On the existence for the free interface 2D Euler equation with a localized vorticity condition

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To the memory of Professor A.V. Balakrishnan

ABSTRACT. We prove a local-in-time existence of solutions result for the two dimensional incompressible Euler equations on a moving boundary, with no surface tension, under the Rayleigh-Taylor stability condition. The main feature of the result is the local regularity assumption on the initial vorticity, namely $H^{1.5+\delta}$ Sobolev regularity in the vicinity of the moving interface in addition to the global regularity assumption on the initial fluid velocity in the $H^{2+\delta}$ space. We use a special change of variables and derive a priori estimates, establishing the local-in-time existence in $H^{2+\delta}$. The assumptions on the initial data constitute the minimal set of assumptions for the existence of solutions to the rotational flow problem to be established in 2D.

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1. Introduction

In this paper, we establish the local-in-time solutions to the incompressible Euler equations for a free moving interface, with no surface tension, for rotational flows under minimal regularity assumptions on the initial data and the Rayleigh-Taylor stability condition.

The Euler equations

$$u_t + u \cdot \nabla u + \nabla p = 0 \quad \text{in } \Omega(t) \times (0,T) \tag{1.1}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega(t) \times (0,T) \tag{1.2}$$

which describe the flow of an ideal inviscid incompressible fluid with a velocity field $u(x,t)$ and a fluid pressure $p(x,t)$ on a moving domain have attracted considerable attention in the mathematical literature. They model propagation of shallow water waves under the influence of gravity. The boundary of the domain $\Omega(t)$ consists of a flexible part $\Gamma_1(t)$, which moves with the fluid velocity, and a stationary part Γ_0. The boundary conditions imposed are

$$u \cdot n = 0 \quad \text{in } \Gamma_0 \times (0,T) \tag{1.3}$$

$$p = \epsilon\sigma \quad \text{in } \Gamma_1(t) \times (0,T) \tag{1.4}$$

where $\sigma(x)$ is the mean curvature of the boundary and $0 \le \epsilon \le 1$ is the surface tension. In this paper we are interested in the no surface tension case, $\epsilon = 0$. In this context, the boundary condition $p = 0$ is also an idealization of the two phase problem involving the interaction of the water waves with the atmosphere. In fact, a relevant problem in this regard has been the study of the two phase problem and showing that the solutions to the system converge to the solutions of the idealized system as the density of the second phase goes to zero [**P**].

Initial results on well-posedness of the system were obtained for small irrotational initial data by Nalimov in 2D [**N**], Yoshihara [**Y1**], and Craig [**Cr**]. Another early result by Shinbrot considered the solution given analytic initial data [**Sh**]. Later, the authors in [**BHL**], established the local existence of solutions to the linearized system under the Taylor condition. The general well-posedness of the system was an open standing problem until Ebin [**E**] showed the ill-posedness of the system for general initial data without the Rayleigh-Taylor sign condition on the initial data.

The major development came in 1996 with the seminal work of Wu, who showed local-in-time existence of solutions for irrotational flows in 2D [**W1**] and 3D [**W2**]. She relied on Clifford analysis to study the evolution of the interface and obtained well-posedness for general smooth initial data under the Rayleigh-Taylor sign condition. The Rayleigh-Taylor sign condition on the initial pressure is given by

$$\frac{\partial q}{\partial N}(x,0) \leq -\frac{1}{C_0} < 0, \qquad x \in \Gamma_1 \tag{1.5}$$

and always holds for irrotational flows for domains with infinite depth or flat bottom (rigid part of the boundary) as was shown by [**W1**], or in case of rigid bottom with small curvature [**L**]. The Taylor-Rayleigh condition is a stability criterion used for the study of two phase ideal fluids and is physically interpreted to mean that the lighter fluid is situated above the heavier one. This condition is connected to a well-known physical phenomenon known as the Rayleigh-Taylor instability, which involves turbulent mixing of two fluids when the heavier fluid happens to be on top of the lighter fluid.

The incorporation of the surface tension into the system serves as a regularization and does not require any additional stability conditions on the initial data. Local-in-time existence results for the system with surface tension were obtained by [**Y2, I, OT, S**]. In [**AM1, AM2**], the authors also showed the convergence of solutions in the limit of vanishing surface tension (as $\epsilon \to 0$) for irrotational flows. In [**SZ1**], the authors used the geometric approach to establish this convergence result under the more general Rayleigh-Taylor condition, by deriving uniform estimates which are independent of the surface tension parameter ϵ. We refer the reader to some other results and references therein [**AD, B, Shn, CL, L, CS1, CS2, EL, KT1, Li1, Li2, ZZ, ABZ1, ABZ2, W3, GMS, IT, CLa, HIT, IP, MR, XZ, T**].

A special attention in the literature has been given to the emergence of singularities on the interface, cf. [**CCFGGS**] for instance. In this context, there has been some recent work on the irrotational case which highlights the dispersive nature of the interface and which relies on singular integrals and Strichartz estimates of the potential flow and stream functions [**ABZ1, ABZ2**]. A particular emphasis is given to the initial regularity of the interface, and the minimal regularity of the initial interface and initial velocity. The authors were able to establish local-in-time existence of solutions but not uniqueness under an initial interface regularity of $H^{d/2+3/2-1/12+\delta}$ and initial interface velocity in $H^{d/2+1-1/12+\delta}$, where $d > 0$ is the interface dimension and $\delta > 0$. Unlike earlier regularity results in the literature, these results suggest that the interface need not have bounded curvature and that the initial velocity need only be Lipschitz continuous. These results, are as far as we know, pose the lowest regularity requirement on the initial data for irrotational flows. However, the tools of analysis in this approach do not apply to the more general case of a rotational flow.

In previous works, we have undertaken the task of obtaining local-in-time solutions to the system under minimal regularity conditions on the initial data for the rotational flow case. In particular, mirroring the regularity results for Euler equations on fixed domains [**MB, Te**], where minimal regularity of initial data required is $H^{n/2+1+\delta}$, where n is the space dimension and $\delta > 0$, two of the authors [**KT2**] proved the local existence of solutions to the 2D system under the Taylor sign condition assuming the minimum requirement on the initial velocity of $H^{2+\delta}$, with the vorticity in $H^{1.5+\delta}$, where $\delta > 0$ using a div-curl type estimate. Later, in [**KTV**], three of the authors addressed the 3D system with rotational initial data, i.e., with the initial velocity in $H^{2.5+\delta}$ and initial vorticity in $H^{2+\delta}$, by resorting to the Cauchy invariance [**C2, FV**]. They also showed uniqueness of solutions. In each case, the interface regularity obtained is $1/2$ space derivative smoother than the interface velocity. We note that in [**SZ2**] Shatah and Zeng also used the Cauchy invariance to analyze the evolution of the vorticity.

The approach we used in [**KTV**] is that of Coutand and Shkoller [**CS1, CS2**], who rely on the Lagrangian formulation of the free surface Euler equations and on div-curl type estimates to prove well-posedness. In particular, Coutand and Shkoller showed the existence and the uniqueness of solutions in 3D assuming the initial velocity belongs to H^3 [**CS1**]. A related geometric approach was developed simultaneously in [**SZ1**], where the authors also obtain a priori estimates showing the local existence in H^3.

In our previous results, we have imposed the condition that the vorticity ω is smoother by $1/2$ space derivative than the gradient of the velocity. In this paper, for the purpose of local existence, we prove that in fact it is sufficient for the vorticity to be smoother only in a neighborhood of the top boundary. In particular, in the 2D result in [**KT2**], the smoother vorticity was used to deduce the regularity of the Lagrangian flow map which in turn was connected to the evolution of the interface. However, it was observed that the use of the Lagrangian flow map on the whole domain can be dispensed with, since it is only the evolution of the interface which drives the tangential regularity of the velocity field. In fact, the choice of the coordinate map away from the boundary is inessential, and it is sufficient to impose the smoother condition on the vorticity only in a neighborhood of the evolving interface.

Our result is then the local existence of solutions for the 2D free-surface Euler equations only assuming that ω_0 lies in $H^{1.5+\delta}(U_{\epsilon_0})$ and v_0 is in $H^{2+\delta}(\Omega)$ with the Taylor condition

$$\frac{\partial q}{\partial N}(x,0) \leq -\frac{1}{C_0} < 0, \qquad x \in \Gamma_1 \tag{1.6}$$

where $C_0 > 0$ is a constant, and where $U_{\epsilon_0} = \{x \in \Omega : \operatorname{dist}(x, \Gamma_1) \leq \epsilon_0\}$ for some fixed $\epsilon_0 > 0$.

