On the local existence for the 3D Euler equation with a free interface

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ABSTRACT. We address the local existence and uniqueness of solutions for the 3D Euler equations with a free interface. We prove the local well-posedness in the rotational case when the initial datum u_0 satisfies $u_0 \in H^{2.5+\delta}$ and $\operatorname{curl} u_0 \in H^{2+\delta}$, where $\delta > 0$ is arbitrarily small, under the Taylor condition on the pressure.

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1. Introduction

The aim of this paper is to address the local existence of solutions in low regularity Sobolev spaces for the rotational free-surface Euler equations

$$u_t + u \cdot \nabla u + \nabla p = 0 \quad \text{in } \Omega(t) \times (0,T) \tag{1.1}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega(t) \times (0,T) \tag{1.2}$$

in a time-dependent domain $\Omega(t) \subseteq \mathbb{R}^3$. The boundary of the domain consists of two parts: the moving part $\Gamma_1(t)$, which is unknown and moves with the fluid velocity field, and the stationary part Γ_0. On the free boundary $\Gamma_1(t)$ we require the vanishing of the pressure, while on Γ_0 we impose the no-flow boundary condition $v \cdot N = 0$.

The earliest work to treat the local existence problem is a paper by Nalimov [**N**], where existence for (1.1)–(1.2) was proven in two space dimensions for small initial data. Other early works [**Y1, Y2, S**] also considered the problem of local existence under a smallness assumption of the data or under the irrotationality assumption, i.e., when the initial vorticity vanishes. For the existence of solutions when the data is rotational, the Taylor stability sign condition $\partial p/\partial N < 0$ must be imposed, as was shown by Ebin [**E**]. Beale, Hou, and Lowengrub then proved in [**BHL**] the local existence of solutions to the linearized system under the Taylor sign condition.

In [**W1, W2**], Wu established local existence of the solution without a smallness assumption on the initial data and under the general Taylor sign condition, in two and three space dimensions. In [**AM1, AM2**], Ambrose and Masmoudi treated the problem in the presence of surface tension. Many other important works treating the problem of local existence and regularity using different methods include [**ABZ1, ABZ2, B, CCFGGS, CLa, Cr, CL, EL, HIT, IT, I, L, Li1, Li2, MR, OT, S, Sh, Shn, T, XZ, ZZ**]. Notably, Coutand and Shkoller provided in [**CS1, CS2**] existence and uniqueness of solutions for H^3 initial velocity with the vorticity in $H^{2.5}$. A similar result but with completely different methods were at the same time obtained by Shatah and Zeng [**SZ**] and Zhang and Zhang [**ZZ**].

Based on the well-posedness results for the classical Euler equation ([**Lic, Te**]), the minimal possible assumption for the velocity one may expect is $H^{n/2+1+\delta}$, where n is the space dimension and $\delta > 0$. In a previous work [**KT2**], two of the authors proved the local existence of solutions to the 2D system under the Taylor sign condition assuming the minimum requirement on the initial velocity of $H^{2+\delta}$, with the vorticity in $H^{1.5+\delta}$, where $\delta > 0$. The proof relies on div-curl type estimates of the velocity and the Lagrangian flow map, which also require tangential estimates on the boundary. In order to establish the curl estimates, [**KT2**] used that in two dimensions the vorticity is invariant in Lagrangian coordinates, which allows us to obtain an estimate for the curl of the Lagrangian map depending only on the initial regularity of the vorticity. This estimate is needed since the tangential estimates produce boundary terms of higher order in the flow map by half a derivative. In [**KT2**], two of the authors also showed the local existence for 3D irrotational initial data with the initial velocity in $H^{2.5+\delta}$. This left the case of the initial velocity in $H^{2.5+\delta}$ with the initial vorticity in $H^{2+\delta}$ open.

The main result of this paper states that the local existence holds when the initial velocity u_0 belongs to $H^{2.5+\delta}$ while the initial vorticity lies in $H^{2+\delta}$, where $\delta > 0$ is arbitrary. We also show full details for the uniqueness of solutions with initial data in this class. We believe that the details of uniqueness are worthwhile to include, since the proof of uniqueness of solutions at this regularity level has not been done in the literature. Moreover, an important feature that we aim to highlight is that the uniqueness (stability) holds for solutions in a Sobolev regularity range in the range between $H^{1.5+\delta}$ and $H^{2+\delta}$; we provide the complete details for the case of uniqueness (stability) in H^2.

The main tool in the existence and uniqueness is the Cauchy invariance, cf. (4.5) below, which yields an identity for the curl of the matrix product of the Jacobian matrix for the Lagrangian flow, with the velocity vector [**Ca, FV, ZF**]. The Cauchy invariance, which follows from the Weber formula [**C1, C2, Web**], provides a three-dimensional analogue to the two-dimensional conservation of vorticity along Lagrangian trajectories, being thus useful for obtaining local in time estimates for the Lagrangian vorticity. The Cauchy invariance also plays a crucial role in the proof of the uniqueness of solutions, for estimating differences of Lagrangian maps, and for bounds on the curl of the differences of Lagrangian velocities.

For irrotational flows, i.e., those with vanishing vorticity, the local existence with optimal regularity assumptions on the initial datum has already been established by Alazard, Burq, and Zuily in [**ABZ2**] in two and three space dimensions, and by Hunter, Ifrim, and Tataru in two dimensions [**HIT**]. In a recent work [**KT2**], two of the authors provided an alternative proof of the optimal regularity for irrotational flow in three dimensions, where the initial data is assumed to be irrotational with $H^{2.5+\delta}$ Sobolev regularity. We also note that in the irrotational case, the delicate problem of global existence of solutions with small initial datum, was settled in both three [**GMS, W3**] and two [**AD, IP, IT**] space dimensions.

In order to make the proof more presentable to the reader, we address the case when the initial boundary free-surface is flat. The proof can be modified to address the case of an initial boundary which is the graph of a function, using a change of variable. The new terms appearing would all be of lower order. The estimates can be justified by the horizontal mollification of the Lagrangian, the device introduced by Coutand and Shkoller in [**CS1**].

The paper is organized as follows. In Section 2, we recall the Lagrangian setting of the problem and state the main theorem. Section 3, we recall the basic estimates for the coefficients a and the pressure estimates from [**KT2**]. The proof of the existence is provided in Section 4, while the uniqueness is proven in Section 5.

2. The main result

Consider the Euler equation on the domain

$$\Omega = \mathbb{R}^2 \times (0,1) \subseteq \mathbb{R}^3 \tag{2.1}$$

with periodic boundary in x_1 and x_2 with period 1. The top

$$\Gamma_1 = \mathbb{R} \times \{x_n = 1\} \tag{2.2}$$

represents the free boundary, while the rigid bottom is represented by

$$\Gamma_0 = \mathbb{R} \times \{x_n = 0\}. \tag{2.3}$$

We denote by $v(x,t) = (v^1, v^2, v^3)$ the Lagrangian velocity, while $q(x,t)$ represents the Lagrangian pressure. The Euler equation in Lagrangian coordinates may be written as

$$v_t^i + a_i^k \partial_k q = 0 \text{ in } \Omega \times (0,T), \qquad i = 1,2,3 \tag{2.4}$$

$$a_i^k \partial_k v^i = 0 \text{ in } \Omega \times (0,T) \tag{2.5}$$

with the initial condition

$$v(0) = v_0. \tag{2.6}$$

Note that the summation convention on repeated indices is used throughout. The matrix a evolves according to

$$a_t = -a : \nabla v : a \tag{2.7}$$

$$a(\cdot, x) = I, \qquad x \in \Omega \tag{2.8}$$

where the symbol : denotes the matrix multiplication. The cofactor matrix represents the inverse

$$a = (\nabla \eta)^{-1} \tag{2.9}$$

where η is defined as

$$\eta_t(x,t) = v(x,t) \tag{2.10}$$

$$\eta(x,0) = x, \qquad x \in \Omega. \tag{2.11}$$

Note that the property

$$a : \nabla \eta = I \tag{2.12}$$

may be deduced directly from the system by checking the evolution of the product $a : \nabla\eta$. (In turn, (2.8) follows from (2.12) by the time differentiation.)

On the top, which represents the free boundary, we impose

$$q = 0 \text{ on } \Gamma_1 \times (0,T) \tag{2.13}$$

while on the bottom boundary we assume

$$v^i N^i = 0 \text{ on } \Gamma_0 \times (0,T) \tag{2.14}$$

where $N = (N^1, N^2, N^3)$ stands for the outward unit normal. Since our domain (2.1) is assumed to be flat, for simplicity, we have $N = (0,0,-1)$ on Γ_0 and $N = (0,0,1)$ on Γ_1.

The following is our main result.

THEOREM 2.1. *Let $\delta > 0$. Assume that $v(\cdot,0) = v_0 \in H^{2.5+\delta}(\Omega)$ is divergence-free with $v \cdot N = 0$ on Γ_0 and*

$$\operatorname{curl} v_0 \in H^{2+\delta}(\Omega). \tag{2.15}$$

Assume that the initial pressure $q(\cdot,0)$ satisfies the Rayleigh-Taylor condition

$$\frac{\partial q}{\partial N}(x,0) \le -\frac{1}{C_0} < 0, \qquad x \in \Gamma_1 \tag{2.16}$$

where $C_0 > 0$ is a constant. Then there exists a unique solution (v, q, a, η) to the free boundary Euler system with the initial condition $v(0) = v_0$ such that

$$\begin{aligned}
&v \in L^\infty([0,T]; H^{2.5+\delta}(\Omega)) \cap C([0,T]; H^{2+\delta}(\Omega))\\
&v_t \in L^\infty([0,T]; H^{2+\delta}(\Omega))\\
&\eta \in L^\infty([0,T]; H^{3+\delta}(\Omega)) \cap C([0,T]; H^{2.5+\delta}(\Omega))\\
&a \in L^\infty([0,T]; H^{2+\delta}(\Omega)) \cap C([0,T]; H^{1.5+\delta}(\Omega))\\
&q \in L^\infty([0,T]; H^{3+\delta}(\Omega))\\
&q_t \in L^\infty([0,T]; H^{2.5+\delta}(\Omega))
\end{aligned} \tag{2.17}$$

for $T > 0$ which depends on the initial data.

The proof of existence is given in Section 4, while the uniqueness is proven in Section 5.

The main emphasis in the statement above is on the fact that $\delta > 0$ is arbitrarily small. However, we emphasize that the statement and the proof holds for *all* positive δ.

