On some electroconvection models

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ABSTRACT. We consider a model of electroconvection motivated by studies of the motion of a two dimensional annular suspended smectic film under the influence of an electric potential maintained at the boundary by two electrodes. We prove that this electroconvection model has global in time unique smooth solutions.
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1. Introduction

Electroconvection is the flow of fluids and particles driven by electrical forces. There are several studies of electroconvection in the physical literature, pertaining to different types of occurrences of the phenomenon. The interaction of electromagnetic fields with condensed matter is a vast and important subject, with applications ranging from solar magnetohydrodynamics to microfluidics.

Here we discuss a particular system, in which a charge distribution interacts with a fluid in a geometrically constrained situation. The fluid is confined to a very thin region, and a voltage difference is maintained by electrodes situated at the boundaries of the region. Physical experiments [**7**] and numerical studies [**11**] consider the flow of an annular suspended smectic film. Despite the non-Newtonian nature of the constituent, the simplest model describes the fluid by Navier-Stokes equations confined to a fixed two dimensional region (an annulus in the cited studies). The Navier-Stokes equations are driven by body forces due to the electrical charge density and the potential. The charge density is transported by the electric potential and by the flow. The electric potential is determined by three dimensional equations, but the fluid is confined to an approximately two dimensional space. This dimensional contrast leads naturally to nonlocal equations and is one of the most natural occurences of the Dirichlet-to-Neumann operator in fluid mechanics. In general charge densities evolve according to equations of the type

$$\partial_t q + \nabla \cdot (uq + \epsilon E) = 0 \tag{1.1}$$

where E is the electric field, ϵ is a 2-tensor and u is an advecting velocity. The electric field obeys the Gauss law

$$\nabla \cdot E = \rho \tag{1.2}$$

where ρ is the total charge density, and the magnetic effects are neglected

$$\nabla \times E = 0. \tag{1.3}$$

Because of the latter, the electric field can be expressed via a potential

$$E_i = \partial_i \Phi, \quad i = 1, 2, 3. \tag{1.4}$$

In our situation the total charge density is confined to a two dimensional region,

$$\rho = 2q\delta_\Omega \tag{1.5}$$

with $\Omega \subset \{(x, 0) \mid \ x \in \mathbb{R}^2\}$, and δ_Ω is the Dirac mass on Ω. The factor 2 is due to the fact that the film has two sides. The term $2q\delta_\Omega$ is the total charge density in the limit of zero thickness of the film. The potential obeys therefore

$$-\Delta_3\Phi = 2q\delta_\Omega \tag{1.6}$$

Here Δ_3 is the 3D Laplacian. All the rest of the derivatives below will be 2D. We tacitly identify $\mathbb{R}^2$ with $\{(x,0) \mid x \in \mathbb{R}^2\}$. Current is supplied by two electrodes maintained at different voltages:

$$\Phi_{\partial K_1} = V, \quad \Phi_{\partial K_2} = 0 \tag{1.7}$$

where K_1 and K_2 are disjoint electrodes, sharing boundaries with the fluid domain Ω. To be more specific, we consider a connected open domain Ω with smooth boundary in $\mathbb{R}^2$. The domain is not simply connected, and $\partial\Omega = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\mathrm{dist}(\Gamma_1, \Gamma_2) > 0$. (An annular region is such a domain). We solve the problem

$$-\Delta_3\Phi_2(x,z) = 2q(x)\delta_\Omega \tag{1.8}$$

for $x \in \Omega$, together with the boundary condition $\Phi_{2|\,\partial\Omega\times\mathbb{R}} = 0$, by setting

$$\Phi_2(x,z) = \begin{cases} (e^{-z\Lambda}\Lambda^{-1}q)(x), & \text{if } z > 0, \\ (e^{z\Lambda}\Lambda^{-1}q)(x), & \text{if } z < 0, \end{cases}$$

where Λ is the square root of the Dirichlet Laplacian in Ω. The potential Φ_2 has a jump singularity in the normal derivative at the points of continuity of q in $x \in \Omega$

$$-[\partial_z\Phi_2]_{|\,z=0} = 2q(x).$$

Here we denote the jump of a function f across $z = 0$ by $[f]_{|\,z=0} = \lim_{z\downarrow 0} f(z) - \lim_{z\uparrow 0} f(z)$. We take $K_1 = \Gamma_1 \times \mathbb{R}$ and $K_2 = \Gamma_2 \times \mathbb{R}$, i.e., we consider vertical electrodes at the boundaries of the domain. The harmonic function $\Phi_1(x)$, solving $\Delta\Phi_1 = 0$ in Ω and $\Phi_{1|\,\Gamma_1} = V$, $\Phi_{1|\,\Gamma_2} = 0$, also solves $\Delta_3\Phi_1 = 0$ in $\Omega \times \mathbb{R}$, with boundary conditions $\Phi_{1|\,K_1} = V$, $\Phi_{1|\,K_2} = 0$ in $\Omega \times \mathbb{R}$. The function Φ_1 is z-independent and harmonic in x. Thus,