Since we only work under a higher regularity of the vorticity around the top boundary, the usual global Lagrangian coordinate does not work in our scenario. What we need here is a new coordinate transform η introduced in Section 2 that captures the evolution of the moving interface and the adjacent flow, but coincides with the Eulerian coordinate system away from the interface. We also use the property that the vorticity ω remains constant along the Lagrangian trajectories in 2D flow. In order to carry out the div-curl estimate, we need a good control of the vorticity near the moving interface represented by the term $\omega(\eta)$, which requires a Sobolev type estimate for a composite of two functions. Another feature of the analysis is that the tangential estimates on the velocity are not global and do not hold for the whole domain due to the nonlocal nature of the pressure terms. In particular, the localized condition on the initial data, leads to a loss of regularity in the time derivative of the pressure q_t by one space derivative below what is required for the global tangential estimates to hold as in [**KT2**]. We note that our regularity result implies that the initial interface velocity is only Lipschitz continuous by the Sobolev embedding theorem. While we consider a flat initial surface to simplify the analysis, the same result can be obtained by a change of variable. In particular, the initial interface $h(x,t)$ can be assumed to be a graph of a function of $H^{3+\delta}(\mathbb{R})$ which also implies an initial interface of class $C^{5/2}$. Throughout this paper, we only provide a priori estimates. The rigorous proof is obtained by smoothing the initial data, estimating under the mollified setting, and passing to the limit.

We would like to dedicate this work to the memory of Professor A.V. Balakrishnan who passed away this last year. Balakrishnan or Bal, as he liked to be known, has been an inspirational figure for the mathematical and engineering community alike for many years. Bal has been celebrated as one of the founding fathers of modern control theory along with major figures such as Pontryagin and Kalman. However, his remarkable contributions to the fields of Semigroups Theory, Stochastic and Distributed systems, and Aerospace and Flight research can not be overstated, and they attest to his broad and in depth knowledge across many fields. His seminal paper on fractional power of operators has been a major development in functional analysis and has had important ramifications in the field of control. To this day, his book on functional analysis is widely read and used by the mathematical and engineering community all around the world.

His work has defined new research directions in communications and control through his original contributions in the field of filtering theory and stochastic optimization. Bal has also been an ardent proponent of integrating mathematical analysis of continuum models to better understand and control systems along with numerical tools, and at times he has decried the trend of excessive reliance on computational methods which offer no deep mathematical insights. His pioneering work in the field of aeroelasticity has offered a fresh new perspective on the study of wing structure stability and control. Moreover, his original work on the Possio integral equation which arises in the study of potential flow in the vicinity of an airfoil, has been instrumental in that regard. As in the potential flow problem that arises in the study of shallow water equations, the potential flow problem around an airfoil is driven by the moving interface which is characterized by "non-usual boundary conditions". Since the prime interest is the stability of the structure, the problem can be reduced to a nonlinear Possio integral equation at the interface relating the pressure jump to the structure velocity "downwash". Although the problem of existence of solution to the nonlinear Possio equation for a general Mach number is still unsettled, Bal has made important advances in the study of this equation. In the last few years, Bal has collected his vast amount of work in this field in a book entitled "Aeroelasticity: The Continuum Theory". The large body of work he has produced over the years will continue to be prized and treasured by the scientific community for many years to come.

The paper is organized as follows. In Section 2, we state our main result of the local existence. Several technical lemmas about the estimates of the coordinates system and the Sobolev norms of a composite of two functions are provided in Section 3. We include our proof of the main results in Section 4 which consists of the div-curl estimates, the pressure estimates, and the tangential estimates.

2. The main result

Consider the Euler equation in the Lagrangian formulation, set in the domain

$$\Omega = \mathbb{R}^1 \times (0,1) \subseteq \mathbb{R}^2 \tag{2.1}$$

with periodic boundary condition in x_1 with period 1. The top

$$\Gamma_1 = \mathbb{R} \times \{x_n = 1\} \tag{2.2}$$

represents the free boundary, while the rigid bottom is given by

$$\Gamma_0 = \mathbb{R} \times \{x_n = 0\}. \tag{2.3}$$

We denote the ϵ-neighborhood of the top boundary by

$$U_\epsilon = \{x \in \Omega : \mathrm{dist}(x, \Gamma_1) \leq \epsilon\}, \tag{2.4}$$

where $\epsilon > 0$. Fix a small constant $\epsilon_0 > 0$ and introduce a smooth cut-off function $\phi = \phi(x_2)$ such that

$$\phi(x_2) = 1, \qquad x_2 \leq 1 - 6\epsilon_0 \tag{2.5}$$

$$\phi(x_2) = 0, \qquad x_2 \geq 1 - 5\epsilon_0 \tag{2.6}$$

with $0 \leq \phi(x_2) \leq 1$ for $1 - 6\epsilon_0 \leq x_2 \leq 1 - 5\epsilon_0$. We use $\widetilde{\eta}$ to denote the standard Lagrangian coordinate of the system, the equations for which read

$$\widetilde{\eta}_t = u(\widetilde{\eta}) \tag{2.7}$$

$$\widetilde{\eta}(0,x) = x, \qquad x \in \Omega, \tag{2.8}$$

and η to denote the coordinate system satisfying

$$\eta_t = (1 - \phi(\eta))u(\eta) \tag{2.9}$$

$$\eta(0,x) = x, \qquad x \in \Omega. \tag{2.10}$$

Let $\widetilde{v}(x,t) = (\widetilde{v}^1, \widetilde{v}^2)$ and $v(x,t) = (v^1, v^2)$ represent the velocities in the Lagrangian and η coordinates respectively, while $\widetilde{q}(x,t)$ and $q(x,t)$ stand for the pressures respectively. Denote by a the inverse matrix of $\nabla\eta$,

$$a = (\nabla\eta)^{-1} \tag{2.11}$$

or in coordinates

$$a_j^k \partial_k \eta^i = \delta_{ij}, \qquad i,j = 1,2. \tag{2.12}$$

Note that the summation convention on repeated indices is used throughout. The equations (1.1)–(1.2) in the η coordinates read

$$v_t^i + \psi a_j^k v^j \partial_k v^i + a_i^k \partial_k q = 0 \text{ in } \Omega \times (0,T), \qquad i = 1,2 \tag{2.13}$$

$$a_i^k \partial_k v^i = 0 \text{ in } \Omega \times (0,T) \tag{2.14}$$

where

$$\psi(x,t) = \phi(\eta(x,t)) \tag{2.15}$$

with the initial condition

$$v(0) = v_0. \tag{2.16}$$

The matrix a evolves according to

$$\begin{aligned} &a_t = -a : \nabla\eta_t : a \\ &a(x,0) = I, \qquad x \in \Omega \end{aligned} \tag{2.17}$$

where the symbol : denotes the matrix multiplication. This follows from $a : \nabla\eta = I$ by time differentiation.

On the top, which represents the free boundary, we impose

$$q = 0 \text{ on } \Gamma_1 \times (0,T) \tag{2.18}$$

while on the bottom boundary we assume

$$v^i N^i = 0 \text{ on } \Gamma_0 \times (0,T) \tag{2.19}$$

where $N = (N^1, N^2)$ stands for the outward unit normal. Since our domain (2.1) is assumed flat, for simplicity, we have $N = (0,-1)$ on Γ_0 and $N = (0,1)$ on Γ_1.

The following is our main result.

THEOREM 2.1. *Let $\delta > 0$. Assume that $v(\cdot,0) = v_0 \in H^{2+\delta}(\Omega)$ is divergence-free with $v_0 \cdot N = 0$ on Γ_0 and*

$$\operatorname{curl} v_0 \in H^{1.5+\delta}(U_{10\epsilon_0}) \tag{2.20}$$

for some $\epsilon_0 > 0$. Assume that the initial pressure $q(\cdot,0)$ satisfies the Rayleigh-Taylor condition

$$\frac{\partial q}{\partial N}(x,0) \le -\frac{1}{C_0} < 0, \qquad x \in \Gamma_1 \tag{2.21}$$

where $C_0 > 0$ is a constant. Then there exists a solution (v,q,a,η) to the free boundary Euler system with the initial condition $v(0) = v_0$ with

$$\begin{aligned} &v \in L^\infty([0,T]; H^{2+\delta}(\Omega)) \cap C([0,T]; H^{1+\delta}(\Omega)) \\ &v_t \in L^\infty([0,T]; H^{1+\delta}(\Omega)) \\ &\eta \in L^\infty([0,T]; H^{2.5+\delta}(\Omega)) \cap C([0,T]; H^{2+\delta}(\Omega)) \\ &a \in L^\infty([0,T]; H^{1.5+\delta}(\Omega)) \cap C([0,T]; H^{1+\delta}(\Omega)) \\ &q \in L^\infty([0,T]; H^{2.5+\delta}(\Omega)) \\ &q_t \in L^\infty([0,T]; H^{1+\delta}(\Omega)) \end{aligned} \tag{2.22}$$

on $[0,T]$, where $T > 0$ depends on the initial data.

