_j/|\xi|$. The boundedness of the Riesz transforms on $L^2(\mathcal{M})$, the Hölder inequality, and the Leibniz rule give the bound

$$\begin{aligned}
\mathcal{P} &\leq Ch^{1/2}\sum_{m=1}^{\infty}\sum_{|\alpha|=m,\alpha_3=0}\sum_{1\leq i,j,k\leq 2}\left\|\partial_i\partial^\alpha\left(\fint_0^h v_jv_k\,dz\right)\right\|_{L^2(\mathcal{M})}\frac{(m+1)^r\tau^m}{m!}\\
&\qquad + h^{1/2}\sum_{m=1}^{\infty}\sum_{|\alpha|=m,\alpha_3=0}\left(f\left\|\partial^\alpha\fint_0^h \boldsymbol{v}\,dz\right\|_{L^2(\mathcal{M})}+g\left\|\partial^\alpha\fint_0^h\nabla\psi\,dz\right\|_{L^2(\mathcal{M})}\right)\frac{(m+1)^r\tau^m}{m!}\\
&\leq C\sum_{m=1}^{\infty}\sum_{|\alpha|=m,\alpha_3=0}\sum_{1\leq i,j,k\leq 2}\|\partial^\alpha(\boldsymbol{v}_j\,\partial_i\boldsymbol{v}_k)\|_{L^2(\mathcal{D})}\frac{(m+1)^r\tau^m}{m!}+f\|\boldsymbol{v}\|_{X_\tau}+g\|\nabla\psi\|_{X_\tau}\\
&\leq C\sum_{m=1}^{\infty}\sum_{|\alpha|=m,\alpha_3=0}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\sum_{1\leq i,j,k\leq 2}\|\partial^\beta\boldsymbol{v}_j\,\partial^{\alpha-\beta}\partial_i\boldsymbol{v}_k\|_{L^2(\mathcal{D})}\frac{(m+1)^r\tau^m}{m!}+f\|\boldsymbol{v}\|_{X_\tau}+g\|\nabla\psi\|_{X_\tau}.
\end{aligned}\tag{7.7}$$

The first term on the right of (7.7) is estimated similarly to $\mathcal{U}(\boldsymbol{v},\boldsymbol{v})$ via Lemma 4.1, the case $d=3$, and we obtain the *a priori* pressure estimate

$$\mathcal{P}\leq C_0(1+\tau^{-5/2})\|\boldsymbol{v}\|_{X_\tau}\|\boldsymbol{v}\|_{Y_\tau}+f\|\boldsymbol{v}\|_{X_\tau}+g\|\nabla\psi\|_{X_\tau}.\tag{7.8}$$

By possibly enlarging C_0, as shown in (3.22) we also have $\|\nabla\psi\|_{X_\tau}\leq C_0\|\rho\|_{Y_\tau}+\|\rho\|_{X_\tau}$, and therefore

$$\mathcal{P}\leq C_0(1+\tau^{-5/2})\|\boldsymbol{v}\|_{X_\tau}\|\boldsymbol{v}\|_{Y_\tau}+f\|\boldsymbol{v}\|_{X_\tau}+g\|\rho\|_{X_\tau}+C_0g\|\rho\|_{Y_\tau}.\tag{7.9}$$

Combining the *a priori* estimate (7.1), the bounds on $\mathcal{U}(\boldsymbol{v},\boldsymbol{v})$ and $\mathcal{V}(w,\boldsymbol{v})$ obtained from Lemma 4.1, and the pressure estimate (7.9), in analogy to (3.23), we obtain the bound

$$\frac{d}{dt}\|\boldsymbol{v}\|_{X_\tau}\leq\left(\dot{\tau}+4C_0(1+\tau^{-5/2})\|\boldsymbol{v}\|_{X_\tau}\right)\|\boldsymbol{v}\|_{Y_\tau}+2C_0g\|\rho\|_{Y_\tau}+f\|\boldsymbol{v}\|_{X_\tau}+2g\|\rho\|_{X_\tau}.\tag{7.10}$$