$$\Phi = \Phi_1 + \Phi_2 \tag{1.9}$$

solves (1.6) with boundary conditions (1.7). The tensor ϵ is anisotropic, and, suppressing constants, it acts like a projection on two dimensions:

$$\epsilon E = (E_1, E_2, 0). \tag{1.10}$$

The charge density q evolves thus in time according to

$$\partial_t q + u \cdot \nabla q = \Delta\Phi_{|\,\Omega}. \tag{1.11}$$

The fluid obeys the two dimensional Navier-Stokes equations forced by the force $q\epsilon E$,

$$\partial_t u + u \cdot \nabla u - \Delta u + \nabla p = -q\nabla\Phi_{|\,\Omega}. \tag{1.12}$$

The fluid is two dimensional, incompressible

$$\nabla \cdot u = 0, \tag{1.13}$$

and adheres to the boundary, $u_{|\,\partial\Omega} = 0$. The system (1.11), (1.12), (1.13) becomes:

$$\begin{cases} \partial_t q + u \cdot \nabla q + \Lambda q = \Delta\Phi_1 \\ \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = -q\nabla\Lambda^{-1}q - q\nabla\Phi_1, \qquad \text{in } \Omega \\ \nabla \cdot u = 0, \end{cases} \tag{1.14}$$

with homogeneous Dirichlet boundary conditions for u on $\partial\Omega$. In the rest of the paper we study the regularity of solutions to (1.14). Our main result, Theorem 3 below, shows that if $u_0 \in H_0^1(\Omega) \cap H^2(\Omega) \cap \mathbb{P}(L^2(\Omega))$ and $q_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, then solutions to (1.14) exist for all time, are smooth, and uniquely determined by initial data.

Because Φ_1 is smooth, time independent, and fixed by the applied voltage, it can only influence the long time behavior and stability of solutions to (1.14), but not their regularity. For simplicity of exposition we take below $\Phi_1 = 0$. The regularity results are valid for arbitrary smooth Φ_1. A study of the forced long time behavior and of different geometric configurations will be conducted elsewhere.

The mathematical challenges are due to the presence of boundaries. Indeed, the system

$$\begin{cases} \partial_t q + u \cdot \nabla q + \Lambda q = 0, \\ \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = -qRq, \\ \nabla \cdot u = 0, \\ R = \nabla \Lambda^{-1}, \end{cases} \tag{1.15}$$

posed in the plane $\mathbb{R}^2$ or the torus $\mathbb{T}^2$ can be studied using tools which are well-established by now [**4**]. The first equation in (1.15) has a maximum principle in L^p [**10**], and thus, if q lies initially in $L^2 \cap L^\infty$, it remains bounded for all time in this space. Because of this bound and boundedness of the Riesz transforms $R = \nabla \Lambda^{-1}$ in L^p spaces, it follows that the forcing term in the second equation in (1.15) is bounded in L^p spaces, for all $p \in [1, \infty)$. The two dimensional Navier-Stokes equations with L^2 forces are known to behave well, and in particular weak solutions are known to be strong solutions

$$u \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2)$$

(cf. [**2**] for the torus; on the whole space this holds under additional mild cancellation conditions). It follows then that u is log-Lipschitz continuous, i.e.

$$|u(x,t) - u(y,t)| \le C(t)|x-y| \left(1 + \sqrt{\log_+ |x-y|^{-1}}\right)$$

with $\int_0^T C^2(t)dt < \infty$. This property feeds back in the first equation of (1.15), and in conjunction with the tools of [**4, 5**] yields the C^α regularity of q, for some $\alpha > 0$. This is done by differentiating the equation for q or by taking finite differences. Once C^α regularity for q is attained, due to the absence of boundaries, the problem becomes subcritical. The forcing term qRq in the 2D Navier-Stokes equation in (1.15) also has C^α regularity, which in turn confers higher regularity to the Navier-Stokes solution u, and so we can obtain even higher regularity for q. For instance, once u is Lipschitz continuous, it follows from the maximum principle obeyed by ∇q, that q is also Lipschitz continuous, and thus uniqueness and higher regularity is immediate.

This approach completely breaks down in the case of bounded domains because Λ is not translation invariant. In order to obtain regularity we need to consider commutators, and these are more expensive than C^α. The equation for q is quasilinear with critical dissipation, and simple fixed point methods do not work. In this paper we prove global regularity using a two-tier approximation procedure. The fundamental property obeyed by the equation for q, a maximum principle, is essential for the global bounds. This leads us to consider a system which couples an ODE, a Galerkin approximation for u, to a PDE, the transport equation for an approximate q. Establishing existence and uniform regularity for this good approximation requires an additional approximation, which however does not have a maximum principle.