3. Preliminary lemmas

Before proving the main result, we state several lemmas needed in the proof. Denote $a \vee b = \max\{a,b\}$ and $\lceil s \rceil = \min\{m \in \mathbb{N} : m \ge s\}$.

LEMMA 3.1. *Let $F\colon \mathbb{R}^n \to \mathbb{R}$ be a function in $\mathcal{S}(\mathbb{R}^n)$ and let $g\colon \Omega \to \mathbb{R}^n$ be C^∞, where $\Omega \subset \mathbb{R}^n$ is a bounded open set. Assume that $\det(\nabla g) \neq 0$. Then*

$$\|F \circ g\|_{H^s(\Omega)} \lesssim \|\det(\nabla g)^{-1}\|_{L^\infty(\Omega)}^{1/2} (1 + \|\nabla g\|_{H^{(s-1)\vee(1+\delta)}(\Omega)})^{\lceil s \rceil} \|F\|_{H^s(\mathbb{R}^n)} \tag{3.1}$$

holds for any $\delta > 0$.

PROOF. We show the proof for the ranges $0 \leq s \leq 1$ and $1 \leq s \leq 2$; the treatment for larger s is similar. First, assume $0 \leq s \leq 1$. Since

$$\begin{aligned}\|F\circ g\|_{L^2(\Omega)}^2 &= \int_\Omega |F\circ g|^2\,dx = \int_{g(\Omega)} |F(y)|^2 |\det(\nabla g)^{-1}|\,dy \\ &\leq \|F\|_{L^2(\mathbb{R}^n)}^2 \|\det(\nabla g)^{-1}\|_{L^\infty(\Omega)} \end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\|F\circ g\|_{H^1(\Omega)}^2 &\lesssim \|F\circ g\|_{L^2(\Omega)}^2 + \int_\Omega |(\nabla F)\circ g|^2 |\nabla g|^2\,dx \\ &\lesssim \|F\|_{L^2(\mathbb{R}^n)}^2 \|\det(\nabla g)^{-1}\|_{L^\infty(\Omega)} + \|\nabla g\|_{L^\infty(\Omega)}^2 \|\nabla F\|_{L^2(\mathbb{R}^n)}^2 \|\det(\nabla g)^{-1}\|_{L^\infty(\Omega)} \\ &\lesssim (1+\|\nabla g\|_{L^\infty(\Omega)}^2)\|F\|_{H^1(\mathbb{R}^n)}^2 \|\det(\nabla g)^{-1}\|_{L^\infty(\Omega)} \\ &\lesssim (1+\|\nabla g\|_{H^{1+\delta}(\Omega)}^2)\|F\|_{H^1(\mathbb{R}^n)}^2 \|\det(\nabla g)^{-1}\|_{L^\infty(\Omega)} \end{aligned} \tag{3.3}$$

hold, we get by complex interpolation for $0 \leq s \leq 1$

$$\|F\circ g\|_{H^s(\Omega)} \lesssim (1+\|\nabla g\|_{H^{1+\delta}(\Omega)}^2)^{1/2} \|\det(\nabla g)^{-1}\|_{L^\infty(\Omega)}^{1/2} \|F\|_{H^1(\mathbb{R}^n)}. \tag{3.4}$$

Next we consider the case $1 \leq s \leq 2$ (the case $s > 2$ is treated similarly). Using the above computation we have

$$\begin{aligned}\|F\circ g\|_{H^s}^2 &\lesssim \|F\circ g\|_{H^1}^2 + \|\nabla(F\circ g)\|_{H^{s-1}}^2 \\ &\lesssim \|F\circ g\|_{H^1}^2 + \|(\nabla F)\circ g\|_{H^{s-1}}^2 \|\nabla g\|_{H^{(s-1)\vee(1+\delta)}}^2 \\ &\lesssim \|\det(\nabla g)^{-1}\|_{L^\infty}(1+\|\nabla g\|_{H^{1+\delta}}^2)(1+\|\nabla g\|_{H^{(s-1)\vee(1+\delta)}}^2)\|F\|_{H^s}^2, \end{aligned} \tag{3.5}$$

where we used that H^r is an algebra for $r > 1$ and

$$\|fg\|_{H^r} \leq \|f\|_{H^r}\|g\|_{H^{1+\delta}}. \tag{3.6}$$

for $r \in [0,1]$. $\square$

In the next lemma, we state a priori estimates for the coefficient matrix a and for the particle map η.

LEMMA 3.2. *Assume that $\|\nabla v\|_{L^\infty([0,T];H^{1+\delta}(\Omega))} \leq M$. For $0 < T < 1/CM$, where C is a sufficiently large constant, the following statements hold:*
(i) $\|\nabla\eta(\cdot,t)\|_{H^{1+\delta}(\Omega)} \lesssim 1$ for $t \in [0,T]$,
(ii) $\|a(\cdot,t)\|_{H^{1+\delta}(\Omega)} \lesssim 1$ (and thus also $\|a(\cdot,t)\|_{L^\infty(\Omega)} \lesssim 1$) for $t \in [0,T]$,
(iii) $\|a_t(\cdot,t)\|_{L^p(\Omega)} \lesssim \|\nabla v(\cdot,t)\|_{L^p(\Omega)}$ for $p \in [1,\infty]$ and $t \in [0,T]$,
(iv) $\|a_t(\cdot,t)\|_{H^r(\Omega)} \lesssim \|\nabla v(\cdot,t)\|_{H^r(\Omega)}$ for $r \in [0,1+\delta)$ and $t \in [0,T]$,
(v) $\|a_{tt}(\cdot,t)\|_{H^\sigma(\Omega)} \lesssim \|\nabla v(\cdot,t)\|_{H^{1+\delta}(\Omega)}\|\nabla v(\cdot,t)\|_{H^\sigma(\Omega)} + \|\nabla v_t(\cdot,t)\|_{H^\sigma(\Omega)}$, for all $t \in [0,T]$ and all $\sigma \in [0,\delta]$,
(vi) For every $\epsilon \in (0,1]$ and all $t \in [0,T']$, where $T' \leq T$ is sufficiently small, we have

$$\|Ra_l^j - \delta_{jl}\|_{H^{1+\delta}(\Omega)} \leq \epsilon \tag{3.7}$$

for $j,l = 1,2$ and

$$\|Ra_l^j a_l^k - \delta_{jk}\|_{H^{1+\delta}(\Omega)} \leq \epsilon \tag{3.8}$$

for $j,k=1,2$, *where* $R=1$ *or* $R=J$.
(vii) $\|J\|_{H^{1+\delta}(\Omega)} \lesssim 1$ *for* $t\in[0,T]$,
(viii) $1/2 \le J = \det(\nabla\eta(x,t)) \le 2$ *for* $(x,t)\in\Omega\times[0,T]$,
(ix) $\|\psi\|_{H^{2+\delta}(\Omega)}, \|\psi_t\|_{H^{2+\delta}(\Omega)} \lesssim 1$ *for* $t\in[0,T]$, *where* $\psi=\phi(\eta)$,
(x) $\|J_t\|_{H^{1+\delta}(\Omega)} \lesssim \|v\|_{H^{1+\delta}(\Omega)}$ *and* $\|J_{tt}\|_{H^{\delta}(\Omega)} \lesssim \|v\|_{H^{1+\delta}(\Omega)}(1+\|v\|_{H^{1+\delta}(\Omega)}) + \|v_t\|_{H^{\delta}(\Omega)}$ *for* $t\in[0,T]$,
(xi) $\|\widetilde{\eta}\|_{H^{2+\delta}(\Omega)} \lesssim 1$ *for* $t\in[0,T]$,
(xii) $\|\eta(\cdot,t)-\eta(\cdot,0)\|_{L^\infty(\Omega)} \le \epsilon_0$ *for* $t\in[0,T]$.