Since the evolution of the density ρ (cf. (1.10)) does not involve the pressure term, using Lemma 4.1, in analogy to (3.24) we have

$$\frac{d}{dt}\|\rho\|_{X_\tau}\leq\dot{\tau}\|\rho\|_{Y_\tau}+2C_0(1+\tau^{-5/2})\|\boldsymbol{v}\|_{X_\tau}\|\rho\|_{Y_\tau}+4C_0(1+\tau^{-5/2})\|\boldsymbol{v}\|_{Y_\tau}\|\rho\|_{X_\tau},\tag{7.11}$$

and therefore, by combining (7.10) and (7.11) we obtain

$$\frac{d}{dt}\|(\boldsymbol{v},\rho)\|_{X_\tau}\leq\left(\dot{\tau}+2C_0g+4C_0(1+\tau^{-5/2})\|(\boldsymbol{v},\rho)\|_{X_\tau}\right)\|(\boldsymbol{v},\rho)\|_{Y_\tau}+C_1\|(\boldsymbol{v},\rho)\|_{X_\tau},\tag{7.12}$$

where $C_1=\max\{f,2g\}$. With $\tau(t)$ defined by $\tau(0)=\tau_0$ and

$$\dot{\tau}+20C_0g+20C_0(1+\tau^{-5/2})\|(\boldsymbol{v}_0,\rho_0)\|_{X_{\tau_0}}e^{C_1t}=0,\tag{7.13}$$

the rest of the proof of Theorem 2.3, namely the estimate (2.7), follows in analogy to the two-dimensional case (cf. Section 3). The uniqueness of x-periodic solutions in the space $L^\infty(0,T_*;X_\tau)\cap L^1(0,T_*;(1+\tau^{-5/2})Y_\tau)$ follows as in Section 6, with the only modification being the power of τ in the estimates is now $-5/2$ instead of -2. The construction of the x-periodic solution is similar to Section 5, with one additional modification: instead of $p^{(n)}$ being defined by (7.6), we define the n^{th} iterate of the pressure via

$$p^{(n+1)}=R_jR_k\fint_0^h v_j^{(n)}v_k^{(n)}\,dz+f(-\Delta)^{-1/2}\fint_0^h(R_1v_2^{(n)}-R_2v_1^{(n)})\,dz+g\fint_0^h\psi^{(n+1)}\,dz.\tag{7.14}$$

To avoid redundancy we omit further details.

7.2. **A domain with boundary: $\mathcal{M}$ is the upper half-plane.** Let $\mathcal{M} = \{\boldsymbol{x} \in \mathbb{R}^2 : x_1 > 0\}$, so that $\Gamma_x = \partial\mathcal{M} \times (0,h) = \{(\boldsymbol{x}, z) \in \mathbb{R}^3 : x_1 = 0,\ 0 < z < h\}$. Therefore the side boundary condition (1.12) is $\int_0^h v_1(0, x_2, z)\, dz = 0$. In order to close the estimates for the pressure (cf. Lemma 7.1), in the case of the half-pane it is necessary to use a modified Sobolev semi-norm (see also Kukavica and Vicol [10]), instead of the classical $|\cdot|_m$ from (2.1). We let

$$[\boldsymbol{v}]_m = \sum_{|\alpha|=m} \frac{1}{2^{\alpha_1}} \|\partial^\alpha \boldsymbol{v}\|_{L^2(\mathcal{D})}, \tag{7.15}$$

and define the corresponding analytic X_τ norm

$$\|\boldsymbol{v}\|_{X_\tau} = \sum_{m=0}^{\infty} [\boldsymbol{v}]_m \frac{(m+1)^r \tau^m}{m!}, \tag{7.16}$$

and respectively the Y_τ semi-norm

$$\|\boldsymbol{v}\|_{Y_\tau} = \sum_{m=1}^{\infty} [\boldsymbol{v}]_m \frac{(m+1)^r \tau^{m-1}}{(m-1)!}. \tag{7.17}$$

