Formation of shocks for 2D isentropic compressible Euler

Tristan Buckmaster[*] **Steve Shkoller**[†] **Vlad Vicol**[‡]

Abstract

We consider the 2D isentropic compressible Euler equations, with pressure law $p(\rho) = (1/\gamma)\rho^\gamma$, with $\gamma > 1$. We provide an elementary constructive proof of shock formation from smooth initial datum of finite energy, with no vacuum regions, and with nontrivial vorticity. We prove that for initial data which has minimum slope $-1/\varepsilon$, for $\varepsilon > 0$ taken sufficiently small relative to the $\mathcal{O}(1)$ amplitude, there exist smooth solutions to the Euler equations which form a shock in time $\mathcal{O}(\varepsilon)$. The blowup time and location can be explicitly computed and solutions at the blowup time are of cusp-type, with Hölder $C^{1/3}$ regularity.

Our objective is the construction of solutions with inherent $\mathcal{O}(1)$ vorticity at the shock. As such, rather than perturbing from an irrotational regime, we instead construct solutions with dynamics dominated by purely azimuthal wave motion. We consider homogenous solutions to the Euler equations and use Riemann-type variables to obtain a system of forced transport equations. Using a transformation to modulated self-similar variables and pointwise estimates for the ensuing system of transport equations, we show the global stability, in self-similar time, of a smooth blowup profile.

Contents

1 Introduction

We consider the Cauchy problem for the two-dimensional isentropic compressible Euler equations

$$\partial_t(\rho u) + \operatorname{div}(\rho\, u \otimes u) + \nabla p(\rho) = 0\,, \tag{1.1a}$$

$$\partial_t \rho + \operatorname{div}(\rho u) = 0\,, \tag{1.1b}$$

[*]Department of Mathematics, Princeton University, Princeton, NJ 08544, buckmaster@math.princeton.edu.

[†]Department of Mathematics, UC Davis, Davis, CA 95616, shkoller@math.ucdavis.edu.

[‡]Courant Institute of Mathematical Sciences, New York University, New York, NY 10012, vicol@cims.nyu.edu.

where $u : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ denotes the velocity vector field, $\rho : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}_+$ denotes the strictly positive density, and the pressure $p : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}_+$ is defined by the ideal gas law

$$p(\rho) = \tfrac{1}{\gamma}\rho^\gamma\,, \qquad \gamma > 1\,.$$

The sound speed $c(\rho) = \sqrt{\partial p/\partial \rho}$ is then given by $c = \rho^\alpha$ where $\alpha = \frac{\gamma-1}{2}$. The Euler equations (1.1) are a system of conservation laws: (1.1a) is the conservation of momentum, which can be equivalently written as $\partial_t u + u \cdot \nabla u + \rho^{\gamma-2}\nabla\rho = 0$, and (1.1b) is the conservation of mass.

This paper is devoted to the construction of solutions to (1.1) which form a shock in finite time: specifically, starting from smooth initial data with $\mathcal{O}(1)$ amplitude and a minimum slope of $-1/\varepsilon$ with $\varepsilon > 0$ sufficiently small, we construct solutions to the 2D Euler equations (1.1) on a time interval $t_0 \leqslant t \leqslant T_*$, $t_0 = -\varepsilon$ and $T_* = \mathcal{O}(\varepsilon^{5/4})$, for which $\rho(\cdot,t)$ and $u(\cdot,t)$ remain bounded, while $|\nabla\rho(\cdot,t)| \to \infty$ and $|\nabla u(\cdot,t)| \to \infty$ as $t \to T_*$; moreover, no other type of singularity can form prior to $t = T_*$, and detailed information on the singularity formation at $t = T_*$ is provided, including blowup time, location, and profile regularity.

We are particularly interested in devising solutions to (1.1) which have *large*[1] *vorticity* at the shock, by which we mean solutions which are not small perturbations of irrotational flows. As such, our strategy will be to construct solutions that are perturbations of purely azimuthal wave motion whose simplest (constant) profiles are of the $x^\perp$-type with $\mathcal{O}(1)$ vorticity at this most basic level. As we shall describe in great detail below, this is in contrast to those solutions which are small perturbations of irrotational simple plane waves.

We are thus motivated to develop a framework of analysis for solutions which are perturbations of purely azimuthal waves. Obviously, polar coordinates provide a natural setting for describing such perturbative solutions, but more fundamentally, we have discovered that the use of homogeneous solutions to (1.1) leads to a remarkable reduction of the Euler dynamics precisely to this nearly-azimuthal wave regime, in which bounded azimuthal waves steepen and then shock, while radial waves (and their slopes) remain bounded. Owing to the inherent vorticity in the most basic wave motion, the solutions are fundamentally two-dimensional in their evolution. We provide a precise description of the shock formation for such Euler solutions, including the *blowup time and location*, by a transformation to self-similar variables that contain dynamically evolving modulation functions that keep track of the location, time, and amplitude of the blowup. At the blowup time $t = T_*$, the wave profile is of Hölder-class $C^{1/3}$. In the special case that the adiabatic exponent γ is equal to 3 and for purely azimuthal initial velocity fields, a series of surprising cancellations reduces the 2D Euler dynamics to an elementary study of the Burgers equation. The solution for the special case that $\gamma = 3$ can be viewed as the purely azimuthal wave motion, and its shock formation is completely characterized for all time.