PROOF. For the proof of (i)–(vii), cf. [**KT2**] and [**KT3**], up to small adjustments.
(viii) From [**BG**] (cf. also [**KT3**]), we know that

$$J_t = Ja_i^k\partial_k\eta_t = Ja_i^k\partial_k((1-\psi)v^i) = -Ja_i^k\partial_k\psi v^i, \tag{3.9}$$

which implies

$$0 \le J \le \exp(T\|a\|_{L^\infty}\|\nabla\psi\|_{L^\infty}\|v\|_{L^\infty}) \le \exp(CTM\|\nabla\psi\|_{L^\infty}) \le \exp(CTM) \tag{3.10}$$

where we used (i). For $T\le 1/CM$ with sufficiently large C, we obtain $J\le 2$. Integrating (3.9) directly gives

$$|J-1| \le 2T\|a\|_{L^\infty}\|\nabla\psi\|_{L^\infty}\|v\|_{L^\infty} \lesssim TM\|\nabla\psi\|_{L^\infty} \lesssim TM \tag{3.11}$$

providing the claimed lower bound for J.
(ix) By Lemma 3.1, we have

$$\|\psi\|_{H^{2+\delta}} \lesssim \|\det(\nabla\eta)^{-1}\|_{L^\infty(\Omega)}^{1/2}(1+\|\nabla\eta\|_{H^{1+\delta}})^3\|\phi\|_{H^{2+\delta}(\mathbb{T}\times(0,1))} \lesssim 1. \tag{3.12}$$

Taking the time derivative of ψ, we arrive at

$$\partial_t\psi = (\nabla\phi)(\eta)\eta_t. \tag{3.13}$$

Since $\phi\in C^\infty$ and $\eta_t\in H^{2+\delta}$, we get $\|\psi_t\|_{H^{2+\delta}(\Omega)}\lesssim 1$.
(x) The first inequality is a consequence of (3.9). Differentiating (3.9), we get

$$J_{tt} = -(J_ta_i^k\partial_k\psi v^i + J\partial_ta_i^k\partial_k\psi v^i + Ja_i^k\partial_k\psi_t v^i + Ja_i^k\partial_k\psi v_t^i). \tag{3.14}$$

From the above identity we obtain

$$\|J_{tt}\|_{H^\delta} \lesssim \|v\|_{H^{1+\delta}(\Omega)}^2 + \|v\|_{H^{1+\delta}(\Omega)}^2 + \|v\|_{H^{1+\delta}(\Omega)} + \|v_t\|_{H^{\delta}(\Omega)} \tag{3.15}$$

providing the second inequality.
(xi) From the equation

$$\widetilde{\eta}_t = u(\widetilde{\eta}) \tag{3.16}$$

we obtain by Lemma 3.1

$$\begin{aligned}\|\widetilde{\eta}\|_{H^{2+\delta}(\Omega)} &\lesssim 1 + \int_0^t \|u(\widetilde{\eta})\|_{H^{2+\delta}(\Omega)}\, ds \\ &\lesssim 1 + \int_0^t \|u\|_{H^{2+\delta}(\Omega(t))} \|\widetilde{\eta}\|^3_{H^{2+\delta}(\Omega)}\, ds \\ &\lesssim 1 + \int_0^t \|v\|_{H^{2+\delta}(\Omega)} \|\eta^{-1}\|^3_{H^{2+\delta}(\Omega)} \|\widetilde{\eta}\|^3_{H^{2+\delta}(\Omega)}\, ds. \end{aligned} \tag{3.17}$$

Applying Gronwall inequality gives $\|\widetilde{\eta}\|_{H^{2+\delta}(\Omega)} \lesssim 1$ provided T is small enough.
(xii) Using the fundamental theorem of calculus, we obtain

$$\|\eta(x,t) - \eta(x,0)\|_{L^\infty} \leq \int_0^t \|\eta_t(x,s)\|_{L^\infty}\, ds \leq \int_0^t \|v\|_{L^\infty}\, ds \leq Mt. \tag{3.18}$$

Choosing $t \leq 1/CM$, where C is sufficiently large, we have proved (xi). □

We also need the following result from [**BM**] addressing the Sobolev norm of the composite of two functions.

LEMMA 3.3. *Given $1 \leq s < \infty$ and $1 < p < \infty$. Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and $f \in C^m$ be such that $f, f', \cdots, f^{(m)} \in L^\infty$ where $m = \lceil s \rceil$. Let*

$$M_f(u) = f \circ u. \tag{3.19}$$

Then M_f is a continuous map from $W^{s,p}(\Omega) \cap W^{1,sp}(\Omega)$ to $W^{s,p}(\Omega)$.

4. Proof of the main result

The proof of the main theorem is divided into three subsections: div-curl estimates, pressure estimate, and tangential estimates.

4.1. Div-curl estimates for η, v, and a.

As in [**KT2**], we introduce the variable curl operator

$$B_a f = a_1^k \partial_k f^2 - a_2^k \partial_k f^1 \tag{4.1}$$

and the variable divergence operator

$$A_a f = a_i^k \partial_k f^i \tag{4.2}$$

where $k, i \in \{1, 2\}$ and f is a vector valued smooth function in $\mathbb{R}^2$. Observe that if $a = I$, then B_I and A_I agree with the usual curl and divergence operators. Define the tangential derivative operator

$$S = (I - \partial_1^2)^{(2+\delta)/2}. \tag{4.3}$$

For technical reasons, we need another cut-off function $\bar{\xi} \colon \Omega \to \mathbb{R}$ such that

$$\begin{aligned} &\bar{\xi} = 1, \qquad x_2 \geq 1 - 2\epsilon_0 \\ &0 \leq \bar{\xi} \leq 1, \qquad 1 - 3\epsilon_0 \leq x_2 \leq 1 - 2\epsilon_0 \\ &\bar{\xi} = 0, \qquad x_2 \leq 1 - 3\epsilon_0. \end{aligned} \tag{4.4}$$

Let $\xi = \bar{\xi}(\eta)$.

Denote by P a generic polynomials in the variables $\|a_t\|_{H^{1+\delta}}$, $\|\eta\|_{H^{2.5+\delta}}$, $\|v\|_{H^{2+\delta}}$, and $\|a\|_{H^{1.5+\delta}}$.

LEMMA 4.1. *Assume that (v,q,a,η) satisfies the Euler equation (2.9)–(2.17) in $\Omega\times[0,T)$ and that we have $\|\nabla v\|_{L^\infty([0,T];H^{1+\delta}(\Omega))}\le M$. Suppose that a satisfies the estimates in Lemma 3.2 for a sufficiently small constant $\epsilon>0$. Then we have*

$$\|\eta\|_{H^{2.5+\delta}}\lesssim\|\eta\|_{L^2}+\int_0^t P\,ds+\|S\eta^2\|_{L^2(\Gamma_1)}+\|\omega_0\|_{H^{1.5+\delta}} \tag{4.5}$$

with

$$\|a\|_{H^{1.5+\delta}}\lesssim\|\eta\|^4_{H^{2.5+\delta}} \tag{4.6}$$

and

$$\|v\|_{H^{2+\delta}}\lesssim\|v\|_{L^2}+\|S(\xi v)\|_{L^2}+\|\omega_0\|_{H^{1+\delta}}, \tag{4.7}$$

for $t\in[0,T]$.