3. Preliminary lemma on the coefficients and pressure estimates

In the next lemma, we recall from [**KT2**] the a priori estimates for the coefficient matrix a and for the particle map η.

LEMMA 3.1. [**KT2**] *Assume that $\|\nabla v\|_{L^\infty([0,T];H^{1.5+\delta}(\Omega))} \le M$. If*

$$T \le \frac{1}{CM} \tag{3.1}$$

where C is a sufficiently large constant, the following statements hold:
(i) $\|\nabla \eta(\cdot,t)\|_{H^{1.5+\delta}(\Omega)} \le C$ *for* $t \in [0,T]$,
(ii) $\det(\nabla \eta(x,t)) = 1$ *for* $(x,t) \in \Omega \times [0,T]$,
(iii) $\|a(\cdot,t)\|_{H^{1.5+\delta}(\Omega)} \le C$ *(and thus also* $\|a(\cdot,t)\|_{L^\infty(\Omega)} \le C$*) for* $t \in [0,T]$,
(iv) $\|a_t(\cdot,t)\|_{L^p(\Omega)} \le C\|\nabla v(\cdot,t)\|_{L^p(\Omega)}$ *for* $p \in [1,\infty]$ *and* $t \in [0,T]$,
(v) $\|a_t(\cdot,t)\|_{H^r(\Omega)} \le C\|\nabla v(\cdot,t)\|_{H^r(\Omega)}$ *for* $r \in [0, 1.5+\delta)$ *and* $t \in [0,T]$,

(vi) $\|a_{tt}(\cdot,t)\|_{H^\sigma(\Omega)} \le C\|\nabla v(\cdot,t)\|_{H^{1.5+\delta}(\Omega)}\|\nabla v(\cdot,t)\|_{H^\sigma(\Omega)} + C\|\nabla v_t(\cdot,t)\|_{H^\sigma(\Omega)}$, *for* $t \in [0,T]$ *and all* $0 < \sigma \le 1.5+\delta$, *and*
(vii) for every $\epsilon \in (0,1]$ *there exists a constant* $C > 0$ *such that for all* $t \in [0,T']$, *where* $T' = \min\{\epsilon/CM, T\} > 0$, *we have*

$$\|a_l^j - \delta_{jl}\|_{H^{1.5+\delta}(\Omega)} \le \epsilon \tag{3.2}$$

for $j,l = 1,2,3$ *and*

$$\|a_l^j a_l^k - \delta_{jk}\|_{H^{1.5+\delta}(\Omega)} \le \epsilon \tag{3.3}$$

for $j,k = 1,2,3$.

For the proof, based on (2.7)–(2.8) and (2.10)–(2.11), cf. [**KT2**].

In the following lemma, we recall the pressure estimates from [**KT2**].

LEMMA 3.2. [**KT2**] *Assume that* (v,q,a,η) *satisfies the Euler equation* (2.4)–(2.11) *in* $\Omega \times [0,T)$ *and that we have* $\|\nabla v\|_{L^\infty([0,T];H^{1.5+\delta}(\Omega))} \le M$. *Assume that* a *satisfies the estimates in Lemma 3.1 for a sufficiently small constant* $\epsilon > 0$. *Then the pressure* q *obeys*

$$\|q(t)\|_{H^{3+\delta}} \le P + P\int_0^t \|q_t(s)\|_{H^{2+\delta}}\,ds, \qquad t \in [0,T] \tag{3.4}$$

where P *is a polynomial in* $\|v\|_{H^{2.5+\delta}}$, $\|\eta\|_{H^{3+\delta}}$, *and* $\|v_0\|_{H^{2.5+\delta}}$, *and*

$$\|q_t(t)\|_{H^{2.5+\delta}} \le P + P\int_0^t \|q_t(s)\|_{H^{2+\delta}}\,ds, \qquad t \in [0,T] \tag{3.5}$$

where P *is a polynomial in* $\|v\|_{H^{2.5+\delta}}$, $\|v_t\|_{H^{2+\delta}}$, $\|q\|_{H^{3+\delta}}$, $\|\eta\|_{H^{3+\delta}}$, *and* $\|v_0\|_{H^{2.5+\delta}}$.

4. Proof of Existence

4.1. Tangential Estimates. Next, we recall the tangential estimates on the solution (v,η,a,q). Denote

$$S = \overline{\partial}^{2.5+\delta} \tag{4.1}$$

where $\overline{\partial} = (I - \Delta_2)^{1/2}$ with $\Delta_2 = \partial_{11} + \partial_{22}$.

LEMMA 4.1. [**KT2**] *For* $t \in [0,T]$, *we have*

$$\begin{aligned}&\|Sv(t)\|_{L^2}^2 + \|a_l^3(t) S\eta^l(t)\|_{L^2(\Gamma_1)}^2 \\ &\qquad \le \int_0^t P(\|v\|_{H^{2.5+\delta}}, \|v_t\|_{H^{2+\delta}}, \|q\|_{H^{3+\delta}}, \|q_t\|_{H^{2.5+\delta}}, \|\eta\|_{H^{3+\delta}})\,ds + Q(\|v_0\|_{H^{2.5+\delta}})\end{aligned} \tag{4.2}$$

where P *and* Q *are polynomials in indicated arguments.*

As in [**KT2**], unless the arguments are specified, the symbol P denotes a generic positive polynomial depending on $\|v\|_{H^{2.5+\delta}}$, $\|v_t\|_{H^{2+\delta}}$, $\|q\|_{H^{3+\delta}}$, $\|q_t\|_{H^{2.5+\delta}}$, and $\|\eta\|_{H^{3+\delta}}$.

For the proof of Lemma 4.1, see [**KT2**].

In case of a non-flat initial boundary, the tangential estimates can be adapted using a change of variable. In particular, we can take the more general domain $\Omega' = \mathbb{R}^2 \times (0, h(x_1, x_2))$ where the top moving boundary Γ_1' is initially given by the graph of a function $h(x_1, x_2) > 0$ or

$$\Gamma_1' = \mathbb{R}^2 \times \{x_3 = h(x_1, x_2)\}$$

while the rigid bottom boundary is flat and described by

$$\Gamma_0 = \mathbb{R}^2 \times \{x_3 = 0\}.$$

We can then use a change of variable from $(x_1, x_2, x_3) \in \Omega'$ to $(y_1, y_2, y_3) \in \Omega$ where

$$\begin{aligned} y_1 &= x_1 \\ y_2 &= x_2 \\ y_3 &= \frac{x_3}{h(x_1, x_2)} \end{aligned}$$

Applying this change of variable, we define

$$\begin{aligned} \bar{v}(y_1, y_2, y_3, t) &= v(x_1, x_2, x_3, t) \\ \bar{q}(y_1, y_2, y_3, t) &= q(x_1, x_2, x_3, t) \\ \bar{\eta}(y_1, y_2, y_3, t) &= \eta(x_1, x_2, x_3, t) \\ \bar{a}(y_1, y_2, y_3, t) &= a(x_1, x_2, x_3, t). \end{aligned}$$

The Euler equations can then be expressed in terms of the new variables on $\Omega = \mathbb{R}^2 \times (0, 1)$ as

$$\bar{v}_t^i + \bar{a}_i^k b_k^j \partial_j \bar{q} = 0 \text{ in } \Omega \times (0, T), \qquad i = 1, 2, 3 \tag{4.3}$$

$$\bar{a}_i^k b_k^j \partial_j \bar{v}^i = 0 \text{ in } \Omega \times (0, T) \tag{4.4}$$

where $b_k^j = \partial y_j / \partial x_k$ are the entries of the Jacobian matrix. Defining $c_i^j = \bar{a}_i^k b_k^j$, we can repeat the same tangential estimates as in [**KT2**] to obtain lemma 4.1. This requires assuming that $h(x_1, x_2)$ has $H^{2+\delta}(\mathbb{R}^2)$ regularity.

4.2. Div-Curl Estimates. Next, differentiating the Cauchy invariance

$$\epsilon_{ijk} \partial_j v^m \partial_k \eta^m = \omega_0^i, \qquad t \geq 0, \qquad i = 1, 2, 3 \tag{4.5}$$

(cf. [**Ca, C2, FV, ZF**] or see the appendix for the proof) we get

$$\epsilon_{ijk} \partial_j v^m \nabla \partial_k \eta^m + \epsilon_{ijk} \partial_k \eta^m \partial_j \nabla v^m = \nabla \omega_0^i, \qquad t \geq 0, \qquad i = 1, 2, 3. \tag{4.6}$$

Here, ϵ_{ijk} is the usual antisymmetric tensor defined by $\epsilon_{123} = 1$ with $\epsilon_{ijk} = -\epsilon_{jik}$ and $\epsilon_{ijk} = \epsilon_{jki}$ for $i, j, k = 1, 2, 3$. Next, using $\partial_j \nabla \eta(0) = 0$, we have

$$\epsilon_{ijk} \partial_k \eta^m \partial_j \nabla \eta^m = \int_0^t \Big(\epsilon_{ijk} \partial_k \eta^m \partial_j \nabla \eta_t^m + \epsilon_{ijk} \partial_k \eta_t^m \partial_j \nabla \eta^m \Big) \, ds, \qquad i = 1, 2, 3. \tag{4.7}$$

For simplicity of notation, we frequently omit the argument t, as well as the argument s inside integrals. The first term inside the integral sign may be rewritten as

$$\begin{aligned} \epsilon_{ijk} \partial_k \eta^m \partial_j \nabla \eta_t^m &= \epsilon_{ijk} \partial_k \eta^m \partial_j \nabla v^m \\ &= -\epsilon_{ijk} \partial_j v^m \nabla \partial_k \eta^m + \nabla \omega_0^i, \qquad i = 1, 2, 3 \end{aligned} \tag{4.8}$$

where we utilized (2.10) in the first and (4.6) in the second equality. Using (4.8) in the first term inside the integral in (4.7), we get

$$\begin{aligned}\epsilon_{ijk}\partial_k\eta^m\partial_j\nabla\eta^m &= \int_0^t \Big(-\epsilon_{ijk}\partial_j v^m\partial_k\nabla\eta^m + \epsilon_{ijk}\partial_k v^m\partial_j\nabla\eta^m\Big)\,ds + t\nabla\omega_0^i, \qquad i=1,2,3\\ &= 2\int_0^t \epsilon_{ijk}\partial_k v^m\partial_j\nabla\eta^m\,ds + t\nabla\omega_0^i, \qquad i=1,2,3 \end{aligned}\tag{4.9}$$

from where, by $\epsilon_{ijk}\partial_j\nabla\eta^k = \nabla((\operatorname{curl}\eta)^i)$ for $i=1,2,3$,

$$\nabla((\operatorname{curl}\eta)^i) = \epsilon_{ijk}(\delta_{km}-\partial_k\eta^m)\partial_j\nabla\eta^m + 2\int_0^t \epsilon_{ijk}\partial_k v^m\partial_j\nabla\eta^m\,ds + t\nabla\omega_0^i, \qquad i=1,2,3. \tag{4.10}$$