Theorem 1.1 (**Rough statement of the main theorem**)**.** *For an open set of smooth initial data with $\mathcal{O}(1)$ amplitude and with minimum initial slope given at initial time t_0 to equal $-1/\varepsilon$, for $\varepsilon > 0$ taken sufficiently small, there exist smooth solutions of the Euler equations with $\mathcal{O}(1)$ vorticity, which form an asymptotically self-similar shock in finite time T_*, such that $T_* - t_0 = \mathcal{O}(\varepsilon)$. The solutions have $\mathcal{O}(1)$ vorticity at the shock, are dominated by azimuthal wave motion, and the location and time of the first singularity can be explicitly computed. The blowup profile at the first singularity is shown to be a cusp with $C^{1/3}$ regularity.*

The precise statement of the main theorem is given in Theorem 4.4, while the special case that $\gamma = 3$ is treated in Theorem 3.1.

[1] Due to the time rescaling symmetry of the Euler equations, by which $u^\beta(x,t) = \beta^{-1}u(x,\beta^{-1}t)$ and $\rho^\beta(x,t) = \beta^{-1/\alpha}\rho(x,\beta^{-1}t)$ are also solutions to (1.1), ∇u can be made smaller or larger by changing the time interval of the evolution.

1.1 A brief history of the analysis of shock formation for the Euler equations

The mathematical analysis of shock formation for the Euler equations has a long and rich history, particularly in the case of one space dimension, which allows the full power of the method of characteristics to be employed. In 1D, the velocity u is a scalar and (1.1) takes the form

$$\partial_t u + u u_x + \rho^{\gamma-2}\rho_x = 0\,, \qquad \partial_t \rho + (\rho u)_x = 0\,.$$

Riemann [41] devised the two *invariant* functions $z = u - {}^{c}\!/\!_{\alpha}$ and $w = u + {}^{c}\!/\!_{\alpha}$ which are constant along the characteristics of the two wave speeds $\lambda_1 = u - c$ and $\lambda_2 = u + c$:

$$\partial_t z + \lambda_1 z_x = 0\,, \qquad \partial_t w + \lambda_2 w_x = 0\,.$$

He proved that from smooth data, shocks can form in finite time. The 1D isentropic Euler equations are an example of a 2×2 system of conversation laws. Using Riemann invariants, Lax [26] proved that finite-time shocks can form from smooth data for general 2×2 genuinely nonlinear hyperbolic systems and Majda [31] gave a geometric proof which also allowed for 2×2 systems with linear degeneracy; John [22] then proved finite-time shock formation for $n \times n$ genuinely nonlinear hyperbolic systems; Liu [27] then generalized this result. Klainerman-Majda [25] proved the formation of singularities for second-order quasilinear wave equations which includes the nonlinear vibrating string. See the book of Dafermos [14] for a more extensive bibliography of 1D results.

In multiple space dimensions, Sideris [42] proved that C^1 regular solutions to (1.1) have a finite lifespan by establishing differential inequalities for certain integrals which lead to a proof by contradiction; in particular, he showed that $\mathcal{O}(\exp(1/\varepsilon))$ is an upper bound for the lifespan (of 3D flows) for data of size ε. The nature of the proof did not, however, reveal the type of singularity that develops, but rather, that some finite-time breakdown must occur.

The first proof of shock formation for the compressible Euler equations in the multi-dimensional setting was given by Christodoulou [7] for relativistic fluids and with the restriction of *irrotational* flow. Later Christodoulou-Miao [10] used the same framework to study shock formation in the non-relativistic setting and also for irrotational flow. Christodoulou's method is based upon a novel eikonal function (see also Christodoulou-Klainerman [9] and Klainerman-Rodnianski [23]), whose level sets correspond to characteristics of the flow; by introducing the inverse foliation density, a function which is inversely proportional to time-weighted derivatives of the eikonal function, Christodoulou proved that shocks form when the inverse foliation density vanishes (i.e., characteristics cross), and that no other breakdown mechanism can occur prior to such shock formation. The proof relies on the use of a geometric coordinate system, along which the solution has long time existence, and remains bounded, so that the shock is constructed by the singular (or degenerate) transformation from geometric to Cartesian coordinates. For the restricted shock development problem, in which the Euler solution is continued past the time of first singularity but vorticity production is neglected, see the discussion in Section 1.6 of [8]. Starting with piecewise regular initial data for which there is a closed curve of discontinuity, across which the density and normal component of velocity experience a jump, Majda [29–31], proved (for more general flows than the 2D isentropic flows) that such a shock can always be continued for a short interval of time, but with derivative loss. For such shock initial data, Métivier [38] later reduced the derivative loss to only a $1/2$-derivative. Gues-Métivier-Williams-Zumbrun [20] studied the existence and stability of this multidimensional shock propagation problem in the vanishing viscosity limit.