2. Preliminaries

We consider the Dirichlet Laplacian in a bounded open domain $\Omega \subset \mathbb{R}^d$ with smooth boundary. We denote by Δ the Laplacian operator with homogeneous Dirichlet boundary conditions. Its $L^2(\Omega)$ - normalized eigenfunctions are denoted ϕ_j, and its eigenvalues counted with their multiplicities are denoted μ_j:

$$-\Delta \phi_j = \mu_j \phi_j. \tag{2.1}$$

It is well known that $0 < \mu_1 \le ... \le \mu_j \to \infty$ and that $-\Delta$ is a positive selfadjoint operator in $L^2(\Omega)$ with domain $\mathcal{D}(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$. Functional calculus is defined using the eigenfunction expansion. In particular

$$(-\Delta)^\alpha f = \sum_{j=1}^\infty \mu_j^\alpha f_j \phi_j \tag{2.2}$$

with

$$f_j = \int_\Omega f(y)\phi_j(y)dy$$

for $f \in \mathcal{D}\left((-\Delta)^{\alpha}\right) = \{f \mid \ (\mu_j^{\alpha} f_j) \in \ell^2(\mathbb{N})\}$. We denote by

$$\Lambda^s = (-\Delta)^{\frac{s}{2}} \tag{2.3}$$

the fractional powers of the Dirichlet Laplacian and by

$$\|f\|_{s,D}^2 = \sum_{j=1}^{\infty} \mu_j^s f_j^2. \tag{2.4}$$

the norms in $\mathcal{D}\left(\Lambda^s\right)$. It is well-known that

$$\mathcal{D}\left(\Lambda\right) = H_0^1(\Omega).$$

We recall the Córdoba-Córdoba inequality [**6**] for bounded domains, established in [**3**]:

PROPOSITION 1. *Let Φ be a C^2 convex function satisfying $\Phi(0) = 0$. Let $f \in C_0^{\infty}(\Omega)$ and let $0 \le s \le 2$. Then*

$$\Phi'(f)\Lambda^s f - \Lambda^s(\Phi(f)) \ge 0 \tag{2.5}$$

holds pointwise almost everywhere in Ω.

We use also the following commutator estimate proven in [**3**]:

THEOREM 1. *Let a vector field v have components in $B(\Omega)$ where $B(\Omega) = W^{2,d}(\Omega) \cap W^{1,\infty}(\Omega)$, if $d \ge 3$, and $B(\Omega) = W^{2,p}(\Omega)$ with $p > 2$, if $d = 2$. Assume that the normal component of the trace of v on the boundary vanishes,*

$$v_{|\,\partial\Omega} \cdot n = 0$$

(i.e the vector field is tangent to the boundary). There exists a constant C such that

$$\|[v\cdot\nabla, \Lambda]f\|_{\frac{1}{2},D} \le C\|v\|_{B(\Omega)}\|f\|_{\frac{3}{2},D} \tag{2.6}$$

holds for any f such that $f \in \mathcal{D}\left(\Lambda^{\frac{3}{2}}\right)$, where

$$\|v\|_{B(\Omega)} = \|v\|_{W^{2,d}(\Omega)} + \|v\|_{W^{1,\infty}(\Omega)}$$

if $d \ge 3$ and

$$\|v\|_{B(\Omega)} = \|v\|_{W^{2,p}(\Omega)}$$

with $p > 2$, if $d = 2$.

The result mentioned above is proved in [**3**] using the method of harmonic extension. It is used to prove an existence theorem for linear equations of transport and nonlocal diffusion [**3**]:

THEOREM 2. *Let $u \in L^2(0, T; B(\Omega)^d)$ be a vector field parallel to the boundary. Then the equation*

$$\partial_t\theta + u\cdot\nabla\theta + \Lambda\theta = 0 \tag{2.7}$$

with initial data $\theta_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ has unique solutions belonging to

$$\theta \in L^{\infty}(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap L^2(0, T; H^{2.5}(\Omega)).$$

If the initial data $\theta_0 \in L^p(\Omega)$, $1 \le p \le \infty$, then

$$\sup_{0\le t\le T} \|\theta(\cdot, t)\|_{L^p(\Omega)} \le \|\theta_0\|_{L^p(\Omega)} \tag{2.8}$$

holds.

We need also the fact that, for $d = 2$,

$$\|f\|_{L^4(\Omega)} \le C\|f\|_{\frac{1}{2},D}. \tag{2.9}$$

Background material and applications of the method of harmonic extension can be found in [**1**].