Now, applying the $H^{1+\delta}$ norms of both sides and writing $\delta_{km}-\partial_k\eta^m$ as a time integral of its derivative, which is $-\partial_k v^m$, we may estimate

$$\begin{aligned}\|\nabla\operatorname{curl}\eta\|_{H^{1+\delta}} &\le C\|\eta\|_{H^{3+\delta}}\|I-\nabla\eta\|_{H^{1.5+\delta}} + C\int_0^t \|v\|_{H^{2.5+\delta}}\|\eta\|_{H^{3+\delta}}\,ds + C\|\omega_0\|_{H^{2+\delta}}\\ &\le C\|\eta\|_{H^{3+\delta}}\int_0^t\|v\|_{H^{2.5+\delta}}\,ds + C\int_0^t \|v\|_{H^{2.5+\delta}}\|\eta\|_{H^{3+\delta}}\,ds + C\|\omega_0\|_{H^{2+\delta}} \end{aligned}\tag{4.11}$$

where we used the multiplicative Sobolev inequality

$$\|fg\|_{H^{1+\delta}} \le C\|f\|_{H^{1+\delta}}\|g\|_{H^{1.5+\delta}} \tag{4.12}$$

which follows from the Kenig-Ponce-Vega product estimate [**KP, KPV**]. Since $\|\eta\|_{H^{2+\delta}} \le C + C\int_0^t\|v\|_{H^{2+\delta}}\,ds$, we get

$$\begin{aligned}\|\operatorname{curl}\eta\|_{H^{2+\delta}} &\le C + C\int_0^t\|v\|_{H^{2+\delta}}\,ds + C\|\eta\|_{H^{3+\delta}}\int_0^t\|v\|_{H^{2.5+\delta}}\,ds\\ &\quad + C\int_0^t\|v\|_{H^{2.5+\delta}}\|\eta\|_{H^{3+\delta}}\,ds + C\|\omega_0\|_{H^{2+\delta}}. \end{aligned}\tag{4.13}$$

On the other hand, from [**KT2**], we recall the estimate on the divergence. First, we write

$$\begin{aligned}\operatorname{div}\eta &= (\delta_{ki}-a_i^k)\partial_k\eta^i + \int_0^t \partial_t(a_i^k\partial_k\eta^i)\,ds + 3\\ &= (\delta_{ki}-a_i^k)\partial_k\eta^i + \int_0^t \partial_t a_i^k\partial_k\eta^i\,ds + 3\end{aligned}$$

where we used (2.5) in the last step. Therefore,

$$\begin{aligned}\|\operatorname{div}\eta\|_{H^{2+\delta}} &\le C\int_0^t\|v\|_{H^{2.5+\delta}}\|\eta\|_{H^{3+\delta}}\,ds + C\int_0^t\|\eta\|_{H^{3+\delta}}^2\|v\|_{H^{2.5+\delta}}\,ds\\ &\quad + C\|\eta\|_{H^{3+\delta}}\int_0^t\|v\|_{H^{2.5+\delta}}\,ds + C\int_0^t\|v\|_{H^{2+\delta}}\,ds + C. \end{aligned}\tag{4.14}$$

Using the inequality

$$\|f\|_{H^s(\Omega)} \le C\|f\|_{L^2(\Omega)} + C\|\operatorname{curl} f\|_{H^{s-1}(\Omega)} + C\|\operatorname{div} f\|_{H^{s-1}(\Omega)} + C\|(\nabla_\tau f)\cdot N\|_{H^{s-1.5}(\partial\Omega)} \tag{4.15}$$

valid for any vector function $f \in H^s(\Omega)$ and $s > 1.5$ ([**CS1, CS2**]) where ∇_τ represents the tangential gradient on the boundary, we get

$$\|\eta\|_{H^{3+\delta}} \leq C\|\eta\|_{L^2} + C\|\operatorname{curl}\eta\|_{H^{2+\delta}} + C\|\operatorname{div}\eta\|_{H^{2+\delta}} + C\|S\eta^3\|_{L^2(\Gamma_1)}. \tag{4.16}$$

Now, we replace (4.13) and (4.14) in (4.16). We also use

$$\begin{aligned}\|S\eta^3\|_{L^2(\Gamma_1)} &\leq \|a_l^3 S\eta^l\|_{L^2(\Gamma_1)} + \|(\delta_{3l} - a_l^3)S\eta^l\|_{L^2(\Gamma_1)} \\ &\leq \|a_l^3 S\eta^l\|_{L^2(\Gamma_1)} + C\epsilon\|S\eta^3\|_{L^2(\Gamma_1)}\end{aligned} \tag{4.17}$$

absorbing the last term assuming that $\epsilon > 0$ is sufficiently small, which we may at the expense of shortening the time interval as in Lemma 3.1. We get

$$\begin{aligned}\|\eta\|_{H^{3+\delta}} \leq C &+ C\int_0^t \|v\|_{H^{2+\delta}}\,ds + C\|\eta\|_{H^{3+\delta}}\int_0^t \|v\|_{H^{2.5+\delta}}\,ds + C\int_0^t \|v\|_{H^{2.5+\delta}}\|\eta\|_{H^{3+\delta}}\,ds \\ &+ C\|\omega_0\|_{H^{2+\delta}} + C\int_0^t \|\eta\|_{H^{3+\delta}}^2\|v\|_{H^{2.5+\delta}}\,ds + C\int_0^t \|v\|_{H^{2+\delta}}\,ds + \|a_l^3 S\eta^l\|_{L^2(\Gamma_1)}.\end{aligned} \tag{4.18}$$

In order to obtain an estimate for $\operatorname{curl} v$, we use the Cauchy invariance (4.5) again. First, we have

$$(\operatorname{curl} v)^i = \epsilon_{ijk}\partial_j v^m = \epsilon_{ijk}\partial_j v^m(\delta_{km} - \partial_k\eta^m) + \omega_0^i, \qquad i = 1,2,3 \tag{4.19}$$

from where, using the algebra property of $H^{1.5+\delta}$,

$$\|\operatorname{curl} v\|_{H^{1.5+\delta}} \leq C\|\nabla v\|_{H^{1.5+\delta}}\sum_{k,m=1}^3 \|\delta_{km} - \partial_k\eta^m\|_{H^{1.5+\delta}} + \|\omega_0\|_{H^{1.5+\delta}}. \tag{4.20}$$

Now, by $\delta_{km} - \partial_k\eta^m = -\int_0^t \partial_k\eta_t^m\,ds = -\int_0^t \partial_k v^m\,ds$ for $k,m = 1,2,3$, we get

$$\|\operatorname{curl} v\|_{H^{1.5+\delta}} \leq C\|v\|_{H^{2.5+\delta}}\int_0^t \|v\|_{H^{2.5+\delta}}\,ds + \|\omega_0\|_{H^{1.5+\delta}}. \tag{4.21}$$

As in [**KT2**], we obtain from here

$$\begin{aligned}\|v\|_{H^{2.5+\delta}} &\leq C\|v\|_{H^2} + C\|Sv\|_{L^2} + C\|v\|_{H^{2.5+\delta}}\int_0^t \|v\|_{H^{2.5+\delta}}\,ds + C\|\omega_0\|_{H^{1.5+\delta}} \\ &\leq C\int_0^t \|v_t\|_{H^2}\,ds + C\|Sv\|_{L^2} + C\|v\|_{H^{2.5+\delta}}\int_0^t \|v\|_{H^{2.5+\delta}}\,ds + C\|\omega_0\|_{H^{1.5+\delta}}.\end{aligned} \tag{4.22}$$

Finally, applying the Gronwall lemma to (4.2), (4.18), (4.22) with Lemma 3.2, as in [**KT2**], concludes the proof of the existence part of Theorem 2.1.

REMARK 4.2. In order to justify the estimates, we can construct solutions using the horizontal mollification procedure introduced in [**CS1**]. Namely, we approximate the Euler system with

$$v_t^i + \widetilde{a}_i^k\partial_k q = 0 \text{ in } \Omega\times(0,T), \qquad i = 1,2,3 \tag{4.23}$$

$$\bar{a}_i^k\partial_k v^i = 0 \text{ in } \Omega\times(0,T) \tag{4.24}$$

where $\widetilde{\eta}$ denotes the horizontal mollification with parameter $\epsilon > 0$ applied to η, and where $\bar{a}$ denotes the inverse of $\nabla\eta$. The Cauchy invariance for the modified system takes the form

$$\partial_t(\epsilon_{ijk}\partial_j v^m\partial_k\eta^m) = -\epsilon_{ijk}\partial_j\overline{a}_m^l\partial_l q\partial_k(\eta^m - \widetilde{\eta}^m) - \epsilon_{ijk}\overline{a}_m^l\partial_{jl}q\partial_k(\eta^m - \widetilde{\eta}^m) \tag{4.25}$$

which can be obtained using the second proof in the appendix. As $\epsilon \to 0$, the identity (4.25) converges to the Cauchy invariance formula (4.5).

5. Proof of uniqueness

Let (v, q, a, η) and $(\widetilde{v}, \widetilde{q}, \widetilde{\eta}, \widetilde{a})$ be two solutions of our system on an interval $[0, T]$ satisfying the bounds provided in the existence part. In addition, assume that the first solution satisfies the Taylor assumption

$$\frac{\partial q}{\partial N}(x,t) \le -1/C, \qquad x \in \Gamma_1, \qquad t \in [0,T]. \tag{5.1}$$

Denote by

$$(V, Q, A, E) = (v, q, a, \eta) - (\widetilde{v}, \widetilde{q}, \widetilde{a}, \widetilde{\eta}) \tag{5.2}$$

the difference of these two solutions. For simplicity of notation, we allow all the constants to depend on the suprema of the norms $\|v\|_{H^{2.5+\delta}}$, $\|v_t\|_{H^{2+\delta}}$, $\|q\|_{H^{3+\delta}}$, $\|q_t\|_{H^{2.5+\delta}}$, $\|\eta\|_{H^{3+\delta}}$, and $\|\widetilde{a}\|_{H^{2+\delta}}$, as well as on the suprema of $\|\widetilde{v}\|_{H^{2.5+\delta}}$, $\|\widetilde{v}_t\|_{H^{2+\delta}}$, $\|\widetilde{q}\|_{H^{3+\delta}}$, $\|\widetilde{q}_t\|_{H^{2.5+\delta}}$, $\|\widetilde{\eta}\|_{H^{3+\delta}}$, and $\|a\|_{H^{2+\delta}}$, over the interval $[0, T]$. For instance, we have $\|v\|_{H^{2.5+\delta}}, \|\widetilde{v}\|_{H^{2.5+\delta}} \le C$ for $t \in [0, T]$.