A special feature of irrotational flows is that the Euler equations can be expressed as a second-order quasilinear wave equation with respect to the velocity potential. The first results on shock formation for 2D quasilinear wave equations which do not satisfy Klainerman's null condition [24] were established by Alinhac [1, 2], wherein a detailed description of the blowup was provided. The geometric framework of [7] has influenced more recent analysis of shock formation for quasilinear wave equations. Holzegel-Klainerman-Speck-Wong [21] have explained the mechanism for stable shock formation for certain types of quasilinear wave equations with small data in three dimensions. Speck [43] generalized and unified earlier work on singularity formation for both covariant and non-covariant scalar wave equations of a certain form. He proved that whenever the nonlinear terms fail Klainerman's null condition [24], shocks develop in solutions arising from an open set of small data, and can thus be viewed as a converse to the well-known result of Christodoulou-Klainerman [9], which showed that when the classic null condition is verified, small-data global existence holds. For quasilinear wave equations that are derived from the least action principle and which satisfy the null condition, Miao-Yu [39] proved shock formation using the so-called short pulse data.

The first proof of shock formation for fluid flows with vorticity was given by Luk-Speck [28], for the 2D isentropic Euler equations with vorticity. The presence of nontrivial vorticity in their analysis does not only allow for a much larger class of data, but also has two families of waves being propagated, sound waves and vorticity waves, thus allowing for multiple characteristics (wave speeds) to interact. Their proof uses Christodoulou's geometric framework from [7, 10], but develops new methods to contend with the aforementioned vorticity waves, establishes new estimates for the regularity of the transported vorticity-divided-by-density, and relies crucially on a new framework for describing the 2D compressible Euler equations as a coupled system of covariant wave and transport equations.

Luk-Speck consider in [28] solutions to Euler which are *small* perturbations of a subclass of outgoing simple plane waves. In the 2D Cartesian plane, with coordinates (x_1, x_2), an outgoing simple plane wave is defined as a solution to the Euler equations (1.1) which moves to the right along the x_1 axis, does not depend on x_2, and has vanishing first Riemann invariant $u^1 - c$. The *smallness* of the perturbation of the plane wave is measured in terms of the ratio of the maximum wave amplitude to the minimum (negative) slope of the initial wave profile. Specifically, they construct solutions which are small perturbations of the irrotational simple plane waves, in which the transverse derivative (to the acoustic characteristics) of u^1 blows up, while the tangential derivatives (to the acoustic characteristics) of (ρ, u^1, u^2) remain bounded, and vorticity is non-vanishing and small at the shock.

1.2 Shock formation with vorticity and the perturbation of purely azimuthal waves

Let us now describe the type of shock wave solutions that we construct and compare them with those of [28]. As noted above, we do not consider perturbations of simple plane waves, but instead construct solutions which are perturbations of azimuthal waves.

Using 2D polar coordinates (r, θ), we denote the velocity components by $u = (u_r(r, \theta, t), u_\theta(r, \theta, t))$. We consider initial conditions $(\rho(\cdot, t_0), u_r(\cdot, t_0), u_\theta(\cdot, t_0))$ which have $\mathcal{O}(1)$ amplitude, but with $\partial_\theta u_\theta(\cdot, t_0)$ and $\partial_\theta \rho(\cdot, t_0)$ having a minimum (negative) value of $-1/\varepsilon$, with $0 < \varepsilon \ll 1$ taken sufficiently small. There are two Riemann invariants for the azimuthal flow, which we write as $\mathcal{R}_\pm = u_\theta \pm \frac{2}{\gamma - 1} \rho^{(\gamma-1)/2}$. The solutions we construct satisfy the following conditions:

(a) solutions (ρ, u_r, u_θ) have $\mathcal{O}(1)$ bounds in L^∞ for $t \in [t_0, T_*)$ with linear variation in the radial r direction for u_r and u_θ and $r^{2/(\gamma-1)}$ variation for ρ;

(b) $|\partial_\theta \mathcal{R}_+|$, $|\partial_\theta u_\theta|$, and $|\partial_\theta \rho|$ are $\mathcal{O}(1/\varepsilon)$ at initial time, and these quantities blow up at time $t = T_*$ with a rate proportional to $1/(T_* - t)$, where $T_* - t_0 = \mathcal{O}(\varepsilon)$;

(c) the blowup profile is of cusp-type with $u_\theta(\cdot, T_*)$ and $\rho(\cdot, T_*)$ in the Hölder space $C^{1/3}$;

(d) $\partial_\theta \mathcal{R}_-$ remains bounded on on $[t_0, T_*)$;

(e) ∂_r of (ρ, u_r, u_θ) and $\partial_\theta u_r$ are bounded on $[t_0, T_*)$;

(f) the vorticity $\partial_r u_\theta - \frac{1}{r} \partial_\theta u_r + \frac{1}{r} u_\theta$ is non-vanishing and bounded at the shock.

There is some correspondence between the properties (a)–(f) of our solutions and the solutions constructed by Luk-Speck [28], in that we are perturbing purely azimuthal wave motion (in the θ-direction), and in [28] they are perturbing simple plane wave motion (in the x_1-direction). A primary difference is that the purely azimuthal wave already has nontrivial vorticity, while the simple plane wave is irrotational, and so we are constructing solutions that are perturbations of flows with nontrivial vorticity. Furthermore, our method allow us to provide a fairly detailed description of the blowup profile for $u_\theta(\cdot, T_*)$ and $\rho(\cdot, T_*)$: the slope becomes infinite along a line segment, and each function is $C^{1/3}$ in space.