As in the periodic case, we have the *a priori* estimate

$$\frac{d}{dt}\|\boldsymbol{v}(t)\|_{X_{\tau(t)}} \le \dot{\tau}(t)\|\boldsymbol{v}(t)\|_{Y_{\tau(t)}} + \mathcal{U}(\boldsymbol{v},\boldsymbol{v}) + \mathcal{V}(w,\boldsymbol{v}) + \mathcal{P} + g\|\nabla\psi\|_{X_\tau}, \tag{7.18}$$

where $\mathcal{U}(\boldsymbol{v},\boldsymbol{v})$ and $\mathcal{V}(w,\boldsymbol{v})$ are defined similarly to (3.18) and (3.19), namely by

$$\mathcal{U}(\boldsymbol{v},\tilde{v}) = \sum_{m=0}^{\infty}\sum_{j=0}^{m}\sum_{|\alpha|=m} \frac{1}{2^{\alpha_1}} \sum_{|\beta|=j,\beta\le\alpha} \binom{\alpha}{\beta} \|\partial^\beta \boldsymbol{v}\cdot\nabla\partial^{\alpha-\beta}\tilde{v}\|_{L^2} \frac{(m+1)^r\tau^m}{m!}, \tag{7.19}$$

and by

$$\mathcal{V}(w,\tilde{v}) = \sum_{m=0}^{\infty}\sum_{j=0}^{m}\sum_{|\alpha|=m} \frac{1}{2^{\alpha_1}} \sum_{|\beta|=j,\beta\le\alpha} \binom{\alpha}{\beta} \|\partial^\beta w\,\partial_z\partial^{\alpha-\beta}\tilde{v}\|_{L^2} \frac{(m+1)^r\tau^m}{m!}, \tag{7.20}$$

and the pressure term is given by

$$\mathcal{P} = h^{1/2}\sum_{m=1}^{\infty}\sum_{|\alpha|=m,\alpha_3=0} \frac{1}{2^{\alpha_1}}\|\nabla\partial^\alpha p\|_{L^2(\mathcal{M})}\frac{(m+1)^r\tau^m}{m!}. \tag{7.21}$$

Recall that the term corresponding to $m = 0$ in (7.21) is missing since the side boundary condition (1.12) implies that $\langle\nabla p, \boldsymbol{v}\rangle = 0$. It is straightforward to check that the proof of Lemma 4.1 also applies to the above defined operators $\mathcal{U}$ and $\mathcal{V}$, and hence we have the three-dimensional bounds

$$\mathcal{U}(\boldsymbol{v},\boldsymbol{v}) + \mathcal{V}(w,\boldsymbol{v}) \le 2C_0(1+\tau^{-5/2})\|\boldsymbol{v}\|_{X_\tau}\|\boldsymbol{v}\|_{Y_\tau}. \tag{7.22}$$

To estimate $\mathcal{P}$, we note that by (7.3) the vertical average of the full pressure

$$\tilde{p}(\boldsymbol{x}) = \fint_0^h P(\boldsymbol{x},z)\,dz = p(\boldsymbol{x}) - g\fint_0^h \psi(\boldsymbol{x},z)\,dz \tag{7.23}$$

satisfies

$$-\Delta\tilde{p} = \partial_k \fint_0^h \Big(v_j\,\partial_j v_k + v_k\,\partial_j v_j\Big)\,dz + f\fint_0^h \Big(\partial_1 v_2 - \partial_2 v_1\Big)\,dz = F \tag{7.24}$$

for all $x \in \mathcal{M}$, where the summation convention over $1 \leq j, k \leq 2$ is used. By applying $\fint_0^h dz$ to (1.7), and taking the inner product with $n = (-1, 0)$, the outward unit normal vector to $\mathcal{M}$, we obtain that (7.24) is supplemented with the boundary condition

$$\begin{aligned} \frac{\partial \tilde{p}}{\partial n} &= -\fint_0^h \Big(v_j \partial_j v_k + w \partial_z v_k \Big)\, dz \cdot n_k - f \fint_0^h v_k^\perp \, dz \cdot n_k \\ &= \fint_0^h \Big(v_1 \partial_j v_j + v_j \partial_j v_1 \Big)\, dz + f \fint_0^h v_2 \, dz = G, \end{aligned} \tag{7.25}$$

where $j \in \{1, 2\}$. We note that as opposed to the Euler equations on a half-space (cf. [10]) the nonlocal boundary condition on the velocity implies that the boundary condition for p is non-homogeneous (i.e., $\partial p / \partial n$ may be nonzero), creating additional difficulties. After subtracting a function of time from the full-pressure we have $\int_{\mathcal{D}} P(\boldsymbol{x}, z)\, d\boldsymbol{x} dz = 0$, and therefore there exists a unique smooth solution to the boundary value problem (7.24)–(7.25).