We recall now some basic notions concerning the Navier-Stokes equations [**2**]. The Stokes operator

$$A = -\mathbb{P}\Delta \tag{2.10}$$

is defined via the Leray-Hodge projector

$$\mathbb{P} : L^2(\Omega)^d \to H \tag{2.11}$$

where H is the closure in $L^2(\Omega)^d$ of the space of divergence-free $C_0^\infty(\Omega)$ vector fields:

$$H = \overline{\{u \mid \ u \in C_0^\infty(\Omega)^d, \ \nabla\cdot u = 0\}}^{L^2(\Omega)^d}. \tag{2.12}$$

The norm in H is the L^2 norm, $\|u\|_H = \|u\|_{L^2(\Omega)^d}$. The domain of A in L^2 is $\mathcal{D}(A) = H_0^1(\Omega)^d \cap H^2(\Omega)^d \cap H$. The operator is positive,

$$(Au, u)_H = \int_\Omega |\nabla u|^2 dx, \quad \forall\, u \in \mathcal{D}(A),$$

elliptic and injective,

$$\|u\|_{H^2(\Omega)^d} \le C\|Au\|_H, \tag{2.13}$$

and its inverse A^{-1} is compact. Functional calculus is defined using eigenfunction expansion. The eigenvalues of A are denoted λ_j, $0 < \lambda_1 \le \dots \lambda_j \le \dots \to \infty$, the eigenfunctions $w_j \in \mathcal{D}(A)$,

$$Aw_j = \lambda_j w_j.$$

The square root $A^{\frac{1}{2}}$ satisfies

$$\|A^{\frac{1}{2}}v\|_H = \|\nabla v\|_{L^2(\Omega)} \tag{2.14}$$

for any $v \in H \cap H_0^1(\Omega)^d$. The nonlinear term in the Navier-Stokes equations is

$$B(u, u) = \mathbb{P}(u\cdot\nabla u). \tag{2.15}$$

It has the property that

$$(B(u, u), u)_H = 0,$$

for all $u \in H \cap H_0^1(\Omega)^d$, $d \le 3$. In addition, for $d = 2$, using

$$\|u\|_{L^4(\Omega)} \le C\|u\|_{L^2(\Omega)}^{\frac{1}{2}}\|\nabla u\|_{L^2(\Omega)}^{\frac{1}{2}}, \tag{2.16}$$

$$\|\nabla u\|_{L^4(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)}^{\frac{1}{2}}\|u\|_{H^2(\Omega)}^{\frac{1}{2}}, \tag{2.17}$$

and the ellipticity of the Stokes operator, we obtain

$$\|B(u, u)\|_H \le C\|u\|_{L^2(\Omega)}^{\frac{1}{2}}\|\nabla u\|_{L^2(\Omega)}\|Au\|_H^{\frac{1}{2}}, \tag{2.18}$$

valid for all $u \in \mathcal{D}(A)$. Also, using elliptic regularity and interpolation, we have

$$\|A^{\frac{1}{2}}B(u, u)\|_H \le C\|u\|_H^{\frac{1}{2}}\|Au\|_H^{\frac{3}{2}} \tag{2.19}$$

for $u \in \mathcal{D}(A)$. Indeed, the right hand side bounds $\nabla(u\cdot\nabla u) = u\cdot\nabla\nabla u + \nabla u\nabla u$ in $L^2(\Omega)$, and, because $B(u, u) = u\cdot\nabla u + \nabla\pi$ with $-\Delta\pi = \nabla(u\cdot\nabla u)$, with homogeneous Neumann boundary condition $\frac{\partial\pi}{\partial n} = 0$ at $\partial\Omega$, it follows by elliptic regularity that $\nabla\nabla\pi$ obeys the same $L^2(\Omega)$ bound.

3. The reduced model

We consider the system formed by the incompressible Navier-Stokes equations

$$\partial_t u + u\cdot\nabla u - \Delta u + \nabla p = -qRq \tag{3.1}$$

with

$$\nabla\cdot u = 0, \tag{3.2}$$

and with

$$Rq = \nabla\Lambda^{-1}q, \tag{3.3}$$

coupled with the evolution of the charge density

$$\partial_t q + u\cdot\nabla q + \Lambda q = 0. \tag{3.4}$$

The equations hold for $x\in\Omega\subset\mathbb{R}^2$, $t\geq 0$, and the initial data u_0 and q_0 are smooth. Our main result is:

THEOREM 3. *Let $u_0\in\mathcal{D}(A)$, $q_0\in H_0^1(\Omega)\cap H^2(\Omega)$ and let T be arbitrary. Then, there exists a unique solution of the problem (3.1), (3.2), (3.3), (3.4) on the time interval $[0,T]$, which obeys*

$$u\in L^\infty(0,T;\mathcal{D}(A))\cap L^2(0,T;\mathcal{D}(A^{\frac{3}{2}})) \tag{3.5}$$

and

$$q\in L^\infty(0,T;\mathcal{D}(-\Delta))\cap L^2(0,T;\mathcal{D}(\Lambda^{\frac{5}{2}})) \tag{3.6}$$

with explicit bounds that depend only on the initial data and not on T. More precisely, there exists an explicit function of one variable, $C[N]$, with double exponential growth in N, such that

$$\begin{aligned}&\|u\|_{L^\infty(0,T;\mathcal{D}(A))\cap L^2(0,T;\mathcal{D}(A^{\frac{3}{2}}))}+\|q\|_{L^\infty(0,T;\mathcal{D}(-\Delta))\cap L^2(0,T;\mathcal{D}(\Lambda^{\frac{5}{2}}))}\\&\leq C\left[\|u_0\|_{\mathcal{D}(A)}+\|q_0\|_{\mathcal{D}(-\Delta)}\right].\end{aligned} \tag{3.7}$$

In order to prove the theorem, we construct solutions by an approximation procedure, prove a priori estimates on the approximants, and pass to the limit via the Aubin-Lions compactness theorem. This is a quasilinear situation with critical dissipation, and fixed point methods are not available.