As we shall next describe, the method we develop to construct shock wave solutions is very different from the methods of [7, 10, 28]; we rely upon a transformation to modulated self-similar variables together with the fact that 2D purely azimuthal wave motion is governed by the dynamics of the Burgers equation; we shall explain how our analysis relies on properties of nonlinear transport equations together with explicit properties of the asymptotically stable self-similar profile.

2 Outline of the proof

2.1 A new class of solutions that shock

In order to study perturbations of purely azimuthal waves, we write the Euler equations (1.1) in polar coordinates for the variables (ρ, u_r, u_θ) as the following system of conservation laws:

$$\left(\partial_t + u_r\partial_r + \tfrac{1}{r}u_\theta\partial_\theta\right) u_r - \tfrac{1}{r}u_\theta^2 + \rho^{\gamma-2}\partial_r\rho = 0\,, \tag{2.1a}$$

$$\left(\partial_t + u_r\partial_r + \tfrac{1}{r}u_\theta\partial_\theta\right) u_\theta + \tfrac{1}{r}u_r u_\theta + \tfrac{1}{r}\rho^{\gamma-2}\partial_\theta\rho = 0\,, \tag{2.1b}$$

$$\left(\partial_t + u_r\partial_r + \tfrac{1}{r}u_\theta\partial_\theta\right) \rho + \rho\left(\tfrac{1}{r}u_r + \partial_r u_r + \tfrac{1}{r}\partial_\theta u_\theta\right) = 0\,. \tag{2.1c}$$

These equations are solved with $\theta \in \mathbb{T} = [-\pi, \pi]$, $r > 0$ and $t \in [t_0, T]$. Defining the fluid vorticity $\omega = \frac{1}{r}\partial_r(ru_\theta) - \frac{1}{r}\partial_\theta u_r$, we shall make use of the fact that ω/ρ is transported as

$$\partial_t \tfrac{\omega}{\rho} + u\cdot\nabla\tfrac{\omega}{\rho} = 0\,. \tag{2.2}$$

For initial density $\rho_0 > 0$ that has no vacuum regions, and for *nontrivial initial vorticity*

$$\omega(r,\theta,t_0) = \partial_r u_\theta(r,\theta,t_0) - \tfrac{1}{r}\partial_\theta u_r(r,\theta,t_0) + \tfrac{1}{r}u_\theta(r,\theta,t_0) \neq 0\,,$$

we construct smooth solutions to (1.1) that form a shock in finite-time. So that our solutions will be perturbations of azimuthal waves, we shall consider homogeneous solutions.

To this end, motivated by the homogeneous solutions introduced for studying singularity formation in incompressible flows by Elgindi and Jeong [18], we consider the new variables $\widetilde{u}$ and $\widetilde{\rho}$ such that

$$u(r,\theta,t) = r\widetilde{u}(r,\theta,t) \text{ and } \rho(r,\theta,t) = r^{\frac{2}{\gamma-1}}\widetilde{\rho}(r,\theta,t)\,,$$

and recalling that $\alpha = \frac{\gamma-1}{2}$, with respect to these new variables, the system (2.1) takes the form:

$$\left(\partial_t + \widetilde{u}_r r\partial_r + \widetilde{u}_\theta\partial_\theta\right)\widetilde{u}_r + \widetilde{u}_r^2 - \widetilde{u}_\theta^2 + \tfrac{1}{\alpha}\widetilde{\rho}^{2\alpha} + \widetilde{\rho}^{2\alpha-1}r\partial_r\widetilde{\rho} = 0\,, \tag{2.3a}$$

$$\left(\partial_t + \widetilde{u}_r r\partial_r + \widetilde{u}_\theta\partial_\theta\right)\widetilde{u}_\theta + 2\widetilde{u}_r\widetilde{u}_\theta + \widetilde{\rho}^{2\alpha-1}\partial_\theta\widetilde{\rho} = 0\,, \tag{2.3b}$$

$$\left(\partial_t + \widetilde{u}_r r\partial_r + \widetilde{u}_\theta\partial_\theta\right)\widetilde{\rho} + \tfrac{\gamma}{\alpha}\widetilde{u}_r\widetilde{\rho} + \widetilde{\rho}\left(r\partial_r\widetilde{u}_r + \partial_\theta\widetilde{u}_\theta\right) = 0\,. \tag{2.3c}$$

Notice that all powers of r have cancelled (expect for the $r\partial_r$ operator which is dimensionless), and hence, if at time $t = t_0$, the initial data is given as

$$\widetilde{u}_r(r,\theta,t_0) = a_0(\theta)\,,\ \ \widetilde{u}_\theta(r,\theta,t_0) = b_0(\theta)\,,\ \ \widetilde{\rho}(r,\theta,t_0) = P_0(\theta)\,, \tag{2.4}$$

where a_0, b_0, and P_0 are independent of r, then $\widetilde{u}$ and $\widetilde{\rho}$ remain independent of r for as long as the solution stays smooth (and hence unique), and thus the system (2.3) reduces to

$$(\partial_t + b\partial_\theta)\, a + a^2 - b^2 + \alpha^{-1} P^{2\alpha} = 0 \tag{2.5a}$$
$$(\partial_t + b\partial_\theta)\, b + 2ab + P^{2\alpha-1}\partial_\theta P = 0 \tag{2.5b}$$
$$(\partial_t + b\partial_\theta)\, P + \tfrac{\gamma}{\alpha} aP + P\partial_\theta b = 0\,, \tag{2.5c}$$

and then the solution to the Euler equations (2.1) is given by

$$u_\theta(r,\theta,t) = rb(\theta,t)\,, \;\; u_r(r,\theta,t) = ra(\theta,t) \text{ and } \rho(r,\theta,t) = r^{1/\alpha}P(\theta,t)\,. \tag{2.6}$$