Lemma 7.1. *The smooth solution $\tilde{p} = p - g \fint_0^h \psi \, dz$ to the elliptic Neumann problem* (7.24)–(7.25)*, satisfies*

$$[\nabla \tilde{p}]_m \leq C_1 [F]_{m-1} + C_1 [G]_m, \tag{7.26}$$

where C_1 is a universal constant, independent of m.

Proof of Lemma 7.1. In order to bound

$$[\nabla \tilde{p}]_m = \sum_{|\alpha| = m,\, \alpha_3 = 0} \frac{1}{2^{\alpha_1}} \| \nabla \partial^\alpha \tilde{p} \|_{L^2(\mathcal{M})}, \tag{7.27}$$

we estimate tangential and normal derivatives separately. To estimate tangential derivatives of the pressure, we note that for any $\alpha_2 \geq 0$, the function $\partial_2^{\alpha_2} \tilde{p}$ is a solution of the elliptic Neumann problem

$$-\Delta (\partial_2^{\alpha_2} \tilde{p}) = \partial_2^{\alpha_2} F \tag{7.28}$$

$$\frac{\partial (\partial_2^{\alpha_2} \tilde{p})}{\partial n} = \partial_2^{\alpha_2} G, \tag{7.29}$$

and hence the classical H^2 regularity theorem, and the trace theorem give that there exists $C_1 > 0$ such that

$$\| \partial_2^{\alpha_2} \tilde{p} \|_{\dot{H}^2(\mathcal{M})} \leq C_1 \| \partial_2^{\alpha_2} F \|_{L^2(\mathcal{M})} + C_1 \| \partial_2^{\alpha_2} G \|_{H^1(\mathcal{M})}. \tag{7.30}$$

To estimate normal derivatives, we note that

$$-\partial_{11} \tilde{p} = -\Delta \tilde{p} + \partial_{22} \tilde{p} = F + \partial_{22} \tilde{p} \tag{7.31}$$

$$(-\partial_{11})^2 \tilde{p} = -\partial_{11} F + \partial_{22} (F + \partial_{22} \tilde{p}) = -\partial_{11} F + \partial_{22} F + \partial_{22}^2 \tilde{p}, \tag{7.32}$$

and by induction one may show that

$$(-\partial_{11})^{k+1} \tilde{p} = \partial_{22}^{k+1} \tilde{p} + \sum_{l=0}^{k} (-1)^l \partial_1^{2l} \partial_2^{2k-2l} F. \tag{7.33}$$

Therefore, if $\alpha = (\alpha_1, \alpha_2, 0) \in \mathbb{N}^3$ is such that $\alpha_1 \geq 2$, and $\alpha_1 = 2k + 2$ is even, then by (7.33) and (7.30) we have

$$\| \nabla \partial^\alpha \tilde{p} \|_{L^2(\mathcal{M})} \leq \| \partial_2^{2k+2+\alpha_2} \nabla \tilde{p} \|_{L^2(\mathcal{M})} + \sum_{l=0}^{k} \| \partial_1^{2l} \partial_2^{2k-2l+\alpha_2} \nabla F \|_{L^2(\mathcal{M})} \tag{7.34}$$

$$\leq C_1 \| \partial_2^{|\alpha|-1} G \|_{H^1(\mathcal{M})} + C_1 \sum_{j=0}^{\alpha_1 - 2} \| \partial_1^j \partial_2^{|\alpha| - j - 2} \nabla F \|_{L^2(\mathcal{M})}. \tag{7.35}$$

Similarly, if $\alpha_1 \geq 3$, and $\alpha_1 = 2k+3$ is odd, then similar arguments show that

$$\|\nabla\partial^\alpha \tilde{p}\|_{L^2(\mathcal{M})} \leq C_1 \|\partial_2^{|\alpha|-1} G\|_{H^1(\mathcal{M})} + C_1 \sum_{j=1}^{\alpha_1-2} \|\partial_1^j \partial_2^{|\alpha|-j-2} \nabla F\|_{L^2(\mathcal{M})}. \tag{7.36}$$