The proof is divided in several subsections corresponding to the estimates for the differences of the pressures, tangential velocities, Lagrangian maps, and the velocity gradients.

5.1. Pressure estimates.

In the following lemma, we derive the pressure estimates satisfied by the difference of two solutions.

LEMMA 5.1. *The difference of pressures Q satisfies*

$$\|Q\|_{H^{2.5}} \le C(\|V\|_{H^2} + \|E\|_{H^{2.5}}) + C\int_0^t \|Q_t\|_{H^2}\, ds \tag{5.3}$$

while for the derivative Q_t we have

$$\|Q_t\|_{H^2} \le C(\|V\|_{H^2} + \|V_t\|_{H^1} + \|A\|_{H^1} + \int_0^t \|Q_t\|_{H^2})\, ds \tag{5.4}$$

for all $t \in [0, T]$.

PROOF. Applying $a_i^j \partial_j$ to $v_t^i + a_i^k \partial_k q = 0$ and $\widetilde{a}_i^j \partial_j$ to $\widetilde{v}_t^i + \widetilde{a}_i^k \partial_k \widetilde{q} = 0$, we obtain

$$\partial_j(a_i^j a_i^k \partial_k q) = \partial_t a_i^j \partial_j v^i \tag{5.5}$$

and

$$\partial_j(\widetilde{a}_i^j \widetilde{a}_i^k \partial_k \widetilde{q}) = \partial_t \widetilde{a}_i^j \partial_j \widetilde{v}^i \tag{5.6}$$

where we used the divergence-free conditions (2.5) and

$$\widetilde{a}_i^k \partial_k \widetilde{v}^i = 0 \text{ in } \Omega \times (0,T) \tag{5.7}$$

as well as the Piola identities $\partial_j a_i^j = 0$ and $\partial_j \widetilde{a}_i^j = 0$ for $i = 1, 2, 3$. Subtracting (5.6) from (5.5) leads to

$$\partial_j(A_i^j a_i^k \partial_k q) + \partial_j(\widetilde{a}_i^j A_i^k \partial_k q) + \partial_j(\widetilde{a}_i^j \widetilde{a}_i^k \partial_k Q) = \partial_t A_i^j \partial_j v^i + \widetilde{a}_i^j \partial_j V_t^i \tag{5.8}$$

from where we get

$$\Delta Q = -\partial_j(A_i^j a_i^k \partial_k q) - \partial_j(\widetilde{a}_i^j A_i^k \partial_k q) + \partial_j(\delta_{jk} - \widetilde{a}_i^j \widetilde{a}_i^k)\partial_k Q$$
$$+ (\delta_{jk} - \widetilde{a}_i^j \widetilde{a}_i^k)\partial_{jk} Q + \partial_t A_i^j \partial_j v^i + \widetilde{a}_i^j \partial_j V_t^i. \tag{5.9}$$

On the other hand, by subtracting the boundary conditions satisfied by q and $\widetilde{q}$ on Γ_0,

$$\partial_i q N^i = (\delta_{ki} - a_i^k)\partial_k q N^i \text{ on } \Gamma_0 \tag{5.10}$$

and

$$\partial_i \widetilde{q} N^i = (\delta_{ki} - \widetilde{a}_i^k)\partial_k \widetilde{q} N^i \text{ on } \Gamma_0 \tag{5.11}$$

we obtain a boundary condition for Q on Γ_0, which reads

$$\partial_i Q N^i = -A_i^k \partial_k q N^i + (\delta_{ki} - a_i^k)\partial_k Q N^i \text{ on } \Gamma_0 \tag{5.12}$$

in addition to the condition

$$Q = 0 \text{ on } \Gamma_1. \tag{5.13}$$

Thus Q satisfies an elliptic problem with mixed Dirichlet/Neumann boundary conditions. In particular the estimate

$$\|Q\|_{H^{2.5}} \le C\|f\|_{H^{0.5}} + C\|g\|_{H^1(\Gamma_0)} \tag{5.14}$$

holds where f is the right side of (5.9) and g is the boundary data on Γ_0 in (5.12). Thus we may write

$$\|Q\|_{H^{2.5}} \le C\|A\|_{H^{1.5}}(\|a\|_{H^{1.5+\delta}} + \|\widetilde{a}\|_{H^{1.5+\delta}})\|q\|_{H^{2.5+\delta}} + C\|a^T : a\|_{H^{2+\delta}}\|Q\|_{H^2}$$
$$+ C\|a^T : a - I\|_{H^{1.5+\delta}}\|Q\|_{H^{2.5}} + C\|A_t\|_{H^1}\|\nabla v\|_{H^{1+\delta}}$$
$$+ C\|\widetilde{a}_t\|_{H^{1+\delta}}\|V\|_{H^2} + C\|I - a\|_{H^{1.5+\delta}}\|Q\|_{H^{2.5}} + C\|A\|_{H^{1.5}}\|q\|_{H^{2.5+\delta}}$$
$$\le C(\|A\|_{H^{1.5}} + \|A_t\|_{H^1} + \|V\|_{H^2}) + C\int_0^t \|Q_t\|_{H^2}\,ds + Ct\|Q\|_{H^{2.5}} \tag{5.15}$$

where we used

$$\|Q\|_{H^2} \le C\int_0^t \|Q_t\|_{H^2}\,ds \tag{5.16}$$

and

$$\|a^T : a - I\|_{H^{1.5+\delta}} \le C\int_0^t \|\partial_t(a^T : a)\|_{H^{1.5+\delta}}\,ds \le Ct \tag{5.17}$$

as well as

$$\|a - I\|_{H^{1.5+\delta}} \le \int_0^t \|a_t\|_{H^{1.5+\delta}}\,ds \le Ct. \tag{5.18}$$

Without loss of generality, we may assume that $T = 1/C$ where C is a constant which is so large that the last term on the far right side (5.15) can be absorbed using the left side. In order to establish the claimed inequality (5.3), we need to bound the norms of A and A_t by those of E. Subtracting $a : \nabla\eta = I$ and $\widetilde{a} : \nabla\widetilde{\eta} = I$, we get $A : \nabla\eta + \widetilde{a} : \nabla E = 0$, from where

$$A = -\widetilde{a} : \nabla E : a. \tag{5.19}$$

Using a multiplicative Sobolev inequality, we get

$$\|A\|_{H^{1.5}} \le C\|\widetilde{a}\|_{H^{1.5+\delta}}\|\nabla E\|_{H^{1.5}}\|a\|_{H^{1.5+\delta}} \le C\|E\|_{H^{2.5}}. \tag{5.20}$$

Next, the identities $a_t = -a : \nabla v : a$ and $\widetilde{a}_t = -\widetilde{a} : \nabla \widetilde{v} : \widetilde{a}$ give

$$A_t = -A : \nabla v : a - \widetilde{a} : \nabla V : a - \widetilde{a} : \nabla \widetilde{v} : A \tag{5.21}$$

and thus

$$\begin{aligned}\|A_t\|_{H^1} &\le C\|A\|_{H^1}\|\nabla v\|_{H^{1.5+\delta}}\|a\|_{H^{1.5+\delta}} + C\|\widetilde{a}\|_{H^{1.5+\delta}}\|\nabla V\|_{H^1}\|a\|_{H^{1.5+\delta}}\\ &\quad + C\|\widetilde{a}\|_{H^{1.5+\delta}}\|\nabla \widetilde{v}\|_{H^{1.5+\delta}}\|A\|_{H^1}\\ &\le C(\|A\|_{H^1} + \|V\|_{H^2}).\end{aligned} \tag{5.22}$$

The inequality (5.3) is thus established.

Similarly, we derive an estimate for Q_t by time differentiating the system (5.9) satisfied by Q. We obtain a Laplace equation

$$\begin{aligned}\Delta Q_t &= -\partial_j(\partial_t A_i^j a_i^k \partial_k q) - \partial_j(A_i^j \partial_t a_i^k \partial_k q) - \partial_j(A_i^j a_i^k \partial_k q_t)\\ &\quad - \partial_j(\partial_t \widetilde{a}_i^j A_i^k \partial_k q) - \partial_j(\widetilde{a}_i^j \partial_t A_i^k \partial_k q) - \partial_j(\widetilde{a}_i^j A_i^k \partial_t q_t)\\ &\quad - \partial_j(\partial_t(\widetilde{a}_i^j \widetilde{a}_i^k))\partial_k Q + \partial_j(\delta_{jk} - \widetilde{a}_i^j \widetilde{a}_i^k)\partial_k Q_t\\ &\quad - \partial_t(\widetilde{a}_i^j \widetilde{a}_i^k)\partial_{jk} Q + (\delta_{jk} - \widetilde{a}_i^j \widetilde{a}_i^k)\partial_{jk} Q_t\\ &\quad + \partial_{tt} A_i^j \partial_j v^i + \partial_t A_i^j \partial_j v_t^i + \partial_{tt}\widetilde{a}_i^j \partial_j V^i + \widetilde{a}_i^j \partial_t \partial_j V_t^i\end{aligned} \tag{5.23}$$

with a boundary condition

$$\partial_i Q_t N^i = -\partial_t A_i^k \partial_k q N^i - A_i^k \partial_k q_t N^i - \partial_t a_i^k \partial_k Q N^i + (\delta_{ki} - a_i^k)\partial_k Q_t N^i \text{ on } \Gamma_0. \tag{5.24}$$

Applying the elliptic estimate in H^2, we obtain after a short computation

$$\begin{aligned}\|Q_t\|_{H^2} &\le C\|A_t\|_{H^1} + C\|A\|_{H^1} + C\|Q_t\|_{H^1} + C\|Q\|_{H^2} + C\|a^T : a - I\|_{H^{1.5+\delta}}\|Q_t\|_{H^2}\\ &\quad + C\|\partial_{tt} A\|_{L^2} + C\|V\|_{H^1} + C\|V_t\|_{H^1} + C\|a - I\|_{H^{1.5+\delta}}\|Q_t\|_{H^2}.\end{aligned} \tag{5.25}$$