The fluid vorticity and fluid divergence corresponding to the ansatz (2.4) are given by

$$\omega(r,\theta,t) = 2b(\theta,t) - \partial_\theta a(\theta,t)\,, \tag{2.7a}$$
$$\operatorname{div} u(r,\theta,t) = 2a + \partial_\theta b\,, \tag{2.7b}$$

so that the vorticity is therefore nontrivial as long as $2b \not\equiv \partial_\theta a$. Setting

$$\varpi = \frac{2b - \partial_\theta a}{P}\,,$$

from equation (2.2), we have that

$$\partial_t \varpi + b\partial_\theta \varpi = \tfrac{a}{\alpha}\varpi\,. \tag{2.8}$$

Next, we define the *Riemann invariants* w and z associated to the tangential velocity b and density P, and their associated wave speeds λ_1, λ_2, as

$$w = b + \frac{1}{\alpha}P^\alpha\,, \qquad z = b - \frac{1}{\alpha}P^\alpha\,, \tag{2.9a}$$
$$\lambda_1 = b - P^\alpha = \frac{1-\alpha}{2}w + \frac{1+\alpha}{2}z\,, \qquad \lambda_2 = b + P^\alpha = \frac{1+\alpha}{2}w + \frac{1-\alpha}{2}z\,. \tag{2.9b}$$

Then, the (a,b,P)-system (2.3) can be written as the following system for the variables (a,z,w):

$$\big(\partial_t + \lambda_2\partial_\theta\big)w + \tfrac{a}{2}\big((1-2\alpha)z + (3+2\alpha)w\big) = 0\,, \tag{2.10a}$$
$$\big(\partial_t + \lambda_1\partial_\theta\big)z + \tfrac{a}{2}\big((1-2\alpha)w + (3+2\alpha)z\big) = 0\,, \tag{2.10b}$$
$$\big(\partial_t + \tfrac{w+z}{2}\partial_\theta\big)a + a^2 - \tfrac{1}{4}(w+z)^2 + \tfrac{\alpha}{4}(w-z)^2 = 0\,. \tag{2.10c}$$

Notice that while z and w are not actual invariants, the advantage of the (a,z,w)-system is that no derivatives appear in the forcing of the transport.

In order to transform the w and z equations into the form of a perturbed Burgers-type equation, we define $t = \frac{1+\alpha}{2}\widetilde{t}$ so that $\partial_t = \frac{1+\alpha}{2}\partial_{\widetilde{t}}$. For notational simplicity, we shall write t for $\widetilde{t}$, in which case (2.10) becomes:

$$\partial_t w + \left(w + \tfrac{1-\alpha}{1+\alpha}z\right)\partial_\theta w = -a\left(\tfrac{1-2\alpha}{1+\alpha}z + \tfrac{3+2\alpha}{1+\alpha}w\right)\,, \tag{2.11a}$$
$$\partial_t z + \left(z + \tfrac{1-\alpha}{1+\alpha}w\right)\partial_\theta z = -a\left(\tfrac{1-2\alpha}{1+\alpha}w + \tfrac{3+2\alpha}{1+\alpha}z\right)\,, \tag{2.11b}$$
$$\partial_t a + \tfrac{1}{1+\alpha}(w+z)\partial_\theta a = -\tfrac{2}{1+\alpha}a^2 + \tfrac{1}{2(1+\alpha)}(w+z)^2 - \tfrac{\alpha}{2(1+\alpha)}(w-z)^2\,. \tag{2.11c}$$

While the local-in-time well-posedness in Sobolev spaces of the system (2.11) follows from the well-posedness of the Euler equations, we shall take the opposite view that solutions to the Euler equations are constructed from solutions of (2.11) together with (2.6) and (2.9).

Lemma 2.1. *For initial data $(w,z,a)|_{t=t_0} = (w_0,z_0,a_0)$ in $C^k(\mathbb{T})$, $k \geqslant 1$, there exists a time T depending on the $C^k(\mathbb{T})$-norm of this data, such that there exists a unique solution $(w,z,a) \in C([t_0,T];C^k(\mathbb{T})$ to (2.11). Furthermore, the solution continues to exist on $[t_0,T_*]$ if*

$$\int_{t_0}^{T_*} \left(\|\partial_\theta w(\cdot,t)\|_{L^\infty(\mathbb{T})} + \|\partial_\theta z(\cdot,t)\|_{L^\infty(\mathbb{T})} + \|a(\cdot,t)\|_{L^\infty(\mathbb{T})} \right) dt < \infty \,. \tag{2.12}$$