PROOF. We start with the estimate for $\operatorname{curl}\eta$. First, we have

$$\operatorname{curl}\eta=0,\qquad x\in\{x_2\le 1-6\epsilon_0\} \tag{4.8}$$

since (2.9) implies that

$$\eta(x,t)=x,\qquad (x,t)\in\{x_2\le 1-6\epsilon_0\}\times[0,T]. \tag{4.9}$$

Therefore, we have

$$\begin{aligned}\|\operatorname{curl}\eta\|_{H^{1.5+\delta}(\Omega)}&\lesssim\|\operatorname{curl}\eta\|_{H^{1.5+\delta}(U_{7\epsilon_0})}\lesssim\|B_I\nabla\eta\|_{H^{0.5+\delta}(U_{7\epsilon_0})}+\|\operatorname{curl}\eta\|_{H^{0.5+\delta}(U_{7\epsilon_0})}\\&\lesssim\|B_I\nabla\eta-B_a\nabla\eta\|_{H^{0.5+\delta}}+\|B_a\nabla\eta\|_{H^{0.5+\delta}}+\|\operatorname{curl}\eta\|_{H^{0.5+\delta}}\\&\lesssim\|I-a\|_{H^{1+\delta}}\|\nabla\eta\|_{H^{1.5+\delta}}+\int_0^t\|B_{a_t}\nabla\eta\|_{H^{0.5+\delta}}\,ds\\&\quad+\int_0^t\|B_a\nabla\eta_t\|_{H^{0.5+\delta}}\,ds+\|\eta\|_{H^{1.5+\delta}}.\end{aligned} \tag{4.10}$$

Since

$$\|B_{a_t}\nabla\eta\|_{H^{0.5+\delta}}\lesssim\|a_t\|_{H^{1+\delta}}\|\nabla\eta\|_{H^{1.5+\delta}}, \tag{4.11}$$

we deduce

$$\int_0^t\|B_{a_t}\nabla\eta\|_{H^{0.5+\delta}}\,ds\lesssim\int_0^t\|a_t\|_{H^{1+\delta}}\|\nabla\eta\|_{H^{1.5+\delta}}\,ds. \tag{4.12}$$

Next we rewrite

$$B_a\nabla\eta_t=B_a\nabla((1-\psi)v)=B_a((1-\psi)\nabla v)+B_a(\nabla(1-\psi)\otimes v), \tag{4.13}$$

from where we get

$$\begin{aligned}&\int_0^t\|B_a\nabla\eta_t\|_{H^{0.5+\delta}}\,ds\\&\quad\lesssim\int_0^t\|1-\psi\|_{H^{1+\delta}}\|B_a\nabla v\|_{H^{0.5+\delta}}+\|a\|_{H^{1+\delta}}\|\nabla\psi\|_{H^{1+\delta}}\|\nabla v\|_{H^{0.5+\delta}}\\&\qquad+\|1-\psi\|_{H^{2+\delta}}\|B_a v\|_{H^{0.5+\delta}}+\|a\|_{H^{1+\delta}}\|D^2\psi\|_{H^{0.5+\delta}}\|v\|_{H^{1.5+\delta}}\,ds.\end{aligned} \tag{4.14}$$

By the identity

$$B_a\nabla v=\nabla(B_a v)-B_{\nabla a}v, \tag{4.15}$$

we further obtain

$$\begin{aligned}\|B_a\nabla v\|_{H^{0.5+\delta}} &\lesssim \|\nabla(B_a v)\|_{H^{0.5+\delta}} + \|B_{\nabla a} v\|_{H^{0.5+\delta}}\\ &\lesssim \|B_a v\|_{H^{1.5+\delta}} + \|\nabla a\|_{H^{0.5+\delta}}\|\nabla v\|_{H^{1+\delta}}\\ &\lesssim \|\omega(\eta)\|_{H^{1.5+\delta}} + \|a\|_{H^{1.5+\delta}}\|v\|_{H^{2+\delta}},\end{aligned} \tag{4.16}$$

where $\omega = \operatorname{curl} u$. By Lemma 3.2, we have

$$\|\eta(x,t) - x\|_{L^\infty} \leq \epsilon_0 \tag{4.17}$$

for small T. From Lemma 3.1 and 3.2, one gets

$$\begin{aligned}\|\omega(\eta)\|_{H^{1.5+\delta}} &\lesssim \|\det(\nabla\eta)^{-1}\|_{L^\infty}^{1/2}(1 + \|\nabla\eta\|_{H^{1+\delta}})^2\|\omega\|_{H^{1.5+\delta}(\eta(U_{7\epsilon_0}))}\\ &\lesssim \|\omega\|_{H^{1.5+\delta}(\eta(U_{7\epsilon_0}))}.\end{aligned} \tag{4.18}$$

Noting $\eta(U_{7\epsilon_0}) \subset \{x : x_2 \geq 1 - 8\epsilon_0\}$ with $\widetilde{\eta}^{-1}\eta(U_{7\epsilon_0}) \subset U_{10\epsilon_0}$ and using Lemma 3.1 again, we arrive at

$$\begin{aligned}\|\omega(\eta)\|_{H^{1.5+\delta}} &\lesssim \|\det(\nabla\widetilde{\eta})\|_{L^\infty}^{1/2}(1 + \|\nabla\widetilde{\eta}^{-1}\|_{H^{1+\delta}})^2\|\omega_0\|_{H^{1.5+\delta}(\widetilde{\eta}^{-1}\eta(U_{7\epsilon_0}))}\\ &\lesssim \|\omega_0\|_{H^{1.5+\delta}(U_{10\epsilon_0})},\end{aligned} \tag{4.19}$$

where we also used that $\|\widetilde{\eta}\|_{H^{2+\delta}} \lesssim 1$. In summary, we get

$$\|\operatorname{curl}\eta\|_{H^{1.5+\delta}} \lesssim \epsilon\|\eta\|_{H^{2.5+\delta}} + \|\eta\|_{H^{1.5+\delta}} + \int_0^t P\,ds + \|\omega_0\|_{H^{1.5+\delta}(U_{10\epsilon_0})}. \tag{4.20}$$

Next we will use similar method to get an estimate for $\operatorname{div}\eta$.

$$\begin{aligned}\|\operatorname{div}\eta\|_{H^{1.5+\delta}} &\lesssim \|(A_I - A_a)\nabla\eta\|_{H^{0.5+\delta}} + \|A_a\nabla\eta\|_{H^{0.5+\delta}} + \|\operatorname{div}\eta\|_{H^{0.5+\delta}}\\ &\lesssim \|I - a\|_{H^{1+\delta}}\|\eta\|_{H^{2.5+\delta}} + \|\operatorname{div}\eta\|_{H^{0.5+\delta}}\\ &\quad + \int_0^t \|A_{a_t}\nabla\eta\|_{H^{0.5+\delta}} + \|A_a\nabla\eta_t\|_{H^{0.5+\delta}}\,ds.\end{aligned} \tag{4.21}$$

Note that

$$A_a\nabla\eta_t = A_a\nabla((1-\psi)v) = A_a(\nabla(1-\psi)\otimes v) + A_a((1-\psi)\nabla v). \tag{4.22}$$

We use product rule in order to get

$$\begin{aligned}\|A_a\nabla\eta_t\|_{H^{0.5+\delta}} &\lesssim \|a\|_{H^{1+\delta}}\|\nabla\psi\|_{H^{1+\delta}}\|\nabla v\|_{H^{0.5+\delta}} + \|1-\psi\|_{H^{2+\delta}}\|A_a v\|_{H^{0.5+\delta}}\\ &\quad + \|a\|_{H^{1+\delta}}\|D^2\psi\|_{H^{0.5+\delta}}\|v\|_{H^{1.5+\delta}},\end{aligned} \tag{4.23}$$

where we use that fact that $A_a\nabla v = 0$ because of the divergence free condition $\operatorname{div} u = 0$. Substituting the above estimate into (4.21) gives

$$\|\operatorname{div}\eta\|_{H^{1.5+\delta}} \lesssim \epsilon\|\eta\|_{H^{2.5+\delta}} + \|\operatorname{div}\eta\|_{H^{0.5+\delta}} + \int_0^t P\,ds. \tag{4.24}$$

Resorting to the inequality

$$\|f\|_{H^s(\Omega)} \lesssim \|f\|_{L^2(\Omega)} + \|\operatorname{curl} f\|_{H^{s-1}(\Omega)} + \|\operatorname{div} f\|_{H^{s-1}(\Omega)} + \|\partial_1 f\cdot N\|_{H^{s-1.5}(\partial\Omega)} \tag{4.25}$$

where f is a vector valued function such that $f \in H^s(\Omega)$ for $s > 1.5$ and N is the outer unite normal vector (cf. [**CS1, CS2**]), we obtain

$$\|\eta\|_{H^{2.5+\delta}(\Omega)} \lesssim \|\eta\|_{L^2(\Omega)} + \|\operatorname{curl} \eta\|_{H^{1.5+\delta}(\Omega)} + \|\operatorname{div} \eta\|_{H^{1.5+\delta}(\Omega)} + \|S\eta^2\|_{L^2(\Gamma_1)}. \tag{4.26}$$