Lastly, when $\alpha_1 \leq 1$, and $|\alpha| \geq 1$, then

$$\|\nabla\partial^\alpha \tilde{p}\|_{L^2(\mathcal{M})} \leq C_1 \left(\|\partial_2^{|\alpha|-1} F\|_{L^2(\mathcal{M})} + \|\partial_2^{|\alpha|-1} G\|_{H^1(\mathcal{M})} \right). \tag{7.37}$$

Summarizing estimates (7.30)–(7.37) we obtain

$$\begin{aligned} \sum_{|\alpha|=m,\alpha_3=0} \frac{1}{2^{\alpha_1}} \|\nabla\partial^\alpha \tilde{p}\|_{L^2(\mathcal{M})} &\leq C_1 \sum_{\alpha_1=0}^{m} \frac{1}{2^{\alpha_1}} \|\partial_2^{m-1} G\|_{H^1(\mathcal{M})} + 2C_1 \|\partial_2^{m-1} F\|_{L^2(\mathcal{M})} \\ &\quad + C_1 \sum_{\alpha_1=2}^{m} \sum_{j=0}^{\alpha_1-2} \left(\frac{1}{2^j} \|\partial_1^j \partial_2^{|\alpha|-j-2} \nabla F\|_{L^2(\mathcal{M})} \right) \frac{1}{2^{\alpha_1-j}}, \end{aligned} \tag{7.38}$$

for all $m \geq 1$. Here we see why the introduction of the normalizing factors $1/2^{\alpha_1}$ was necessary. Without them the constant C_1 in the above estimates would depend linearly on m. However since $\sum_k 1/2^k < \infty$, by possibly enlarging C_1 (which is independent of m) we have

$$\sum_{|\alpha|=m,\alpha_3=0} \frac{1}{2^{\alpha_1}} \|\nabla\partial^\alpha \tilde{p}\|_{L^2(\mathcal{M})} \leq C_1 \|\partial_2^{m-1} G\|_{H^1(\mathcal{M})} + C_1 \sum_{|\alpha|=m-1,\, \alpha_3=0} \frac{1}{2^{\alpha_1}} \|\partial^\alpha F\|_{L^2(\mathcal{M})}, \tag{7.39}$$

concluding the proof of the lemma. □

Remark 7.2. If C_1 would depend on m, and would grow unboundedly as $m \to \infty$, then the additional loss of one full derivative coming from estimating w in terms of $\boldsymbol{v}$, prevents the estimates from closing. The normalizing weights $1/2^{\alpha_1}$ may be viewed as a suitable combinatorial encoding of the transfer of normal to tangential derivatives in (7.33).

Lemma 7.3. *Let $(\boldsymbol{v}, \rho) \in X_\tau$, and $\tilde{p}$ be the unique smooth solution of the elliptic Neumann-problem (7.24)–(7.25). Then the term $\mathcal{P}$ as defined in (7.21), with $p = \tilde{p} + g \int_0^h \psi$, is bounded by*

$$\mathcal{P} \leq C_1 (1 + \tau^{-5/2}) \|\boldsymbol{v}\|_{X_\tau} \|\boldsymbol{v}\|_{Y_\tau} + g \|\rho\|_{X_\tau} + C_1 g \|\rho\|_{Y_\tau} + C_1 f \|\boldsymbol{v}\|_{X_\tau}, \tag{7.40}$$

for some positive universal constat C_1.

Proof of Lemma 7.3. By the triangle inequality and the definition of $\tilde{p}$ cf. (7.23), we have that

$$\sum_{|\alpha|=m,\, \alpha_3=0} \frac{1}{2^{\alpha_1}} \|\nabla\partial^\alpha p\|_{L^2(\mathcal{M})} \leq \sum_{|\alpha|=m,\, \alpha_3=0} \frac{1}{2^{\alpha_1}} \|\nabla\partial^\alpha \tilde{p}\|_{L^2(\mathcal{M})} + g \sum_{|\alpha|=m,\, \alpha_3=0} \frac{1}{2^{\alpha_1}} \|\nabla\partial^\alpha \int_0^h \psi \, dz\|_{L^2(\mathcal{M})}.$$