Proof of Lemma 2.1. We set $\beta_0 = \frac{1-\alpha}{1+\alpha}$, $\beta_1 = \frac{1-2\alpha}{1+\alpha}$, $\beta_2 = \frac{3+2\alpha}{1+\alpha}$, $\beta_3 = \frac{1}{1+\alpha}$, and define the characteristics of the three wave-speeds in (2.11) by

$$\begin{aligned}
\partial_t \psi_w &= w \circ \psi_w + \beta_0 z \circ \psi_w \,, \\
\partial_t \psi_z &= z \circ \psi_z + \beta_0 w \circ \psi_z \,, \\
\partial_t \psi_a &= \beta_3 (w \circ \psi_a + z \circ \psi_a) \,,
\end{aligned}$$

which are the identity at time t_0. Letting $\mathcal{W} = w \circ \psi_w$, $\mathcal{Z} = z \circ \psi_z$, and $\mathcal{A} = a \circ \psi_a$, the system (2.11) is then equivalent to

$$\begin{aligned}
\partial_t \mathcal{W} &= -\mathcal{A} \circ \psi_a^{-1} \circ \psi_w \left(\beta_1 \mathcal{Z} \circ \psi_z^{-1} \circ \psi_w + \beta_2 \mathcal{W} \right) \,, \\
\partial_t \mathcal{Z} &= -\mathcal{A} \circ \psi_a^{-1} \circ \psi_z \left(\beta_1 \mathcal{W} \circ \psi_w^{-1} \circ \psi_z + \beta_2 \mathcal{Z} \right) \,, \\
\partial_t \mathcal{A} &= \beta_3 \left[-2\mathcal{A}^2 + \tfrac{1}{2} \left(\mathcal{W} \circ \psi_w^{-1} \circ \psi_a + \mathcal{Z} \circ \psi_z^{-1} \circ \psi_a \right)^2 - \tfrac{\alpha}{2} \left(\mathcal{W} \circ \psi_w^{-1} \circ \psi_a - \mathcal{Z} \circ \psi_z^{-1} \circ \psi_a \right)^2 \right] \,, \\
\partial_t \psi_w &= \mathcal{W} + \beta_0 \mathcal{Z} \circ \psi_z^{-1} \circ \psi_w \,, \\
\partial_t \psi_z &= \mathcal{Z} + \beta_0 \mathcal{W} \circ \psi_w^{-1} \circ \psi_z \,, \\
\partial_t \psi_a &= \beta_3 (\mathcal{W} \circ \psi_w^{-1} \circ \psi_a + \mathcal{Z} \circ \psi_z^{-1} \circ \psi_a) \,,
\end{aligned}$$

with initial data $(\mathcal{W},\mathcal{Z},\mathcal{A})_{t=t_0} = (w_0,z_0,a_0) \in C^k(\mathbb{T})$ and $(\psi_w,\psi_z,\psi_a)|_{t=t_0} = (\mathrm{Id},\mathrm{Id},\mathrm{Id})$. Since this system does not present derivative losses, a standard Picard iteration argument proves the existence, uniqueness, and well-posedness of this coupled system on some time interval $[t_0,T]$, in the class $C([t_0,T],C^k(\mathbb{T}))$. This local in time solution may be continued as long as the transport velocities remain bounded in $L^1_t \mathrm{Lip}_x$ – indeed, this condition ensures the unique solvability and invertibility of the Lagrangian maps (ψ_w,ψ_z,ψ_a) introduced above. Lastly, we have excluded $\|w\|_{L^\infty}$ and $\|z\|_{L^\infty}$ from (2.12) because these remain finite if $a \in L^1_t L^\infty_x$, while $\|\partial_\theta a\|_{L^\infty}$ remains bounded due to the boundedness of $\|\varpi\|_{L^\infty}$ from (2.8). □

From a solution (w,z,a) of (2.11), we obtain a solution to the Euler equations (1.1) using that $b = \frac{w+z}{2}$, $P = \left(\frac{\alpha(w-z)}{2}\right)^{1/\alpha}$ and defining (u,ρ) using (2.6). Given the Euler velocity field u, we define the Lagrangian flow η_u as the solution to $\partial_t \eta_u = u \circ \eta_u$ for $t > t_0$ with $\eta_u(r,\theta,t_0) = (r,\theta)$. We shall consider annular regions

$$A_{\underline{r},\overline{r}} = \{(r,\theta) \colon \underline{r} < r < \overline{r}, \theta \in \mathbb{T}\}$$

for radii $0 < \underline{r} < \overline{r} < \infty$. Given $0 < R_0 < r_0 < r_1 < R_0$, we consider a small annulus A_{r_0,r_1} properly contained in a large annulus A_{R_0,R_1}. We define the time-dependent domain

$$\Omega(t) = \eta_u(A_{r_0,r_1},t) \subset A_{R_0,R_1} \quad \text{ for } \quad t \in [t_0,T_*] \,, \tag{2.13}$$

where the inclusion holds for T_* sufficiently small whenever $u \in L^\infty_t L^\infty_x$.

We shall construct solutions to (2.11) which form a shock in finite time and satisfy properties (a)-(f) listed above. Before describing our method of construction which is based on a transformation into self-similar variables, there is a singularly interesting choice for the adiabatic parameter γ which allows for a particularly simple construction of shock formation. When $\gamma = 3$, and hence $\alpha = 1$, it will be shown that the system (2.11) can be reduced exactly to $\partial_t w + w\partial_\theta w = 0$ with $a = 0$ and $z = 0$, in which case we have a purely azimuthal wave solution $(\rho, u_r, u_\theta) = \frac{1}{2}(rw, 0, rw)$ with a precise time and location for the shock formation, coming from the well-known solution to the Burgers equation. As noted above, we view this purely azimuthal wave as the polar analogue of the simple plane wave, because the radial velocity component vanishes as does the first Riemann invariant.