We consider Galerkin approximations for u. These are defined using the projectors $\mathbb{P}_m$:

$$\mathbb{P}_m u = \sum_{j=1}^m (u,w_j)_H\, w_j. \tag{3.8}$$

We also consider Galerkin approximations for q, given by

$$P_n(f) = \sum_{j=1}^n (f,\phi_j)_{L^2(\Omega)}\,\phi_j. \tag{3.9}$$

The *final* approximate system is

$$\partial_t u_m + Au_m + \mathbb{P}_m(B(u_m,u_m)) = -\mathbb{P}_m(qRq) \tag{3.10}$$

for $u_m\in\mathbb{P}_mH$, coupled with

$$\partial_t q + u_m\cdot\nabla q + \Lambda q = 0 \tag{3.11}$$

with initial data $q_0\in H_0^1(\Omega)\cap H^2(\Omega)$. The initial data for u_m are

$$u_m(0) = \mathbb{P}_m u_0 \tag{3.12}$$

where $u_0\in\mathcal{D}(A)$ is the initial velocity in our problem. The system (3.10)–(3.11) is thus a system of nonlinear ordinary differential equations coupled to a linear transport and nonlocal diffusion partial differential equation. By Theorem 2 (see [**3**]) we would have that the linear equation (3.11) has unique solutions in

$$q\in L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega))\cap L^2(0,T;H^{2.5}(\Omega))$$

if $q_0\in H_0^1(\Omega)\cap H^2(\Omega)$, as long as $u_m\in L^2(0,T;B(\Omega)^d)$ is a vector field wich is parallel to the boundary. Because u_m is a finite linear combination of eigenfunctions of the Stokes operator, smooth and vanishing at the boundary, the only issue would be whether some norm of u_m stays finite and square integrable in time. In the final approximate system for q we choose the full PDE, rather than some approximation, in order to obtain the same a priori uniform L^∞ bounds for q as in the linear case. However, the system formed with (3.10) and (3.11) is nonlinear: u_m is not given, but rather computed. In order to investigate the *final* approximate system we need to consider a *preliminary* approximate system, which unfortunately does not behave well with respect to L^∞. This involves taking an additional approximation, at fixed m of (3.11) by eigenfunction expansions of the Laplacian. We prove energy bounds for the preliminary approximate systems, and pass to the limit to the final approximate system.

The preliminary approximate system has two parameters m and n, and is given by

$$\partial_t u_m + Au_m + \mathbb{P}_m(B(u_m, u_m)) = -\mathbb{P}_m(q_n R q_n) \tag{3.13}$$

for $u_m \in \mathbb{P}_m H$, with initial data $u_m(0) = \mathbb{P}_m u_0$, coupled with

$$\partial_t q_n + P_n(u_m \cdot \nabla q_n) + \Lambda q_n = 0 \tag{3.14}$$

with initial data $q_n(0) = P_n q_0$. The preliminary approximate system (3.13)–(3.14) is thus a system of ODEs which has solutions on a maximal time interval. In fact, the solutions are defined for all time because the good structure of the equations, which provides a cancellation and an a priori bound.

LEMMA 1. *Let T be arbitrary, $q_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_0 \in H \cap H_0^1(\Omega) \cap H^2(\Omega)$. Let m, n be positive integers and let (u_m, q_n) solve the system (3.13-(3.14) with initial data $u_m(0) = \mathbb{P}_m u_0$, $q_n(0) = P_n q_0$. Then, the solutions obey for all $t \in [0, T]$*

$$\begin{aligned} &\tfrac{1}{2}\left(\|u_m(t)\|_H^2 + \|q_n\|_{-\frac{1}{2},D}^2\right) + \textstyle\int_0^t \left(\|u_m(s)\|_{H^1}^2 + \|q_n\|_{L^2}^2\right) ds \\ &\leq \tfrac{1}{2}\left(\|u_0\|_H^2 + \|q_0\|_{-\frac{1}{2},D}^2\right). \end{aligned} \tag{3.15}$$

Moreover, there exists a constant C, independent of T, m, n and the initial data, such that

$$\sup_{0\leq t\leq T} \|\Lambda^2 q_n(\cdot, t)\|_{L^2(\Omega)}^2 + \int_0^T \|\Lambda^{\frac{5}{2}} q_n\|_{L^2(\Omega)}^2 dt \leq C\|\Lambda^2 q_0\|_{L^2(\Omega)}^2 e^{C\int_0^T (\|u_m\|_{B(\Omega)} + \|u_m\|_{B(\Omega)}^2)dt} \tag{3.16}$$

holds on $[0, T]$.