Now we replace $\operatorname{curl} \eta$ and $\operatorname{div} \eta$ by (4.20) and (4.24) in (4.26), absorbing the ϵ-term to the left side, to get

$$\|\eta\|_{H^{2.5+\delta}(\Omega)} \lesssim \|\eta\|_{L^2} + \int_0^t P\, ds + \|\eta\|_{H^{1.5+\delta}} + \|S\eta^2\|_{L^2(\Gamma_1)}. \tag{4.27}$$

We note that by the Gagliardo-Nirenberg inequality and Young's inequality

$$\|\eta\|_{H^{1.5+\delta}} \lesssim \|\eta\|_{L^2}^{1/(2.5+\delta)} \|\eta\|_{H^{2.5+\delta}}^{(1.5+\delta)/(2.5+\delta)} \lesssim \|\eta\|_{L^2} + \epsilon \|\eta\|_{H^{2.5+\delta}}, \tag{4.28}$$

which proves (4.5) by absorbing the ϵ term to the left side. Similarly, we estimate $\|v\|_{H^{2+\delta}}$ as

$$\begin{aligned}\|v\|_{H^{2+\delta}(\Omega)} &\lesssim \|v\|_{L^2(\Omega)} + \|\operatorname{curl} v\|_{H^{1+\delta}(\Omega)} + \|\operatorname{div} v\|_{H^{1+\delta}(\Omega)} + \|\partial_1 v \cdot N\|_{H^{0.5+\delta}(\Gamma_1)} \\ &\lesssim \|v\|_{L^2} + \|(B_I - B_a)v\|_{H^{1+\delta}} + \|B_a v\|_{H^{1+\delta}} + \|(A_I - A_a)v\|_{H^{1+\delta}} \\ &\quad + \|A_a v\|_{H^{1+\delta}} + \|\partial_1 v^2\|_{H^{0.5+\delta}(\Gamma_1)}.\end{aligned} \tag{4.29}$$

By the trace theorems, we are allowed to estimate

$$\begin{aligned}\|\partial_1 v^2\|_{H^{0.5+\delta}(\Gamma_1)} &= \|\partial_1(\xi v^2)\|_{H^{0.5+\delta}(\Gamma_1)} \lesssim \|\partial_1(\xi v^2)\|_{H^{1+\delta}} \\ &\lesssim \|\partial_1(\xi v^2)\|_{L^2} + \|\partial_1 \nabla(\xi v^2)\|_{H^\delta} \lesssim \|S(\xi v^2)\|_{L^2} + \|\partial_1\partial_2(\xi v^2)\|_{H^\delta}.\end{aligned} \tag{4.30}$$

A direct computation shows that

$$\begin{aligned}\partial_1\partial_2(\xi v^2) &= \partial_1(\partial_2\xi v^2 + \xi\partial_2 v^2) = \partial_1(\partial_2\xi v^2) + \partial_1(\xi(\operatorname{div} v - \partial_1 v^1)) \\ &= \partial_1(\partial_2\xi v^2) + \partial_1(\xi(A_{I-a} v - \partial_1 v^1))\end{aligned} \tag{4.31}$$

where we used that $A_a v = 0$. Therefore, we obtain by Lemma 3.2

$$\begin{aligned}\|\partial_1\partial_2(\xi v^2)\|_{H^\delta} &\lesssim \|v\|_{H^{1+\delta}} + \|A_{I-a} v\|_{H^{1+\delta}} + \|\xi\partial_1^2 v^1\|_{H^\delta} \\ &\lesssim \|v\|_{H^{1+\delta}} + \epsilon\|v\|_{H^{2+\delta}} + \|\xi\partial_1^2 v^1\|_{H^\delta}.\end{aligned} \tag{4.32}$$

By the identity

$$\xi\partial_1^2 v^1 = \partial_1^2(\xi v^1) - (\partial_1^2\xi v^1 + 2\partial_1\xi\partial_1 v^1) \tag{4.33}$$

it follows

$$\|\xi\partial_1^2 v^1\|_{H^\delta} \lesssim \|S(\xi v)\|_{L^2} + \|v\|_{H^{1+\delta}}. \tag{4.34}$$

From (4.30) we further deduce that

$$\|\partial_1 v^2\|_{H^{0.5+\delta}(\Gamma_1)} \lesssim \|S(\xi v)\|_{L^2} + \|v\|_{H^{1+\delta}} + \epsilon\|v\|_{H^{2+\delta}}. \tag{4.35}$$

On the other hand, by the Gagliardo-Nirenberg inequality and Young's inequality, we have

$$\|v\|_{H^{1+\delta}} \lesssim \|v\|_{L^2}^{1/(2+\delta)} \|v\|_{H^{2+\delta}}^{(1+\delta)/(2+\delta)} \lesssim \|v\|_{L^2} + \epsilon\|v\|_{H^{2+\delta}}. \tag{4.36}$$

Therefore, by absorbing the ϵ term to the left side, we obtain from (4.29) that

$$\|v\|_{H^{2+\delta}} \lesssim \|v\|_{L^2} + \|S(\xi v)\|_{L^2} + \|\omega_0\|_{H^{1+\delta}}, \tag{4.37}$$

which proves (4.7). To finish the proof, a is the only term left to treat. Noting that

$$a = (\nabla\eta)^{-1} = \frac{1}{J}\mathrm{Cof}(\nabla\eta) \tag{4.38}$$

we get

$$\|a\|_{H^{1.5+\delta}} \lesssim \left\|\frac{1}{J}\right\|_{H^{1.5+\delta}} \|\eta\|_{H^{2.5+\delta}}^2 \lesssim \|J\|_{H^{1.5+\delta}} \|\eta\|_{H^{2.5+\delta}}^2 \lesssim \|\eta\|_{H^{2.5+\delta}}^4 \tag{4.39}$$

where we used Lemma 3.3. In fact, since $1/2 \leq J \leq 2$, we have

$$\frac{1}{J} = \frac{\rho(J)}{J} \tag{4.40}$$

holds for a smooth cut-off function $\rho\colon \mathbb{R} \to [0,1]$ such that

$$\begin{aligned} &\rho(\tau) = 0, && \tau \leq 1/4 \text{ or } \tau \geq 4 \\ &\rho(\tau) = 1, && \tau \in [1/2, 2] \\ &0 \leq \rho(\tau) \leq 1, && \text{otherwise.} \end{aligned} \tag{4.41}$$

Therefore, by Lemma 3.3, $\|1/J\|_{H^{1.5+\delta}} = \|\rho(J)/J\|_{H^{1.5+\delta}} \lesssim \|J\|_{H^{1.5+\delta}}$ holds, which concludes the proof. □

4.2. Pressure estimates. In the following lemma, we obtain the elliptic estimates for the pressure q.

LEMMA 4.2. *For $t \in [0,T]$, the pressure q obeys*

$$\|q\|_{H^{2.5+\delta}} \leq P + P\int_0^t \|q_t\|_{L^2}\, ds \tag{4.42}$$

and

$$\|q_t\|_{H^{1+\delta}} \leq P + P\int_0^t \|q_t(s)\|_{L^2}\, ds \tag{4.43}$$

with

$$\|q_t\|_{H^{2+\delta}(U_{4\epsilon_0})} \leq P + P\int_0^t \|q_t(s)\|_{L^2}\, ds \tag{4.44}$$

where P is a polynomial in $\|v\|_{H^{2+\delta}}$, $\|v_t\|_{H^{1+\delta}}$, $\|\eta\|_{H^{2.5+\delta}}$, and $\|v_0\|_{H^{2+\delta}}$.