2.2 A transformation to self-similar variables with modulation functions

Turning to the case of general adiabatic exponent $\gamma > 1$ for the Euler system (1.1), we shall next introduce a self-similar transformation [19] with dynamic modulation variables [33]. Let

$$x(\theta, t) := \frac{\theta - \xi(t)}{(\tau(t) - t)^{\frac{3}{2}}}, \quad s := -\log(\tau(t) - t),$$

and define the new variables (A, Z, W) by

$$w(\theta, t) = e^{-\frac{s}{2}} W(x, s) + \kappa(t), \quad z(\theta, t) = Z(x, s), \quad a(\theta, t) = A(x, s).$$

This is a self-similar transformation[2] with three dynamic *modulation variables*, $\xi(t)$, $\tau(t)$, and $\kappa(t)$, each satisfying relatively simple ordinary differential equations. This technique was developed in the context of the Schrödinger equation [33–35] the nonlinear heat equation [36], the generalized KdV equation [32], the nonlinear wave equation [37] and other dispersive problems, and it has recently been applied to solve problems in fluid dynamics [6, 11–13, 15, 17]. In all these cases, the role of the modulation variables is to enforce certain orthogonality conditions required to study perturbations of the self-similar blowup. In our context, the modulation variables $\xi(t)$, $\tau(t)$, and $\kappa(t)$, respectively, control precisely the shock location, blowup time, and wave amplitude. In the absence of these dynamic variables, the above rescaling coincides with the well-known self-similar transformation for the Burgers equation (see [3, 12, 16, 40]), but the use of the modulation variables allows us to impose constraints on W and its first and second derivatives at $x = 0$.

Upon switching to self-similar variables, the (a, z, w)-system (2.11) is transformed to self-similar evolution equations for (A, Z, W) detailed below in (4.15). As we have noted above, for the special case that $\gamma = 3$, this system of self-similar equations reduces to the self-similar Burgers evolution, and a key feature of our proof is that the construction of shocks which are perturbations of purely azimuthal waves exactly coincides with the self-similar perturbation of the Burgers equation. Of paramount importance to our analysis, then, is the explicit representation of the stable, steady-state, self-similar Burgers profile [3]

$$\overline{W}(x) = \left(-\frac{x}{2} + \left(\frac{1}{27} + \frac{x^2}{4}\right)^{1/2}\right)^{1/3} - \left(\frac{x}{2} + \left(\frac{1}{27} + \frac{x^2}{4}\right)^{1/2}\right)^{1/3}, \tag{2.14}$$

solving the steady self-similar Burgers equation

$$-\frac{1}{2}\overline{W} + \left(\frac{3x}{2} + \overline{W}\right)\partial_x \overline{W} = 0. \tag{2.15}$$

[2]We note that our use of self-similar variables to construct the blowup is in some ways analogous to the use of geometric coordinates in the construction scheme of [7, 10, 28] wherein the long time existence in geometric coordinates leads to a finite-time blowup by the singular transformation back to Cartesian coordinates. We also note that self-similar variables have been used in a very different way to study the problem of self-similar 2D shock reflection off a wedge [4, 5].

Our proof of finite-time blowup for $\partial_\theta u_\theta$ and $\partial_\theta \rho$ relies upon showing that $\partial_\theta w$ has finite-time blowup, which in turn relies upon the global existence of solutions to the $(A(x,s), Z(x,s), W(x,s))$-system (4.15) for $x \in \mathbb{R}$ and $s \in [-\log(-t_0), \infty)$. Since

$$\partial_\theta w(\theta,t) = e^s \partial_x W(x,s)\,, \qquad e^s = \frac{1}{\tau(t)-t}\,, \tag{2.16}$$

by letting the blowup time modulation variable $\tau(t)$ satisfy $\tau(0) = 0$ and $\tau(T_*) = T_*$ and the blowup location modulation variable $\xi(t)$ satisfy $\xi(0) = 0$ and $\xi(T_*) = \theta_*$, we see that as $s \to \infty$, $|\partial_\theta w(\theta_*, t)| \to \infty$ at a rate proportional to $1/(T_* - t)$. Note, that all points θ which are not equal to θ_*, when converted to the self-similar variable x, are sent to $\pm\infty$ as $s \to +\infty$. In the proof, we show that $|\partial_x W| \lesssim (1+x^2)^{-1/3}$ and hence from this bound, it follows that $\left|W_x(e^{3s/2}(\theta-\xi), s)\right| \lesssim e^{-s}(\theta-\theta_*)^{-2/3}$, and from (2.16), $\partial_\theta w(\theta,t)$ does not blowup as $t \to T_*$.