PROOF. Taking the scalar product of (3.10) with u_m in L^2, the scalar product of (3.14) with $\Lambda^{-1} q_n$, adding the resulting equations, integrating by parts and using the fact that u_m is divergence-free we obtain, after using the cancellation

$$\int_\Omega \mathbb{P}_m(q_n R q_n) \cdot u_m dx + \int_\Omega P_n(u_m \cdot \nabla q_n)\, \Lambda^{-1} q_n dx = 0$$

that

$$\frac{1}{2}\frac{d}{dt}\left(\|u_m\|_H^2 + \|q_n\|_{-\frac{1}{2},D}^2\right) \leq 0. \tag{3.17}$$

Therefore the ODEs (3.13)–(3.14) have global solutions and, checking the equation we obtain for any t the uniform bound (3.15) . We use now energy bounds employing the commutator estimate (2.6) and the proof of Theorem 2 to obtain a priori estimates. The basic estimate concerns the H^2 norms. We apply Λ to (3.14), and use the commutator:

$$\partial_t \Lambda q_n + \Lambda(\Lambda q_n) + P_n(u_m \cdot \nabla \Lambda q_n) + P_n[\Lambda, u_m \cdot \nabla] q_n = 0. \tag{3.18}$$

We take the scalar product with $\Lambda^3 q_n$:

$$\begin{aligned}
&\int_\Omega (u_m \cdot \nabla\Lambda q_n)\Lambda^3 q_n dx = \int_\Omega \Lambda^2(u_m\cdot\nabla\Lambda q_n)\Lambda q_n dx\\
&= \int_\Omega \left[(-\Delta u_m)\cdot\nabla\Lambda q_n - 2\nabla u_m\cdot\nabla\nabla\Lambda q_n\right]\Lambda q_n dx + \int_\Omega (u_m\cdot\nabla\Lambda^3 q_n)\Lambda q_n dx\\
&= \int_\Omega \left[(-\Delta u_m)\cdot\nabla\Lambda q_n - 2\nabla u_m\cdot\nabla\nabla\Lambda q_n\right]\Lambda q_n dx - \int_\Omega \Lambda^3 q_n(u_m\cdot\nabla\Lambda q_n)dx\\
&= \int_\Omega \left[((-\Delta u_m)\cdot\nabla\Lambda q_n)\Lambda q_n + 2\nabla u_m\nabla\Lambda q_n\nabla\Lambda q_n\right]dx - \int_\Omega (u_m\cdot\nabla\Lambda q_n)\Lambda^3 q_n dx.
\end{aligned}$$

In the first integration by parts we used the fact that Λq_n is a finite linear combination of eigenfunctions which vanish at the boundary. Then we use the fact that $\Lambda^2 = -\Delta$ is local. In the last equality we integrated by parts using the fact that Λq_n is a finite linear combination of eigenfunctions which vanish at the boundary and the fact that u_m is divergence-free. It follows that

$$\int_\Omega (u_m\cdot\nabla\Lambda q_n)\Lambda^3 q_n dx = \frac{1}{2}\int_\Omega \left[((-\Delta u_m)\cdot\nabla\Lambda q_n)\Lambda q_n + 2\nabla u_m\nabla\Lambda q_n\nabla\Lambda q_n\right]dx$$

and consequently

$$\left|\int_\Omega (u_m\cdot\nabla\Lambda q_n)\Lambda^3 q_n dx\right| \le C\|u_m\|_{B(\Omega)}\|\Lambda^2 q_n\|^2_{L^2(\Omega)}.$$

We estimate the commutator (2.6):

$$\|[\Lambda, u_m\cdot\nabla]q_n\|_{\frac{1}{2},D} \le C\|u_m\|_{B(\Omega)}\|q_n\|_{\frac{3}{2},D}$$

and we follow this by a Young inequality and use of the dissipative term $\|\Lambda^{2.5}q_n\|^2_{L^2(\Omega)}$. We obtain thus (3.16) on any time interval $[0,T]$. This ends the proof of the lemma. □

Passing to the limit $n\to\infty$ at fixed m, we obtain solutions (q, u_m) of (3.10)–(3.11) on $[0,T]$.