Differentiating (5.21) in t, we obtain

$$\begin{aligned}A_{tt} &= -A_t : \nabla v : a - A : \nabla v_t : a - A : \nabla v : a_t\\ &\quad - \widetilde{a}_t : \nabla V : a - \widetilde{a} : \nabla V_t : a - \widetilde{a} : \nabla V : a_t\\ &\quad - \widetilde{a}_t : \nabla \widetilde{v} : A - \widetilde{a} : \nabla \widetilde{v}_t : A - \widetilde{a} : \nabla \widetilde{v} : A_t.\end{aligned} \tag{5.26}$$

Applying a multiplicative Sobolev inequality, we obtain

$$\|A_{tt}\|_{L^2} \le C(\|A_t\|_{L^2} + \|A\|_{H^1} + \|V\|_{H^1} + \|V_t\|_{H^1}) \tag{5.27}$$

from where, using (5.22),

$$\|A_{tt}\|_{H^1} \le C(\|A\|_{H^1} + \|V\|_{H^2} + \|V_t\|_{H^1}). \tag{5.28}$$

The sum of the fifth and the last term in (5.25) is dominated by $Ct\|Q_t\|_{H^2}$, which can be absorbed if $T = 1/C$ with C sufficiently large. We get

$$\begin{aligned}\|Q_t\|_{H^2} &\le C(\|V\|_{H^1} + \|V_t\|_{H^1} + \|A\|_{H^1} + \|Q\|_{H^2})\\ &\le C\left(\|V\|_{H^1} + \|V_t\|_{H^1} + \|A\|_{H^1} + \int_0^t \|Q_t\|_{H^2}\,ds\right)\end{aligned} \tag{5.29}$$

on $[0,T]$, and the proof of Lemma 5.1 is complete. □

5.2. Tangential estimates. In this subsection, we derive the tangential estimates for V and E. Denote

$$R = -\Delta_2 = -\sum_{m=1}^{2} \partial_{mm}. \tag{5.30}$$

LEMMA 5.2. *For $t \in [0,T]$, we have*

$$\begin{aligned} &\|RV(t)\|_{L^2}^2 + \|a_l^3(t) RE^l(t)\|_{L^2(\Gamma_1)}^2 \\ &\quad \le C\|E\|_{H^{2.5}} \int_0^t \|V\|_{H^2}\,ds + C\int_0^t \left(\|V\|_{H^2}^2 + \|E\|_{H^{2.5}}^2 + \|Q\|_{H^{2.5}}^2\right) ds \end{aligned} \tag{5.31}$$

for $t \in [0,T]$.

Proof of Lemma 5.2. The difference V of the two solutions satisfies the equation

$$V_t^i + A_i^k \partial_k q + \widetilde{a}_i^k \partial_k Q = 0. \tag{5.32}$$

Applying R to the equation (5.32) and multiplying it scalarly by RV, we get

$$\frac{1}{2}\frac{d}{dt}\|RV\|_{L^2}^2 = -\int_\Omega R(A_i^k \partial_k q) RV^i\,dx - \int_\Omega R(\widetilde{a}_i^k \partial_k Q) RV^i\,dx \tag{5.33}$$

from where

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|RV\|_{L^2}^2 &= -\int_\Omega RA_i^k \partial_k q RV^i\,dx - \int_\Omega \widetilde{a}_i^k \partial_k RQRV^i\,dx \\ &\quad - \int_\Omega \left(R(A_i^k\partial_k q) - RA_i^k \partial_k q\right) RV^i\,dx \\ &\quad - \int_\Omega \left(R(\widetilde{a}_i^k \partial_k Q)RV^i - \widetilde{a}_i^k \partial_k RQRV^i\right) dx \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{5.34}$$

In order to bound the first term I_1, we need the identity

$$\partial_m a_i^k = -a_l^k \partial_s \partial_m \eta^l a_i^s, \qquad m = 1,2 \tag{5.35}$$

which is obtained by applying ∂_m to $a : \nabla\eta = I$. Differentiating (5.35) leads to

$$\partial_{mm} a_i^k = -\partial_m a_l^k \partial_s \partial_m \eta^l a_i^s - a_l^k \partial_s \partial_{mm} \eta^l a_i^s - a_l^k \partial_s \partial_m \eta^l \partial_m a_i^s, \qquad m = 1,2. \tag{5.36}$$

Subtracting the analogous equation for $\widetilde{a}_{mm}$ we obtain

$$\begin{aligned} \partial_{mm} A_i^k = &-\partial_m A_l^k \partial_s \partial_m \eta^l a_i^s - \partial_m \widetilde{a}_l^k \partial_s \partial_m E^l a_i^s - \partial_m \widetilde{a}_l^k \partial_s \partial_m \widetilde{\eta}^l A_i^s \\ &- A_l^k \partial_s \partial_{mm} \eta^l a_i^s - \widetilde{a}_l^k \partial_s \partial_{mm} E^l a_i^s - \widetilde{a}_l^k \partial_s \partial_{mm} \widetilde{\eta}^l A_i^s \\ &- A_l^k \partial_s \partial_m \eta^l \partial_m a_i^s - \widetilde{a}_l^k \partial_s \partial_m E^l \partial_m a_i^s - \widetilde{a}_l^k \partial_s \partial_m \widetilde{\eta}^l \partial_m A_i^s \end{aligned} \tag{5.37}$$

for every fixed $m \in \{1,2\}$. Only the fifth term on the right side, $-\widetilde{a}_l^k \partial_s \partial_{mm} E^l a_i^s$, needs special treatment. Therefore, we write

$$\partial_{mm} A_i^k = -\widetilde{a}_l^k \partial_s \partial_{mm} E^l a_i^s + R_{ikm}, \qquad m = 1,2 \tag{5.38}$$

where R_{ikm} denotes the sum of all the terms on the right side of (5.37) other than fifth. Using (5.38), we rewrite $I_1 = -\int_\Omega RA_i^k \partial_k q RV^i \, dx = \sum_{m=1}^2 \int_\Omega \partial_{mm} A_i^k \partial_k q RV^i \, dx$ as

$$\begin{aligned} I_1 &= -\sum_{m=1}^2 \int_\Omega \widetilde{a}_l^k \partial_s \partial_{mm} E^l a_i^s \partial_k q RV^i \, dx + \sum_{m=1}^2 \int_\Omega R_{ikm} \partial_k q RV^i \, dx \\ &= I_{11} + I_{12}. \end{aligned} \tag{5.39}$$

In order to treat the first term, I_{11}, we integrate by parts in the x_s variable and obtain

$$\begin{aligned} I_{11} &= \sum_{m=1}^2 \int_\Omega \partial_s \widetilde{a}_l^k \partial_{mm} E^l a_i^s \partial_k q RV^i \, dx + \sum_{m=1}^2 \int_\Omega \widetilde{a}_l^k \partial_{mm} E^l a_i^s \partial_s \partial_k q RV^i \, dx \\ &\quad + \sum_{m=1}^2 \int_\Omega \widetilde{a}_l^k \partial_{mm} E^l a_i^s \partial_k q \partial_s RV^i \, dx - \sum_{m=1}^2 \int_{\Gamma_1} \widetilde{a}_l^k \partial_{mm} E^l a_i^s \partial_k q RV^i N^s \, d\sigma(x) \\ &\quad - \sum_{m=1}^2 \int_{\Gamma_0} \widetilde{a}_l^k \partial_{mm} E^l a_i^s \partial_k q RV^i N^s \, d\sigma(x) \\ &= I_{111} + I_{112} + I_{113} + I_{114} + I_{115}. \end{aligned} \tag{5.40}$$

For I_{111}, we use a multiplicative Hölder inequality to obtain

$$I_{111} \leq C \|\nabla a\|_{H^{1.5+\delta}} \|RE\|_{L^2} \|a\|_{H^{1.5+\delta}} \|\nabla q\|_{H^{1.5+\delta}} \|RV\|_{L^2} \leq C \|E\|_{H^2} \|V\|_{H^2} \tag{5.41}$$

according to the convention on the generic constant C at the beginning of Section 5. Similarly, for I_{112}, we have

$$I_{112} \leq C \|\widetilde{a}_l^k\|_{H^{1.5+\delta}} \|\partial_{mm} E^l\|_{H^{0.5}} \|a_i^s\|_{H^{1.5+\delta}} \|\partial_s \partial_k q\|_{H^{1+\delta}} \|RV^i\|_{L^2} \leq C \|E\|_{H^{2.5}} \|V\|_{H^2}. \tag{5.42}$$

The third term I_{113} requires more care due to the extra derivative on RV; thus we use the divergence-free condition to reduce its order. First, we write

$$\begin{aligned} I_{113} &= -\sum_{m,n=1}^2 \int_\Omega \widetilde{a}_l^k \partial_{mm} E^l \partial_k q a_i^s \partial_{snn} (v^i - \widetilde{v}^i) \, dx \\ &= -\sum_{m,n=1}^2 \int_\Omega \widetilde{a}_l^k \partial_{mm} E^l \partial_k q \partial_{nn} (a_i^s \partial_s (v^i - \widetilde{v}^i)) \, dx + \sum_{m,n=1}^2 \int_\Omega \widetilde{a}_l^k \partial_{mm} E^l \partial_k q \partial_{nn} a_i^s \partial_s (v^i - \widetilde{v}^i) \, dx \\ &\quad + 2 \sum_{m,n=1}^2 \int_\Omega \widetilde{a}_l^k \partial_{mm} E^l \partial_k q \partial_n a_i^s \partial_{ns} (v^i - \widetilde{v}^i) \, dx \\ &= I_{1131} + I_{1132} + I_{1133}. \end{aligned} \tag{5.43}$$