From (7.39) and the above estimate it follows that the pressure term is bounded by

$$\begin{aligned} \mathcal{P} \leq C_1 h^{1/2} \sum_{m=1}^{\infty} \sum_{|\alpha|=m-1,\, \alpha_3=0} \frac{1}{2^{\alpha_1}} \left(\|\partial^\alpha F\|_{L^2(\mathcal{M})} + \|\partial^\alpha G\|_{H^1(\mathcal{M})} \right) \frac{(m+1)^r \tau^m}{m!} \\ + g h^{1/2} \sum_{m=1}^{\infty} \sum_{|\alpha|=m,\, \alpha_3=0} \frac{1}{2^{\alpha_1}} \|\nabla\partial^\alpha \int_0^h \psi \, dz\|_{L^2(\mathcal{M})}. \end{aligned} \tag{7.41}$$

Using Hölder's inequality and the definitions of F and G, we obtain

$$
\begin{aligned}
\mathcal{P} &\le C_1 \sum_{m=0}^{\infty} \sum_{|\alpha|=m,\,\alpha_3=0} \frac{1}{2^{\alpha_1}} \|\partial^\alpha (v_j\,\partial_j v_k + v_k \partial_j v_j)\|_{L^2(\mathcal{D})} \frac{(m+1)^r \tau^m}{m!} \\
&\qquad + g\|\nabla\psi\|_{X_\tau} + C_1 f \sum_{m=0}^{\infty} \sum_{|\alpha|=m,\,\alpha_3=0} \frac{1}{2^{\alpha_1}} \|\partial^\alpha \boldsymbol{v}\|_{L^2(\mathcal{D})} \frac{(m+1)^r \tau^m}{m!} \\
&\le C_1 \sum_{m=0}^{\infty} \sum_{j=0}^{m} \sum_{|\alpha|=m,\,\alpha_3=0} \frac{1}{2^{\alpha_1}} \sum_{|\beta|=j,\,\beta\le\alpha} \binom{\alpha}{\beta} \|\partial^\beta \boldsymbol{v}\cdot\nabla\partial^{\alpha-\beta}\boldsymbol{v}\|_{L^2(\mathcal{D})} \frac{(m+1)^r \tau^m}{m!} \\
&\qquad + g\|\rho\|_{X_\tau} + C_1 g\|\rho\|_{Y_\tau} + C_1 f\|\boldsymbol{v}\|_{X_\tau}.
\end{aligned} \tag{7.42}
$$

The first term on the right of the above term is bounded as $\mathcal{U}(\boldsymbol{v},\boldsymbol{v})$ using the three-dimensional case of Lemma 4.1, concluding the proof of the lemma. □

We now conclude the proof of Theorem 2.3 in the case when $\mathcal{M}$ is a half-space. Combining (7.40) with (7.18) and (7.22), we obtain the analytic *a priori* estimate

$$
\frac{d}{dt}\|\boldsymbol{v}\|_{X_\tau} \le \left(\dot\tau + C_2(1+\tau^{-5/2})\|\boldsymbol{v}\|_{X_\tau}\right)\|\boldsymbol{v}\|_{Y_\tau} + g\|\rho\|_{X_\tau} + C_2 g\|\rho\|_{Y_\tau} + C_2 f\|\boldsymbol{v}\|_{X_\tau}, \tag{7.43}
$$

for some positive constant $C_2 = C_2(C_0, C_1)$. Since the equation for the evolution of the density ρ does not contain a pressure term, similarly to (7.11) we have

$$
\frac{d}{dt}\|\rho\|_{X_\tau} \le \dot\tau\|\rho\|_{Y_\tau} + C_2(1+\tau^{-5/2})\|\boldsymbol{v}\|_{X_\tau}\|\rho\|_{Y_\tau} + C_2(1+\tau^{-5/2})\|\boldsymbol{v}\|_{Y_\tau}\|\rho\|_{X_\tau}, \tag{7.44}
$$

and therefore, by combining the above with (7.43) we obtain

$$
\frac{d}{dt}\|(\boldsymbol{v},\rho)\|_{X_\tau} \le \left(\dot\tau + C_2 g + C_2(1+\tau^{-5/2})\|(\boldsymbol{v},\rho)\|_{X_\tau}\right)\|(\boldsymbol{v},\rho)\|_{Y_\tau} + C_3\|(\boldsymbol{v},\rho)\|_{X_\tau}, \tag{7.45}
$$