LEMMA 2. *Let T be arbitrary, $q_0 \in H^1_0(\Omega)\cap H^2(\Omega)$, $u_0\in H\cap H^1_0(\Omega)\cap H^2(\Omega)$. Let m be a positive integer. There exists a unique solution (u_m, q) of the system (3.10)–(3.11) with initial data $u_m(0) = \mathbb{P}_m u_0$, $q(0) = q_0$. The solutions are bounded uniformly, independently of T and m, with bounds depending only on the initial data*

$$u_m \in L^\infty(0,T;\mathcal{D}(A))\cap L^2(0,T;\mathcal{D}(A^{\frac{3}{2}}))$$

and

$$q\in L^\infty(0,T;\mathcal{D}(-\Delta))\cap L^2(0,T;\mathcal{D}(\Lambda^{\frac{5}{2}})).$$

PROOF. Passage to the limit $n\to\infty$ in the preliminary approximate system and use of the Aubin-Lions lemma provides a solution (u_m, q) of the final approximate system. By lower semicontinuity we obtain in the weak limit the bound

$$\begin{aligned}
&\|u_m(t)\|^2_H + \|q(t)\|^2_{-\frac{1}{2},D} + \textstyle\int_0^t \left(\|u_m(s)\|^2_{H^1} + \|q(s)\|^2_{L^2}\right)ds\\
&\le \tfrac{1}{2}\left(\|u_0\|^2_H + \|q_0\|^2_{-\frac{1}{2},D}\right).
\end{aligned}\tag{3.19}$$

The right hand side is obviously uniform in m. A bound of the H^2 norm of q follows from (3.16) by passage to weak limit and lower semicontinuity. There exists a constant γ_m, depending on m, but independent of T and the initial data such that the solutions obey

$$\sup_{0\le t\le T}\|\Lambda^2 q(\cdot,t)\|^2_{L^2(\Omega)} + \int_0^T \|\Lambda^{\frac{5}{2}}q\|^2_{L^2(\Omega)}dt \le C\|\Lambda^2 q_0\|^2_{L^2(\Omega)}e^{\gamma_m[\|u_0\|^2_H + \|q_0\|^2_{-\frac{1}{2},D}]}\tag{3.20}$$

The right hand side of (3.20) not uniformly bounded in m. Estimate (3.20) follows from the bound (3.16) using the bound (3.15) and the fact that, for functions $u_m\in\mathbb{P}_m H$ bounds in H^1 imply bounds in $B(\Omega)$. We have therefore obtained the qualitative fact that q is very regular, but uniform in m bounds will be obtained only later in the proof, after we obtain other m- independent apriori uniform bounds.

A priori uniform L^p estimates on q follow from Proposition 1:

$$\sup_{0\le t\le T}\|q\|_{L^p(\Omega)}\le \|q_0\|_{L^p(\Omega)}. \tag{3.21}$$

These are valid for any p, $1\le p\le\infty$. These inequalities are justified in our situation by the energy bounds for u_m (3.19) above. Indeed, because $u_m\in\mathbb{P}_mH$, it follows that they are smooth divergence-free and the L^p bounds follow from Proposition 1. More precisely, we can bound the L^p norms $1\le p<\infty$ directly by integration by parts, and pass $p\to\infty$. It is here where we use the full PDE for q.

Using classical methods for $2D$ Navier-Stokes equations it follows next that u_m are uniformly in m bounded

$$u_m\in L^\infty(0,T,H_0^1(\Omega)^2)\cap L^2(0,T;H^2(\Omega)^2). \tag{3.22}$$

Indeed, we take the scalar product of (3.10) in H with Au_m, and then, using (2.18) and Young's inequality we obtain the evolution inequality

$$\frac{d}{dt}\|\nabla u_m\|^2_{L^2(\Omega)}+\|Au_m\|^2_{L^2(\Omega)}\le C\|qRq\|^2_{L^2(\Omega)}+C\|u_m\|^2_H\|\nabla u_m\|^4_{L^2(\Omega)}. \tag{3.23}$$

The fact that q is bounded apriori in L^∞ is used now, together with the boundedness of R in L^2. A Gronwall inequality and the bounds from the energy inequality (3.19), we obtain (3.22) with bounds independent of m. The bounds depend only on the norms of initial data, but do not depend on T. Note that at this stage we do not yet have uniform bounds in m for $u_m\in L^2(0,T;B(\Omega))$ although obviously we do have each $u_m\in L^2(0,T;B(\Omega))$ in view of the weaker bounds (3.22) and the fact that u_m are functions in $\mathbb{P}_mH$. We revisit now an H^1 bound for q: we take the scalar product of the equation (3.11) with $-\Delta q$.

$$\frac{1}{2}\frac{d}{dt}\|\nabla q\|^2_{L^2(\Omega)}+\|q\|^2_{\frac{3}{2},D}=\int_\Omega(u_m\cdot\nabla q)\Delta q dx.$$

We integrate by parts, and using the facts that u_m vanish on the boundary and are divergence-free, we obtain

$$\int_\Omega(u_m\cdot\nabla q)\Delta q dx=-\int_\Omega\nabla q(\nabla u_m)\nabla q dx.$$