Using $a_i^s \partial_s v^i = 0$ and $\widetilde{a}_i^s \partial_s \widetilde{v}^i = 0$, we get

$$\begin{aligned}
I_{1131} &= \sum_{m,n=1}^{2} \int_\Omega \widetilde{a}_l^k \partial_{mm} E^l \partial_k q \partial_{nn} a_i^s \widetilde{v}^i \, dx = \sum_{m,n=1}^{2} \int_\Omega \widetilde{a}_l^k \partial_{mm} E^l \partial_k q \partial_{nn} (a_i^s \widetilde{v}^i - \widetilde{a}_i^s \widetilde{v}^i) \, dx \\
&= \sum_{m,n=1}^{2} \int_\Omega \widetilde{a}_l^k \partial_{mm} E^l \partial_k q \partial_{nn} A_i^s \widetilde{v}^i \, dx + 2 \sum_{m,n=1}^{2} \int_\Omega \widetilde{a}_l^k \partial_{mm} E^l \partial_k q \partial_n A_i^s \partial_n \widetilde{v}^i \, dx \\
&\quad + \sum_{m,n=1}^{2} \int_\Omega \widetilde{a}_l^k \partial_{mm} E^l \partial_k q A_i^s \partial_{nn} \widetilde{v}^i \, dx \\
&= \sum_{m,n=1}^{2} \int_\Omega (I - \Delta_2)^{1/4} (\widetilde{a}_l^k \partial_{mm} E^l \partial_k q \widetilde{v}^i) (I - \Delta_2)^{-1/4} \partial_{nn} A_i^s \, dx \\
&\quad + 2 \sum_{m,n=1}^{2} \int_\Omega \widetilde{a}_l^k \partial_{mm} E^l \partial_k q \partial_n A_i^s \partial_n \widetilde{v}^i \, dx \\
&\quad + \sum_{m,n=1}^{2} \int_\Omega \widetilde{a}_l^k \partial_{mm} E^l \partial_k q A_i^s \partial_{nn} \widetilde{v}^i \, dx. \qquad (5.44)
\end{aligned}$$

All three terms on the far right side are bounded by $C\|E\|_{H^{2.5}}\|A\|_{H^{1.5}}$ using multiplicative Sobolev inequalities. Therefore,

$$I_{1131} \leq C\|E\|_{H^{2.5}}\|A\|_{H^{1.5}} \leq C\|E\|_{H^{2.5}}^2 \qquad (5.45)$$

where we used (5.20) in the last step. The term $I_{114} = \int_{\Gamma_1} \widetilde{a}_l^k R E^l a_i^s \partial_k q R E_t^i N^s \, d\sigma(x)$ requires the use of the Taylor condition. First, by

$$\widetilde{a}_l^k R E^l = a_l^k R E^l - A_l^k R \eta^l + A_l^k R \widetilde{\eta}^l \qquad (5.46)$$

we have

$$\begin{aligned}
I_{114} &= \int_{\Gamma_1} a_l^k R E^l a_i^s \partial_k q R E_t^i N^s \, d\sigma(x) - \int_{\Gamma_1} A_l^k R \eta^l a_i^s \partial_k q R E_t^i N^s \, d\sigma(x) \\
&\quad + \int_{\Gamma_1} A_l^k R \widetilde{\eta}^l a_i^s \partial_k q R E_t^i N^s \, d\sigma(x) \\
&= J_1 + J_2 + J_3. \qquad (5.47)
\end{aligned}$$

We may easily check that

$$J_2, J_3 \leq \|A\|_{H^{1.5}}\|E\|_{H^{2.5}}. \qquad (5.48)$$

The first term on the right side of (5.47) may be rewritten as

$$\begin{aligned}
J_1 &= \frac{1}{2}\frac{d}{dt} \int_{\Gamma_1} a_l^k R E^l a_i^s \partial_k q R E^i N^s \, d\sigma(x) - \int_{\Gamma_1} \partial_t a_l^k R E^l a_i^s \partial_k q R E^i N^s \, d\sigma(x) \\
&\quad - \frac{1}{2} \int_{\Gamma_1} a_l^k R E^l a_i^s \partial_k q_t R E^i N^s \, d\sigma(x) \\
&= J_{11} + J_{12} + J_{13}. \qquad (5.49)
\end{aligned}$$

Using trace and Sobolev multiplicative inequalities, we have

$$J_{12} + J_{13} \leq C\|E\|_{H^{2.5}}^2. \tag{5.50}$$

On the other hand, we have

$$\begin{aligned}\int_0^t J_{11}(x,s)\,ds &= \frac{1}{2}\int_{\Gamma_1} a_l^3 RE^l a_i^3 \partial_3 q RE^i\,d\sigma(x)\,\Big|_t - \frac{1}{2}\int_{\Gamma_1} a_l^3 RE^l a_i^3 \partial_3 q RE^i\,d\sigma(x)\,\Big|_0\\ &= \frac{1}{2}\int_{\Gamma_1} a_l^3 RE^l a_i^3 \partial_3 q RE^i\,d\sigma(x)\,\Big|_t\,. \end{aligned}\tag{5.51}$$

Using the Taylor sign condition (5.1), we get

$$J_1 \leq -\frac{1}{C}\|a_l^3 RE^l\|_{L^2}^2. \tag{5.52}$$

which yields the second term on the left side of the inequality (5.31). Next, as in [**KT2**], we have

$$I_{115} = \sum_{m=1}^{2}\int_{\Gamma_0} \widetilde{a}_l^k \partial_{mm} E^l a_i^s \partial_k q RV^i N^s\,d\sigma(x) = 0 \tag{5.53}$$

since $a_1^3 = a_2^3 = 0$ and $v_3 = 0$ on Γ_0. In order to complete the treatment of I_1, we estimate

$$I_{12} \leq C(\|E\|_{H^{2.5}} + \|A\|_{H^{1.5}})\|V\|_{H^2} \leq C\|E\|_{H^{2.5}}\|V\|_{H^2} \tag{5.54}$$

with the help of (5.20). Therefore, we conclude

$$\int_0^t I_1\,ds \leq -\frac{1}{C}\|a_l^3 RE^l\|_{L^2}^2 + C\|E\|_{H^{2.5}}\int_0^t \|V\|_{H^2}\,ds + C\int_0^t \big(\|V\|_{H^2}^2 + \|E\|_{H^{2.5}}^2\big)\,ds. \tag{5.55}$$

Next, we consider the second term in (5.34),

$$I_2 = -\int_\Omega \widetilde{a}_i^k \partial_k RQRV^i\,dx = \int_\Omega \widetilde{a}_i^k RQ\partial_k RV^i\,dx \tag{5.56}$$

where we used the Piola identity $\partial_k \widetilde{a}_i^k = 0$ for $i = 1, 2, 3$. Using $a_i^s \partial_s v^i = 0$ and $\widetilde{a}_i^s \partial_s \widetilde{v}^i = 0$, we have

$$\widetilde{a}_i^k \partial_k V^i = -A_i^k \partial_k v^i \tag{5.57}$$

and thus we may rewrite

$$\begin{aligned}\widetilde{a}_i^k \partial_k RV^i &= R(\widetilde{a}_i^k \partial_k V^i) - R\widetilde{a}_i^k \partial_k V^i + 2\sum_{m=1}^{2} \partial_m \widetilde{a}_i^k \partial_{mk} V^i\\ &= -R(A_i^k \partial_k v^i) - R\widetilde{a}_i^k \partial_k V^i + 2\sum_{m=1}^{2} \partial_m \widetilde{a}_i^k \partial_{mk} V^i\\ &= -RA_i^k \partial_k v^i + 2\sum_{m=1}^{2} \partial_m A_i^k \partial_{mk} v^i - A_i^k R\partial_k v^i\\ &\quad - R\widetilde{a}_i^k \partial_k V^i + 2\sum_{m=1}^{2} \partial_m \widetilde{a}_i^k \partial_{mk} V^i. \end{aligned}\tag{5.58}$$

where we also used the product rule in the first and the third step. We split I_2 into a sum of five terms according to the far right side of (5.58) and name the terms I_{21}, I_{22}, I_{23}, I_{24}, and I_{25}. For $I_{21} = -\int_\Omega RQRA_i^k\partial_k v^i\,dx$, we write

$$I_{21} = -\int_\Omega (I-\Delta_2)^{1/4}(RQ\partial_k v^i)(I-\Delta_2)^{-1/4}RA_i^k\,dx \le C\|Q\|_{H^{2.5}}\|A\|_{H^{1.5}}. \tag{5.59}$$

For I_{22}, we simply use a multiplicative Sobolev inequality to obtain

$$I_{22} = 2\sum_{m=1}^{2}\int_\Omega RQ\partial_m A_i^k\partial_{mk}v^i\,dx \le C\|Q\|_{H^{2.5}}\|A\|_{H^{1.5}} \le C\|Q\|_{H^{2.5}}\|E\|_{H^{2.5}} \tag{5.60}$$

where we also used (5.20) in the last step. The third term, $I_{23} = -\int_\Omega RQA_i^kR\partial_k v^i\,dx$, is treated similarly to I_{21}, i.e.,

$$I_{23} = -\int_\Omega (I-\Delta_2)^{1/4}(RQA_i^k)(I-\Delta_2)^{-1/4}R\partial_k v^i\,dx \le C\|Q\|_{H^{2.5}}\|A\|_{H^{1.5}} \le C\|Q\|_{H^{2.5}}\|E\|_{H^{2.5}}. \tag{5.61}$$

Similarly

$$I_{24} = -\int_\Omega RQR\widetilde{a}_i^k\partial_k(v^i-\widetilde{v}^i)\,dx \le C\|Q\|_{H^{2.5}}\|V\|_{H^2} \tag{5.62}$$

and

$$I_{25} = 2\sum_{m=1}^{2}\int_\Omega RQ\partial_m\widetilde{a}_i^k\partial_{mk}V^i\,dx \le C\|Q\|_{H^{2.5}}\|V\|_{H^2}. \tag{5.63}$$

The terms I_3 and I_4 are lower order and all the terms which result may be treated using multiplicative Sobolev inequalities. We obtain

$$I_3 \le C\|A\|_{H^{1.5}}\|V\|_{H^2} \le C\|E\|_{H^{2.5}}\|V\|_{H^2} \tag{5.64}$$

and

$$I_4 \le C\|Q\|_{H^{2.5}}\|V\|_{H^2}. \tag{5.65}$$

Combining all the estimates on I_1 through I_4 we obtain the desired inequality (5.31). $\square$