PROOF. Applying $a_i^k\partial_k$ operator to Equation (2.13) gives

$$a_i^k\partial_k v_t^i + a_i^k\partial_k(\psi a_j^k v^j \partial_k v^i) + a_i^k\partial_k(a_i^k\partial_k q) = 0, \tag{4.45}$$

from where we obtain

$$-\partial_{kk}q = -\partial_t a_i^k \partial_k v^i + a_i^k\partial_k(\psi a_j^l \partial_l v^i v^j) + (a_i^k a_i^l - \delta_{kl})\partial_{kl}q + a_i^k\partial_k a_i^l \partial_l q. \tag{4.46}$$

By the boundary conditions (2.18)–(2.19), we get that q satisfies

$$q = 0 \text{ on } \Gamma_1 \times (0,T) \tag{4.47}$$

and

$$\partial_i q N^i = 0 \text{ on } \Gamma_0 \times (0,T), \tag{4.48}$$

where we also used that

$$\nabla\eta = I \text{ on } \Gamma_0 \times (0,T). \tag{4.49}$$

By the elliptic regularity estimate, we have

$$\begin{aligned}\|q\|_{H^{2.5+\delta}} &\lesssim \|\partial_t a\|_{H^{1+\delta}}\|v\|_{H^{2+\delta}} + \|a_i^k\partial_k(\psi a_j^l\partial_l v^i v^j)\|_{H^{0.5+\delta}} \\ &\quad + \|I - a : a^T\|_{H^{1+\delta}}\|q\|_{H^{2.5+\delta}} + \|a\|_{H^{1+\delta}}\|a\|_{H^{1.5+\delta}}\|q\|_{H^{2+\delta}}.\end{aligned} \tag{4.50}$$

Noting the identity

$$\begin{aligned}a_i^k\partial_k(\psi a_j^l\partial_l v^i v^j) &= a_i^k\partial_k(\psi a_j^l v^j)\partial_l v^i + a_i^k(\psi a_j^l v^j)\partial_{lk}v^i \\ &= a_i^k\partial_k(\psi a_j^l v^j)\partial_l v^i - \partial_l a_i^k\partial_k v^i(\psi a_j^l v^j)\end{aligned} \tag{4.51}$$

where we used the divergence free condition (2.14), we arrive at

$$\begin{aligned}\|a_i^k\partial_k(\psi a_j^l\partial_l v^i v^j)\|_{H^{0.5+\delta}} &\lesssim \|a\|_{H^{1+\delta}}\|\psi a_j^l v^j\|_{H^{1.5+\delta}}\|v\|_{H^{2+\delta}} \\ &\quad + \|a\|_{H^{1.5+\delta}}\|\psi\|_{H^{1+\delta}}\|a\|_{H^{1+\delta}}\|v\|^2_{H^{2+\delta}} \\ &\lesssim \|a\|_{H^{1.5+\delta}}\|a\|_{H^{1+\delta}}\|\psi\|_{H^{1+\delta}}\|v\|^2_{H^{2+\delta}}.\end{aligned} \tag{4.52}$$

Then from (4.50) and by Lemma 3.2, it follows

$$\begin{aligned}\|q\|_{H^{2.5+\delta}} &\lesssim \|v\|_{H^{2+\delta}} + \|a\|_{H^{1.5+\delta}}\|\psi\|_{H^{1+\delta}}\|v\|^2_{H^{2+\delta}} \\ &\quad + \epsilon\|q\|_{H^{2.5+\delta}} + \|a\|_{H^{1.5+\delta}}\|q\|_{H^{2+\delta}},\end{aligned} \tag{4.53}$$

which implies that

$$\|q\|_{H^{2.5+\delta}} \lesssim \|v\|_{H^{2+\delta}} + \|a\|_{H^{1.5+\delta}}\|v\|^2_{H^{2+\delta}} + \|a\|_{H^{1.5+\delta}}\|q\|_{L^2}^{0.5/(2.5+\delta)}\|q\|_{H^{2.5+\delta}}^{(2+\delta)/(2.5+\delta)}. \tag{4.54}$$

By Young's inequality, we further obtain

$$\begin{aligned}\|q\|_{H^{2.5+\delta}} &\lesssim \|v\|_{H^{2+\delta}} + \|a\|_{H^{1.5+\delta}}\|v\|^2_{H^{2+\delta}} + \|a\|^{5+2\delta}_{H^{1.5+\delta}}\|q\|_{L^2} \\ &\lesssim P + P\int_0^t \|q_t\|_{L^2}\,ds\end{aligned} \tag{4.55}$$

where P is a polynomial of $\|\eta\|_{H^{2.5+\delta}}$, $\|v\|_{H^{2+\delta}}$, $\|v_0\|_{H^{2+\delta}}$ and $\|v_t\|_{H^{1+\delta}}$. This accomplished the proof of (4.42). Next we achieve the global and interior estimate for q_t. We first get another equation for q as

$$-\partial_{kk}q = -J\partial_t a_i^k\partial_k v^i + Ja_i^k\partial_k(\psi a_j^l\partial_l v^i v^j) + \partial_k(Ja_i^k a_i^l\partial_l q - \partial_k q). \tag{4.56}$$

Taking the t derivative of the above equation we obtain

$$-\partial_{kk}q_t = -\partial_k\partial_t(\partial_t(Ja_i^k)v^i) + \partial_k\partial_t(Ja_i^k\psi a_j^l\partial_l v^i v^j) + \partial_k\partial_t((Ja_i^k a_i^l\partial_l q - \delta_{kl})\partial_l q) \tag{4.57}$$

where we used the divergence free condition (2.14) and the Piola identity

$$\partial_k(Ja_i^k) = 0, \qquad i = 1,2. \tag{4.58}$$

Naturally, the boundary condition reads

$$q_t = 0 \text{ on } \Gamma_1 \times (0,T) \tag{4.59}$$

and

$$\partial_i q_t N^i = 0 \text{ on } \Gamma_0 \times (0,T). \tag{4.60}$$

The elliptic estimate gives

$$\begin{aligned}\|q_t\|_{H^{1+\delta}} &\lesssim \|\partial_t(\partial_t(Ja_i^k)v^i)\|_{H^\delta} + \|\partial_t(Ja_i^k\psi a_j^l\partial_l v^i v^j)\|_{H^\delta}\\ &\quad + \|\partial_t((Ja_i^k a_i^l\partial_l q - \delta_{kl})\partial_l q)\|_{H^\delta}\\ &\lesssim \|\partial_{tt}(Ja_i^k)v^i\|_{H^\delta} + \|\partial_t(Ja_i^k)\partial_t v^i\|_{H^\delta}\\ &\quad + \|\partial_t(Ja_i^k)\psi a_j^l\partial_l v^i v^j\|_{H^\delta} + \|Ja_i^k\partial_t(\psi a_j^l)\partial_l v^i v^j\|_{H^\delta} + \|Ja_i^k\psi a_j^l\partial_t(\partial_l v^i v^j)\|_{H^\delta}\\ &\quad + \|\partial_t((Ja_i^k a_i^l\partial_l q - \delta_{kl}))\partial_l q\|_{H^\delta} + \|(Ja_i^k a_i^l\partial_l q - \delta_{kl})\partial_l q_t\|_{H^\delta}. \end{aligned} \tag{4.61}$$

Note by Lemma 3.2

$$\begin{aligned}\|\partial_{tt}(Ja_i^k)\|_{H^\delta} &\lesssim \|\partial_{tt}Ja_i^k\|_{H^\delta} + \|\partial_t J\partial_t a_i^k\|_{H^\delta} + \|J\partial_{tt}a_i^k\|_{H^\delta}\\ &\lesssim \|v\|_{H^{1+\delta}}(1+\|v\|_{H^{1+\delta}}) + \|v_t\|_{H^\delta}\\ &\quad + \|v\|_{H^{1+\delta}}^2 + \|v\|_{H^{2+\delta}}\|v\|_{H^{1+\delta}} + \|\nabla v_t\|_{H^\delta}\end{aligned} \tag{4.62}$$

and

$$\begin{aligned}\|\partial_t(Ja_i^k)\|_{H^{1\delta}} &\lesssim \|\partial_t J\|_{H^{1+\delta}}\|a\|_{H^{1+\delta}} + \|J\|_{H^{1+\delta}}\|\partial_t a\|_{H^{1+\delta}}\\ &\lesssim \|\nabla v\|_{H^{1+\delta}} + \|v\|_{H^{1+\delta}}.\end{aligned} \tag{4.63}$$

We further get from (4.61)

$$\begin{aligned}\|q_t\|_{H^{1+\delta}} &\lesssim ((1+\|v\|_{H^{2+\delta}})\|v\|_{H^{1+\delta}} + \|v_t\|_{H^{1+\delta}})\,\|v\|_{H^{1+\delta}}\\ &\quad + \|v\|_{H^\delta} + \|v\|_{H^{2+\delta}}^2\|v\|_{H^{1+\delta}} + \|v_t\|_{H^\delta}\|\nabla v\|_{H^{1+\delta}}\\ &\quad + (\|\nabla v\|_{H^{1+\delta}} + \|v\|_{H^{1+\delta}})\|v\|_{H^{2+\delta}}\|v\|_{H^{1+\delta}} + \|\partial_t\psi\|_{H^{1+\delta}}\|\eta\|_{H^{2+\delta}}^2\|v\|_{H^{2+\delta}}^2\\ &\quad + \|\psi\|_{H^{1+\delta}}\|\eta\|_{H^{2+\delta}}^2\|v\|_{H^{2+\delta}}^3 + \|\eta\|_{H^{2+\delta}}^2\|\nabla v_t\|_{H^\delta}\\ &\quad + \|\partial_t(Ja_i^k a_i^l - \delta_{kl})\|_{H^\delta}\|q\|_{H^{1+\delta}} + \epsilon\|q_t\|_{H^{1+\delta}}.\end{aligned} \tag{4.64}$$