for some fixed positive constant $C_2 > 0$, where $C_3 = g + C_2 f$. Lastly, we let $\tau(t)$ be the solution of the ordinary differential equation

$$
\dot\tau + 2C_2 g + 2C_2(1+\tau^{-5/2})\|(\boldsymbol{v}_0,\rho_0)\|_{X_{\tau_0}} e^{C_3 t} = 0, \tag{7.46}
$$

with initial data τ_0. Arguments similar to those for the periodic case and to those for the two-dimensional case, give the existence and uniqueness of solutions satisfying

$$
\begin{aligned}
&\|(\boldsymbol{v}(t),\rho(t))\|_{X_{\tau(t)}} + C_2 g\int_0^t e^{C_3(t-s)}\,\|(\boldsymbol{v}(s),\rho(s))\|_{Y_{\tau(s)}}\,ds \\
&\qquad + C_2\|(\boldsymbol{v}_0,\rho_0)\|_{X_{\tau_0}} e^{C_3 t}\int_0^t \left(1+\tau^{-5/2}(s)\right)\|(\boldsymbol{v}(s),\rho(s))\|_{Y_{\tau(s)}}\,ds \le \|(\boldsymbol{v}_0,\rho_0)\|_{X_{\tau_0}} e^{C_3 t},
\end{aligned} \tag{7.47}
$$

where $C_3 = C_3(C_2, f, g)$ is a fixed constant, for all $t \in [0, T_*)$, where T_* can be estimated from the data. We point out that these *a priori* estimates can be made rigorous using verbatim arguments to those in Sections 5 and 6. The only difference is that in the construction of the solutions, the n^{th} iterate $p^{(n)}$ is defined here by

$$
p^{(n+1)}(\boldsymbol{x}) = \tilde p^{(n)}(\boldsymbol{x}) + g\fint_0^h \psi^{(n+1)}(\boldsymbol{x},z)\,dz, \tag{7.48}
$$

where $\tilde p^{(n)}$ is the unique smooth mean-free solution of the elliptic Neumann-problem

$$
-\Delta\tilde p^{(n)} = \partial_k \fint_0^h \left(v_j^{(n)}\,\partial_j v_k^{(n)} + v_k^{(n)}\,\partial_j v_j^{(n)}\right)dz + f\fint_0^h \left(\partial_1 v_2^{(n)} - \partial_2 v_1^{(n)}\right)dz \tag{7.49}
$$

$$
\frac{\partial\tilde p^{(n)}}{\partial n} = \fint_0^h \left(v_1^{(n)}\partial_j v_j^{(n)} + v_j^{(n)}\partial_j v_1^{(n)}\right)dz + f\fint_0^h v_2^{(n)}\,dz. \tag{7.50}
$$

We recall that the velocity iterate $\boldsymbol{v}^{(n+1)}$ is defined via cf. (5.5), and hence after taking the derivative in time, the average value in z, and integrating by parts in z, it satisfies

$$\partial_t \fint_0^h \boldsymbol{v}^{(n+1)}\,dz + \fint_0^h \left(\boldsymbol{v}^{(n)}\cdot\nabla + \operatorname{div}\boldsymbol{v}^{(n)}\right)\boldsymbol{v}^{(n)}\,dz + \nabla\tilde{p}^{(n)} + f\fint_0^h \boldsymbol{v}^{(n)\perp}\,dz = 0. \tag{7.51}$$

By taking the dot product of (7.51) with the outward unit normal n to Γ_x, and using (7.50), we obtain that

$$\int_0^h v_1^{(n+1)}(0,x_2,z,t)\,dz = \int_0^h v_1^{(n+1)}(0,x_2,z,0)\,dz = \int_0^h v_{01}(0,x_2,z)\,dz. \tag{7.52}$$

Therefore, the boundary condition $\int_0^h v_1^{(n)}(0,x_2,z,t)\,dz = 0$ (cf. (1.12)) is satisfied by all iterates if it is satisfied by the initial data. Similarly, by taking the two-dimensional divergence of (7.51), and using (7.49), we obtain that the compatibility condition $\int_0^h \operatorname{div}\boldsymbol{v}^{(n)}(\boldsymbol{x},z,t)\,dz = 0$ (cf. (1.14)) is satisfied by all iterates if it is satisfied by the initial data. We omit further details.

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