5.3. Gradient estimates for E. Now, we obtain curl estimates for E using the Cauchy invariance. Subtracting (4.10) and the analogous equation for $\widetilde{\eta}$, i.e.,

$$\begin{aligned}\nabla(\operatorname{curl}\widetilde{\eta})^i &= \epsilon_{ijk}(\delta_{km}-\partial_k\widetilde{\eta}^m)\partial_j\nabla\widetilde{\eta}^m \\ &\quad + \int_0^t \Big(-\epsilon_{ijk}\partial_j\widetilde{v}^m\partial_k\nabla\widetilde{\eta}^m + \epsilon_{ijk}\partial_k\widetilde{v}^m\partial_j\nabla\widetilde{\eta}^m\Big)\,ds + (t\nabla\omega_0)^i\end{aligned} \tag{5.66}$$

we get

$$\begin{aligned}\nabla(\operatorname{curl}E)^i &= -\epsilon_{ijk}\partial_k E^m\partial_j\nabla\eta^m + \epsilon_{ijk}(\delta_{km}-\partial_k\widetilde{\eta}^m)\partial_j\nabla E^m \\ &\quad + \int_0^t \Big(-\epsilon_{ijk}\partial_j V^m\partial_k\nabla\eta^m + \epsilon_{ijk}\partial_k V^m\partial_j\nabla\eta^m\Big)\,ds \\ &\quad + \int_0^t \Big(-\epsilon_{ijk}\partial_j\widetilde{v}^m\partial_k\nabla E^m + \epsilon_{ijk}\partial_k\widetilde{v}^m\partial_j\nabla E^m\Big)\,ds.\end{aligned} \tag{5.67}$$

Applying the $H^{0.5}$ norms on both sides, we get

$$\begin{aligned}\|\nabla(\operatorname{curl} E)^i\|_{H^{0.5}} \le C\|E\|_{H^{2.5}} \sum_{j,k=1}^{3} \|\partial_{jk}\eta\|_{H^{0.5+\delta}} + C\|I-\nabla\eta\|_{H^{2.5+\delta}}\|E\|_{H^{2.5}} \\ + C\int_0^t \Big(\|V\|_{H^2}\|\eta\|_{H^{3+\delta}} + \|\widetilde{v}\|_{H^{2.5+\delta}}\|E\|_{H^{2.5}}\Big)\, ds. \qquad (5.68)\end{aligned}$$

The second factor in the first term is bounded by $\sum_{j,k=1}^3 \int_0^t \|\partial_{jk}\eta_t\|_{H^{0.5+\delta}}\, ds \le Ct$ and thus

$$\begin{aligned}\|\operatorname{curl} E\|_{H^{1.5}} &\le C\|\nabla E\|_{H^{0.5}} + Ct\|E\|_{H^{2.5}} + C\int_0^t \Big(\|V\|_{H^2} + \|E\|_{H^{2.5}}\Big)\, ds \\ &\le Ct\|E\|_{H^{2.5}} + C\int_0^t \Big(\|V\|_{H^2} + \|E\|_{H^{2.5}}\Big)\, ds \qquad (5.69)\end{aligned}$$

where we used $E_t = V$ in order to estimate $\|\nabla E\|_{H^{0.5}} \le \|E\|_{H^2} \le C\int_0^t \|V\|_{H^2}\, ds$.

Now, we proceed to estimate the divergence of E using the Lagrangian divergence conditions (2.5) and (5.7). From [**KT2**, p. 350], we get

$$(D_a\nabla\eta) = \int_0^t (D_{a_t}\nabla\eta + D_a\nabla v)\, ds \qquad (5.70)$$

where we denoted

$$D_a f = a_i^k \partial_k f^i. \qquad (5.71)$$

Writing the analogous equation for $\widetilde{v}$ and $\widetilde{\eta}$ and subtracting it from (5.70), we obtain

$$A_i^k\partial_k\nabla\eta^i + \widetilde{a}_i^k\partial_k\nabla E = \int_0^t \Big(\partial_t A_i^k\partial_k\nabla\eta^i + \partial_t\widetilde{a}_i^k\partial_k\nabla E^i + A_i^k\partial_k\nabla v^i + \widetilde{a}_i^k\partial_k\nabla V^i\Big)\, ds. \qquad (5.72)$$

The last term in the integral may be rewritten as

$$\begin{aligned}\widetilde{a}_i^k\partial_k\nabla V^i &= \widetilde{a}_i^k\partial_k\nabla v_i - \widetilde{a}_i^k\partial_k\nabla\widetilde{v}^i \\ &= \nabla\big(\widetilde{a}_i^k\partial_k v^i - \widetilde{a}_i^k\partial_k\widetilde{v}^i\big) - \partial_k v^i\nabla\widetilde{a}_i^k + \partial_k\widetilde{v}^i\nabla\widetilde{a}_i^k \\ &= \nabla\big((\widetilde{a}_i^k - a_i^k)\partial_k v^i\big) - \partial_k V^i\nabla\widetilde{a}_i^k. \qquad (5.73)\end{aligned}$$

Also, note that

$$\widetilde{a}_i^k\partial_k\nabla E^i = \nabla\operatorname{div} E + (\widetilde{a}_i^k - \delta_{ik})\partial_k\nabla E^i. \qquad (5.74)$$

Using (5.21), (5.73), and (5.74) in (5.72), we get

$$\begin{aligned}\nabla\operatorname{div} E &= \widetilde{a}_i^k\partial_k\nabla E^i - (\widetilde{a}_i^k - \delta_{ik})\partial_k\nabla E^i \\ &= -A_i^k\partial_k\nabla\eta^i - \left(\int_0^t \partial_t(\widetilde{a}_i^k)\, ds\right)\partial_k\nabla E^i \\ &\quad + \int_0^t \Big(\partial_t A_i^k\partial_k\nabla\eta^i + \partial_t\widetilde{a}_i^k\partial_k\nabla E^i + A_i^k\partial_k\nabla v^i + \widetilde{a}_i^k\partial_k\nabla V^i\Big)\, ds \qquad (5.75)\end{aligned}$$

and thus, taking the $H^{0.5}$ norms of both sides

$$\begin{aligned}\|\operatorname{div} E\|_{H^{1.5}} &\leq C\|\nabla E\|_{H^{0.5}} + C\|A\|_{H^1} + Ct\|E\|_{H^{2.5}} + \int_0^t \bigl(\|A_t\|_{H^1} + \|E\|_{H^{2.5}} + \|V\|_{H^2}\bigr)\,ds \\ &\leq C\int_0^t \|\nabla E_t\|_{H^{0.5}}\,ds + C\int_0^t \|A_t\|_{H^1}\,ds + Ct\|E\|_{H^{2.5}} + \int_0^t \bigl(\|E\|_{H^{2.5}} + \|V\|_{H^2}\bigr)\,ds \\ &\leq Ct\|E\|_{H^{2.5}} + \int_0^t \bigl(\|E\|_{H^{2.5}} + \|V\|_{H^2}\bigr)\,ds \end{aligned} \tag{5.76}$$

where we used (5.20) and (5.22) in the last step.

In order to obtain an estimate for $\|E\|_{H^{2.5}}$, we use (4.15) with $f = E$ and $s = 2.5$. For the boundary term, we write

$$\|\nabla_2 E \cdot N\|_{H^1(\Gamma_0\cup\Gamma_1)} = \|\nabla_2 E^3\|_{H^1(\Gamma_0\cup\Gamma_1)} \leq C\|RE^3\|_{L^2(\Gamma_0\cup\Gamma_1)}. \tag{5.77}$$

Since $a_l^3 RE^l = RE^3 + (a_l^3 - \delta_{l3})RE^l$, we get

$$\begin{aligned}\|RE^3\|_{L^2(\Gamma_0\cup\Gamma_1)} &\leq \|a_l^3 RE^l\|_{L^2(\Gamma_0\cup\Gamma_1)} + \|(a_l^3 - \delta_{l3})RE^l\|_{L^2(\Gamma_0\cup\Gamma_1)} \\ &\leq \|a_l^3 RE^l\|_{L^2(\Gamma_0\cup\Gamma_1)} + \|a - I\|_{L^\infty}\|RE\|_{L^2(\Gamma_0\cup\Gamma_1)} \\ &\leq \|a_l^3 RE^l\|_{L^2(\Gamma_0\cup\Gamma_1)} + Ct\|E\|_{H^{2.5}} \end{aligned} \tag{5.78}$$

where we used $\|RE\|_{L^2(\Gamma_0\cup\Gamma_1)} \leq C\|E\|_{H^2(\Gamma_0\cup\Gamma_1)} \leq C\|E\|_{H^{2.5}}$ in the last step. By (4.15), we then get

$$\|E\|_{H^{2.5}} \leq Ct\|E\|_{H^{2.5}} + C\int_0^t \Bigl(\|V\|_{H^2} + \|E\|_{H^{2.5}}\Bigr)\,ds + C\|a_l^3 RE^l\|_{L^2(\Gamma_0\cup\Gamma_1)}. \tag{5.79}$$

Assuming that $T = 1/C$ with C sufficiently large, we get

$$\|E\|_{H^{2.5}} \leq C\int_0^t \Bigl(\|V\|_{H^2} + \|E\|_{H^{2.5}}\Bigr)\,ds + C\|a_l^3 RE^l\|_{L^2(\Gamma_0\cup\Gamma_1)} \tag{5.80}$$

on $[0, T]$.