From (2.13), it is easy to see by the algebra property of $H^{1+\delta}$ that

$$\|v_t\|_{H^{1+\delta}} \lesssim \|v\|_{H^{2+\delta}} + \|\partial_k q\|_{H^{1+\delta}}. \tag{4.65}$$

Combining the estimate (4.61) and (4.65) and absorbing the ϵ term, we get

$$\|q_t\|_{H^{1+\delta}} \lesssim P + P\int_0^t \|q_t\|_{L^2}\,ds \tag{4.66}$$

where P is a generic polynomial as above and which proves (4.43). Noting that $\psi = 0$ on $U_{4\epsilon_0}\times(0,T)$, we apply an interior elliptic estimate to get

$$\begin{aligned}\|q_t\|_{H^{2+\delta}(U_{4\epsilon_0})} &\lesssim \|\partial_t(J\partial_t a_i^k\partial_k v^i)\|_{H^\delta(U_{4\epsilon_0})} + \|\partial_t(Ja_i^k\psi a_j^l\partial_l v^i v^j)\|_{H^{1+\delta}(U_{4\epsilon_0})}\\ &\quad + \|\partial_t((Ja_i^k a_i^l\partial_l q - \delta_{kl})\partial_l q)\|_{H^{1+\delta}(U_{4\epsilon_0})} + \|q_t\|_{H^{1+\delta}(\Omega)}\\ &\lesssim \left(\|\nabla v\|_{H^{1+\delta}(\Omega)}^2 + \|\nabla v_t\|_{H^\delta(\Omega)}\right)\|v\|_{H^{2+\delta}(\Omega)}\\ &\quad + \|\partial_t(Ja_i^k a_i^l\partial_l q - \delta_{kl})\|_{H^{1+\delta}(U_{4\epsilon_0})}\|q\|_{H^{2+\delta}(U_{2\epsilon_0})}\\ &\quad + \epsilon\|q_t\|_{H^{2+\delta}(U_{4\epsilon_0})} + \|q_t\|_{H^{1+\delta}}.\end{aligned} \tag{4.67}$$

The inequality (4.44) is obtained by absorbing the ϵ term in the above estimate. □

4.3. Tangential estimates and the conclusion. In this section, the tangential estimates are given. To be more specific, we obtain a bound for $\|S(\xi v(t))\|_{L^2}^2$, which already has appeared in the divergence-curl estimates and where S and ξ are as in Subsection 4.1. The result is stated as follows.

LEMMA 4.3. *For $t \in [0,T]$, we have*

$$\begin{aligned}
&\|S(\xi v(t))\|_{L^2}^2 + \|a_l^2(t) S\eta^l(t)\|_{L^2(\Gamma_1)}^2 \\
&\quad \leq \int_0^t P(\|v\|_{H^{2+\delta}}, \|v_t\|_{H^{1+\delta}}, \|q\|_{H^{2.5+\delta}}, \|q_t\|_{H^{1+\delta}}, \|\eta\|_{H^{2.5+\delta}})\, ds + Q(\|v_0\|_{H^{2+\delta}})
\end{aligned} \tag{4.68}$$

where P and Q are polynomials in indicated arguments.

PROOF. We begin with a simple observation that $\psi\xi = 0$ for $t \in [0,T]$ due to the fact $\|\eta(\cdot,t) - \eta(\cdot,0)\|_{L^\infty(\Omega)} \leq \epsilon_0$. In fact we have that $\xi = 0$ on $\{x : x_2 \leq 1 - 4\epsilon_0\}$ and $\psi = 0$ on $\{x : x_2 \geq 1 - 4\epsilon_0\}$. We multiply (2.13) by ξ in order to get

$$(\xi v^i)_t + \xi a_i^k \partial_k q = \xi_t v^i. \tag{4.69}$$

Applying the operator S to the above equation, we get

$$S(\xi v^i)_t + S(\xi a_i^k \partial_k q) = S(\xi_t v^i). \tag{4.70}$$

We multiply this equation by $S(\xi v^i)$, integrating it and summing over $i = 1, 2$, to obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|S(\xi v)\|_{L^2}^2 &= -\int S(\xi a_i^k \partial_k q) S(\xi v^i)\, dx + \int S(\xi_t v^i) S(\xi v^i)\, dx \\
&= -\int S(a_i^k)\xi \partial_k q S(\xi v^i)\, dx - \int a_i^k S(\xi \partial_k q) S(\xi v^i)\, dx \\
&\quad - \int (S(a_i^k \xi \partial_k q) - S(a_i^k)\xi\partial_k q - a_i^k S(\xi \partial_k q)) S(\xi v^i)\, dx \\
&\quad + \int S(\xi_t v^i) S(\xi v^i)\, dx.
\end{aligned} \tag{4.71}$$

For the last term on the right side, we have

$$\int S(\xi_t v^i) S(\xi v^i)\, dx \lesssim \|S(\xi_t v^i)\|_{L^2} \|S(\xi v^i)\|_{L^2} \lesssim \|v\|_{H^{2+\delta}}^2, \tag{4.72}$$

where we used the algebra property of $H^{2+\delta}$ and $\|\xi\|_{H^{2+\delta}} \lesssim 1$ by Lemma 3.1 and Lemma 3.2. The rest three terms are treated similarly as in [**KT2**] except one difference. Namely, as in that paper, we get the boundary term

$$\begin{aligned}
\int_0^t \int_{\Gamma_1} \xi a_l^2 S\eta^l a_i^2 \partial_2 q S(\xi v^i)\, d\sigma\, ds &\lesssim -\int_{\Gamma_1} a_l^2 S\eta^l a_i^2 S\eta^i\, d\sigma \Big|_t - \int_{\Gamma_1} a_l^2 S\eta^l a_i^2 S\eta^i \partial_2 q\, d\sigma \Big|_0 \\
&\quad - \int_0^t \int_{\Gamma_1} \partial_t a_l^2 S\eta^l a_i^2 S\eta^i \partial_2 q\, d\sigma\, ds - \frac{1}{2}\int_0^t \int_{\Gamma_1} a_l^2 S\eta^l a_i^2 S\eta^i \partial_{2t} q\, d\sigma\, ds
\end{aligned} \tag{4.73}$$

where we used the fact $\xi = 1$ on Γ_1 and the Rayleigh-Taylor condition

$$\frac{\partial q}{\partial N} \leq -\frac{1}{C_0} < 0 \quad \text{on } \Gamma_1. \tag{4.74}$$

Since q_t is not in $H^{2+\delta}(\Omega)$, we are not able to carry out the estimate in the same manner as in [**KT2**] for the last term. However, we have $q_t \in H^{2+\delta}(U_{4\epsilon_0})$, which allow us to estimate as

$$\begin{aligned} \int_0^t \int_{\Gamma_1} a_l^2 S\eta^l a_i^2 S\eta^i \partial_{2t} q \, d\sigma \, ds &\lesssim \int_0^t \|a\|^2_{L^\infty(\Gamma_1)} \|\eta\|^2_{H^{2+\delta}(\Gamma_1)} \|\partial_{2t} q\|_{L^\infty(\Gamma_1)} \, ds \\ &\lesssim \int_0^t \|\eta\|^2_{H^{2.5+\delta}(\Omega)} \|\partial_t q\|_{H^{2+\delta}(U_{4\epsilon_0})} \, ds \lesssim \int_0^t P \, ds \end{aligned} \tag{4.75}$$

where P is a polynomial in $\|v\|_{H^{2+\delta}}, \|v_t\|_{H^{1+\delta}}, \|q\|_{H^{2.5+\delta}}, \|q_t\|_{H^{1+\delta}}$, and $\|\eta\|_{H^{2.5+\delta}}$. The proof is completed. $\square$

Now we conclude the proof of the main theorem by the same Gronwall type argument as in section 8 of [**KT2**].

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