We use a Hölder inequality to bound

$$\left|\int_\Omega\nabla q(\nabla u_m)\nabla q dx\right|\le\|\nabla q\|_{L^2(\Omega)}\|\nabla q\|_{L^4(\Omega)}\|\nabla u_m\|_{L^4(\Omega)}.$$

Now, we claim that

$$\|\nabla q\|_{L^4(\Omega)}\le C\|q\|_{\frac{3}{2},D}.$$

This is true because $R:L^4(\Omega)\to L^4(\Omega)$ is bounded [**8**], and because $\Lambda q\in\mathcal{D}(\Lambda^{\frac{1}{2}})$; in fact $q\in\mathcal{D}(\Delta)$ by [**3**]. (We remark here that since $\partial\Omega$ is smooth, the boundedness of the associated Riesz transforms $R=\nabla(-\Delta_D)^{-1/2}$ follows by a classical argument: flattening of the boundary, maximal elliptic L^p regularity of the operator $L=\mathrm{div}A\nabla$ when the matrix A is smooth and uniformly elliptic, and complex interpolation. This argument carries over to the case of Lipschitz domains, albeit only for a restricted range of p, see [**8**] for details.) Thus, $\nabla q=R\Lambda q$ is bounded in L^4 using (2.9). Using (2.13) and (2.17), we deduce

$$\left|\int_\Omega\nabla q(\nabla u_m)\nabla q dx\right|\le C\|\nabla q\|_{L^2(\Omega)}\|q\|_{\frac{3}{2},D}\|\nabla u_m\|^{\frac{1}{2}}_{L^2(\Omega)}\|Au_m\|^{\frac{1}{2}}_H.$$

Consequently, after a Young inequality, because of (3.22), it follows that

$$q\in L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;\mathcal{D}(\Lambda^{\frac{3}{2}})) \tag{3.24}$$

is bounded a priori, independently of time T or m, in terms only of initial data.

We now take (3.10), apply A and take the scalar product with Au_m. We obtain the differential inequality

$$\frac{1}{2}\frac{d}{dt}\|Au_m\|_H^2 + \|A^{\frac{3}{2}}u_m\|_H^2 \le C\|A^{\frac{1}{2}}\mathbb{P}_m(qRq)\|_H^2 + C\|A^{\frac{1}{2}}\mathbb{P}_m B(u_m,u_m)\|_H^2.$$

Now

$$\|A^{\frac{1}{2}}\mathbb{P}_m(qRq)\|_H^2 \le \|A^{\frac{1}{2}}(qRq)\|_H^2 = \|\nabla(qRq)\|_{L^2(\Omega)}^2$$

and

$$\|\nabla(qRq)\|_{L^2(\Omega)}^2 \le \|Rq\|_{L^4(\Omega)}^2\|\nabla q\|_{L^4(\Omega)}^2 + \|q\|_{L^\infty(\Omega)}^2\|\nabla q\|_{L^2(\Omega)}^2 \in L^1((0,T))$$

because Riesz transforms are bounded in $L^4(\Omega)$ (cf. [**8**]) and because of the bounds (3.21) and (3.24). The other term obeys, using (2.19)

$$\|A^{\frac{1}{2}}\mathbb{P}_m B(u_m,u_m)\|_H^2 \le C\|u_m\|_H\|Au_m\|_H^3.$$

Using (3.22) and a Gronwall inequality we obtain

$$u_m \in L^\infty(0,T;\mathcal{D}(A)) \cap L^2(0,T;\mathcal{D}(A^{\frac{3}{2}})) \tag{3.25}$$

with uniform bounds in m, independent of T and depending only on initial data. It is only now that we attained by interpolation the uniform bounds for $u_m \in L^2(0,T,B(\Omega))$. We take now (3.11) apply Δ, multiply by Δq and integrate. We are allowed to do so because of (3.20) which guarantees q is smooth enough. We obtain, after a cancellation using integration by parts and vanishing of u_m on the boundary

$$\begin{aligned}&\tfrac{d}{dt}\|\Delta q\|_{L^2(\Omega)}^2 + \|\Lambda^{2.5}q\|_{L^2(\Omega)}^2\\ &\le 2\left[\int_\Omega |\Delta u_m||\nabla q||\Delta q|dx + 2\int_\Omega |\nabla u_m||\nabla\nabla q||\Delta q|dx\right].\end{aligned}$$

Because of the uniform bounds (3.25), interpolation and Gronwall, we finally obtain that

$$q \in L^\infty(0,T;\mathcal{D}(-\Delta)) \cap L^2(0,T;H^{2.5}(\Omega)) \tag{3.26}$$

is bounded independently of m and T in terms only of the initial data. □

This concludes the uniform bounds, which are (3.25) and (3.26). Passage to the limit $m \to \infty$ is done using an Aubin-Lions lemma [**9**] and the bounds are inherited by the solutions of the limit equations. We omit further details.

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