5.4. Gradient estimates for V. From (4.19) and the analogous estimate

$$(\operatorname{curl}\widetilde{v})^i = \epsilon_{ijk}\partial_j \widetilde{v}^m(\delta_{km} - \partial_k \widetilde{\eta}^m) + \omega_0^i, \qquad i = 1,2,3 \tag{5.81}$$

we obtain

$$(\operatorname{curl} V)^i = \epsilon_{ijk}\partial_j V^m(\delta_{km} - \partial_k \eta^m) - \epsilon_{ijk}\partial_j \widetilde{v}^m \partial_k E^m, \qquad i = 1,2,3 \tag{5.82}$$

from where

$$\|\operatorname{curl} V\|_{H^1} \leq C\|V\|_{H^2}\|I - \nabla\eta\|_{H^{1.5+\delta}} + C\|\nabla\widetilde{v}\|_{H^{2+\delta}}\|E\|_{H^{2.5}}. \tag{5.83}$$

Since $\|I - \nabla\eta\|_{H^{1.5+\delta}} \leq \int_0^t \|\nabla\eta_t\|_{H^{1.5+\delta}} \leq Ct$, we get

$$\|\operatorname{curl} V\|_{H^1} \leq Ct\bigl(\|V\|_{H^2} + \|E\|_{H^{2.5}}\bigr). \tag{5.84}$$

Next, we need an estimate for the divergence of V. Subtracting $a_i^k \partial_k v^i = 0$ and $\widetilde{a}_i^k \partial_k \widetilde{v}^i = 0$ we obtain

$$A_i^k \partial_k v^i + \widetilde{a}_i^k \partial_k V^i = 0 \tag{5.85}$$

from where

$$\operatorname{div} V = (\delta_{ki} - \widetilde{a}_i^k)\partial_k V^i - A_i^k \partial_k v^i. \tag{5.86}$$

Applying the H^1 norm on both sides, we get

$$\|\operatorname{div} V\|_{H^1} \le C\|I-\widetilde{a}\|_{H^{1.5+\delta}}\|V\|_{H^2} + C\|A\|_{H^{1.5}}\|v\|_{H^{2+\delta}} \tag{5.87}$$

and this leads to

$$\|\operatorname{div} V\|_{H^1} \le Ct\big(\|V\|_{H^2} + \|A\|_{H^{1.5}}\big) \tag{5.88}$$

since $\|1-\widetilde{a}\|_{H^{1.5+\delta}} \le \int_0^t \|\widetilde{a}_t\|_{H^{1.5+\delta}}\,ds \le Ct$ and $\|v\|_{H^{2+\delta}} \le C\int_0^t \|v_t\|_{H^{2+\delta}} \le Ct$. Using (4.15) with $s=2$, we get

$$\|V\|_{H^2} \le C\|V\|_{L^2(\Omega)} + C\|\operatorname{curl} V\|_{H^1(\Omega)} + C\|\operatorname{div} V\|_{H^1(\Omega)} + C\|\nabla_2 V_3\|_{H^{0.5}(\Gamma_0\cup\Gamma_1)}. \tag{5.89}$$

Since

$$\begin{aligned}\|\nabla_2 V_3\|_{H^{0.5}(\Gamma_0\cup\Gamma_1)} &\le \|\nabla_2 V_3\|_{H^1} \le C\|\nabla_2 V_3\|_{L^2} + C\|RV_3\|_{L^2} + C\|\partial_3\nabla_2 V_3\|_{L^2}\\ &\le C\|RV\|_{L^2} + C\|\nabla_2\partial_3 V_3\|_{L^2}\end{aligned} \tag{5.90}$$

we get

$$\|V\|_{H^2} \le C\|\operatorname{curl} V\|_{H^1(\Omega)} + C\|\operatorname{div} V\|_{H^1(\Omega)} + C\|RV\|_{L^2} + C\|\nabla_2\partial_3 V_3\|_{L^2}. \tag{5.91}$$

When estimating the term analogous to $\nabla_2\partial_3 v_3$ while establishing existence, the proof in [**KT2**] used the divergence condition. Here we write instead

$$\begin{aligned}\|\nabla_2\partial_3 V_3\|_{L^2} &\le \|\nabla_2 \operatorname{div} V\|_{L^2} + \sum_{m=1}^{2}\|\nabla_2\partial_m V_m\|_{L^2}\\ &\le C\|RV\|_{L^2} + Ct\big(\|V\|_{H^2} + \|A\|_{H^{1.5}}\big) + C\|RV\|_{L^2}\end{aligned} \tag{5.92}$$

where we used (5.88) in the last step. Applying the inequalities (5.84), (5.88), and (5.92) in (5.91) leads to

$$\|V\|_{H^2} \le Ct\big(\|V\|_{H^2} + \|E\|_{H^{2.5}}\big) + C\|RV\|_{L^2}. \tag{5.93}$$

Since we have assumed that $T=1/C$ with C sufficiently large constant C, we obtain

$$\|V\|_{H^2} \le Ct\|E\|_{H^{2.5}} + C\|RV\|_{L^2} \tag{5.94}$$

on $[0,T]$.

5.5. Conclusion of the proof of uniqueness. We are still missing an estimate for the L^2 norm of V. From (5.32), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|V\|_{L^2}^2 &\le C\|A\|_{L^2}\|V\|_{L^2} + C\|\nabla Q\|_{L^2}\|V\|_{L^2}\\ &\le C\|V\|_{L^2}\int_0^t \|A_t\|_{L^2}\,ds + \|Q\|_{H^1}\int_0^t \|\nabla V_t\|_{L^2}\,ds\\ &\le C\|V\|_{L^2}\int_0^t \|A_t\|_{L^2}\,ds + \|Q\|_{H^1}\int_0^t \|Q\|_{H^2}\,ds.\end{aligned} \tag{5.95}$$

Now, we are ready to collect all the estimates and conclude the proof of uniqueness. Introduce the the quantities

$$X(t) = \|V\|_{H^2}^2 + \|E\|_{H^{2.5}}^2 \tag{5.96}$$

and

$$Y(t) = \|Q\|^2_{H^{2.5}} + \|Q_t\|^2_{H^2}. \tag{5.97}$$

Without loss of generality, we may assume $T \leq 1$. First, using (5.3) and (5.4), we get

$$Y(t) \leq CX(t) + C\int_0^t Y(s)\,ds. \tag{5.98}$$

On the other hand, (5.31) together with (5.80), (5.94), and (5.95), assuming that $T \leq 1/C$ to absorb the first term on the right side of (5.94), gives

$$X(t) \leq CX(t)^{1/2}\int_0^t X(s)^{1/2}\,ds + C\int_0^t \bigl(X(s)+Y(s)\bigr)\,ds \tag{5.99}$$

from where

$$X(t) \leq C\int_0^t \bigl(X(s)+Y(s)\bigr)\,ds. \tag{5.100}$$

Let $\epsilon \in (0,1]$. Multiplying (5.98) by ϵ and adding to (5.100), we get

$$X(t)+\epsilon Y(t) \leq C\epsilon X(t) + C\int_0^t \bigl(X(s)+Y(s)\bigr)\,ds \leq C\epsilon X(t) + \frac{C}{\epsilon}\int_0^t \bigl(X(s)+\epsilon Y(s)\bigr)\,ds. \tag{5.101}$$

Choosing ϵ so that the first term on the far right side can be absorbed and then applying the standard Gronwall argument leads to $X(t)=Y(t)=0$ for $t\in[0,T]$, and the proof of uniqueness is completed.

Appendix A. Appendix

For convenience, we provide here two proofs of the Cauchy invariance identity

$$\epsilon_{ijk}\partial_j v^m \partial_k \eta^m = \omega_0^i, \qquad t \geq 0 \tag{A.1}$$

The first proof, which establishes also the Weber formula, is from [**Ca, FV, ZF**], however rewritten in the coordinate notation used in the present paper. The second proof, which we believe is new, is shorter and bypasses the Weber formula.

Proof 1: We start with the Weber formula [**C2, Web**]

$$\partial_t(v^j\partial_k\eta^j) = \partial_k\left(\frac{1}{2}|v|^2 - q\right), \qquad i=1,2,3 \tag{A.2}$$

which is proven as follows. The left side equals

$$\begin{aligned} v_t^j\partial_k\eta^j + v^j\partial_k\eta_t^j &= -a_j^m\partial_m q\partial_k\eta^j + v^j\partial_k v^j \\ &= -\partial_k q + \frac{1}{2}\partial_k(|v|^2) = \partial_k\left(\frac{1}{2}|v|^2 - q\right) \end{aligned} \tag{A.3}$$

where we used

$$a_j^m\partial_k\eta^j = \delta_{jk}, \qquad j,k=1,2,3. \tag{A.4}$$

Applying the curl operator to the identity (A.2), rewritten as

$$\partial_t\begin{pmatrix} v^j\partial_1\eta^j \\ v^j\partial_2\eta^j \\ v^j\partial_3\eta^j \end{pmatrix} = \nabla\left(\frac{1}{2}|v|^2 - q\right), \tag{A.5}$$

we get

$$\partial_t\left(\epsilon_{ijk}\partial_j(v^m\partial_k\eta^m)\right) = 0. \tag{A.6}$$

Note that

$$\partial_t \left(\epsilon_{ijk} v^m \partial_{jk} \eta^m\right) = 0 \tag{A.7}$$

since

$$\epsilon_{ijk} v^m \partial_{jk} \eta^m = \epsilon_{ijk} v^m \partial_{kj} \eta^m = -\epsilon_{ikj} v^m \partial_{kj} \eta^m \tag{A.8}$$

using $\partial_{jk} = \partial_{kj}$ in the first equality and $\epsilon_{ijk} = -\epsilon_{ikj}$ in the second. By (A.6) and (A.7), we obtain

$$\partial_t \left(\epsilon_{ijk} \partial_j v^m \partial_k \eta^m\right) = 0. \tag{A.9}$$

The expression in parentheses at $t = 0$ equals

$$\epsilon_{ijk} \partial_j v^m(0) \partial_k \eta^m(0) = \omega_0^i, \qquad t \geq 0 \tag{A.10}$$

and thus (A.1) follows.

Proof 2: Taking the time derivative, we obtain

$$\begin{aligned} \partial_t(\epsilon_{ijk} \partial_j v^m \partial_k \eta^m) &= \epsilon_{ijk} \partial_j v^m \partial_k v^m + \epsilon_{ijk} \partial_j v_t^m \partial_k \eta^m \\ &= 0 - \epsilon_{ijk} \partial_j (a_m^l \partial_l q) \partial_k \eta^m \end{aligned} \tag{A.11}$$

where we replaced v_t^m by $-a_m^l \partial_l q$ using the Euler equation. Now, by $\partial_j a_m^l = a_s^l \partial_{jr} \eta^s a_m^r$, which follows by differentiating $a : \nabla \eta = I$, we get

$$\begin{aligned} \partial_t(\epsilon_{ijk} \partial_j v^m \partial_k \eta^m) &= -\epsilon_{ijk} a_m^l \partial_{jl} q \partial_k \eta^m - \epsilon_{ijk} a_s^l \partial_{jr} \eta^s a_m^r \partial_l q \partial_k \eta^m \\ &= -\epsilon_{ijk} \partial_{jl} q \delta_{kl} - \epsilon_{ijk} a_s^l \partial_{jr} \eta^s \partial_l q \delta_{kr} \\ &= -\epsilon_{ijk} \partial_{jk} q - \epsilon_{ijk} a_s^l \partial_{jk} \eta^s \partial_l q \\ &= 0 + 0 = 0 \end{aligned} \tag{A.12}$$

where we used in the second equality $a : \nabla \eta = I$. Hence,

$$\epsilon_{ijk} \partial_j v^m \partial_k \eta^m = \epsilon_{ijk} \partial_j v_0^m \partial_k \eta^m(0) = \omega_0^i$$

and the proof of (A.1) is concluded.

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