The (A, Z, W)-system (4.15) consists of transport type equations, which allow us to use L^∞-type estimates to construct global-in-time solutions in C^4. We view the W equation (4.15a) as producing the dominant dynamics, and the key to our analysis is a careful comparison of $W(x,s)$ with $\overline{W}(x)$. In particular, differentiation of the system (4.15) shows that the equations satisfied by $\partial_x^n W$, $\partial_x^n Z$, and $\partial_x^n A$ for $n = 0, 1, 2, 3, 4$, have either damping or anti-damping terms that depend on the solutions and their derivatives. It is only when $n = 4$ that a clear damping term emerges, while for $n = 1$ and $n = 2$, a very subtle analysis must be made for the evolution equations of both $\partial_x W - \partial_x \overline{W}$ and $\partial_x^2 W - \partial_x^2 \overline{W}$; a very delicate analysis allows us to find lower-bounds for the damping terms in these equations by specially constructed rational functions that are found with the help of Taylor expansions of $\partial_x \overline{W}$ near $x = 0$ and $x = \infty$ (see, in particular, (4.54) and (4.65)). A bootstrap procedure is employed wherein we assume bounds for $(A, Z, W, \tau, \xi, \kappa)$ as well as their derivatives, and then proceed to close the bootstrap argument with even better bounds.

2.3 Paper outline

In Section 3, we consider the case that $\gamma = 3$, and we have the simple example of purely azimuthal shock formation. In this special case, the dynamics are reduced entirely to those of the Burgers equation. The formation of shocks for the 2D Euler equations with general adiabatic exponent $\gamma > 1$ is then treated in Section 4; a detailed description of the data is given, the main theorem is stated, and the proof of is given. Concluding remarks are stated in Section 5. We include Appendix A which contains some important maximum-principle-type lemmas for solutions of non-locally forced and damped transport equations.

3 Purely azimuthal waves and shocks: a simple example

In the case that $\gamma = 3$, some remarkable cancellations occur in the homogeneous solutions of the Euler equations which allow for an exceedingly simple mechanism of shock formation, in which a smooth purely azimuthal wave travels around the circle, steepens and forms a shock wave which can be continued for all time. Our general construction of shock waves for all $\gamma > 1$ will be a perturbation of this purely azimuthal shock wave solution, but we shall first describe this simple solution.

For the most concise presentation, we shall consider the Euler equations posed on a two-dimensional annular domain A_{r_0,r_1} where $0 < r_0 < r_1 < \infty$ with the standard no-flux boundary conditions $u_r|_{r=r_0} = u_r|_{r=r_1} = 0$.

In view of (2.4), the no-flux boundary condition requires that $a \equiv 0$ for all time. Therefore, from equation (2.5a), we must have the relation

$$b^2 = \frac{2}{\gamma-1} P^{\gamma-1} \tag{3.1}$$

for all time. If we impose condition (3.1) at $t = 0$, an explicit computation verifies that the evolution equations (2.5) preserve the constraint (3.1) if and only if $\gamma = 3$, in which case, we have that $b = P$, and hence from (2.9), the Riemann invariants are given by

$$w = 2b \quad \text{and} \quad z = 0\,.$$

Thus, with $a = 0$ and $z = 0$, the system (2.11) reduces to a single equation for the unknown w, which we identify as the 1D Burgers equation,

$$\partial_t w + w\partial_\theta w = 0 \qquad w(\theta, 0) = w_0(\theta)\,, \qquad \theta \in \mathbb{T} = [-\pi, \pi]\,, \tag{3.2}$$

solved on $\mathbb{T}$ with periodic boundary conditions. It is well known that any initial datum w_0 which has a negative slope at a point forms a shock (or infinite slope) in finite time. Note that for $\gamma = 3$, the formula (2.7a) shows that the vorticity $\omega = 2b = w$ and hence, ω is nontrivial even for the purely azimuthal wave. We shall sometimes use w' to denote $\partial_\theta w$.

Theorem 3.1 (Construction of the purely azimuthal shock). *For $\gamma = 3$, let $0 < r_0 < r_1$ be arbitrary, and consider initial datum $u_r = 0$, $u_\theta = \rho_0 = \frac{1}{2} r w_0$, in A_{r_0,r_1}, where $w_0 \in C^\infty(\mathbb{T})$ is such that $w_0 \geqslant \nu_0 > 0$. Suppose that*

$$\|w_0\|_{L^\infty} \leqslant 1\,, \tag{3.3}$$

and that there is a single point $\theta_0 \in \mathbb{T}$ such that $w_0'(\theta_0) = \min_{\theta\in\mathbb{T}} w_0(\theta)$, and that

$$\partial_\theta w_0(\theta_0) = -\frac{1}{\varepsilon} \tag{3.4}$$

for some $\varepsilon > 0$. Then the solution w of (3.2), develops a singularity at time $T_* = \varepsilon$ and angle $\theta_* = \theta_0 + \varepsilon w_0(\theta_0)$. Moreover, the functions $u_r = 0$, $u_\theta = \frac{1}{2} r w(\theta, t)$, and $\rho = \frac{1}{2} r w(\theta, t)$ form the unique smooth solution to the initial value problem for the Euler system (2.1) in the domain A_{r_0,r_1}, on the time interval $[0, \varepsilon)$. This solution satisfies the bounds

$$\sup_{t\in[0,T_*)} \left(\|\rho(\cdot, t)\|_{L^\infty(A_{r_0,r_1})} + \|u(\cdot, t)\|_{L^\infty(A_{r_0,r_1})} \right) \leqslant 2r_1\,, \tag{3.5}$$

$$\sup_{t\in[0,T_*)} \left(\|\partial_r\rho(\cdot, t)\|_{L^\infty(A_{r_0,r_1})} + \|\partial_r u(\cdot, t)\|_{L^\infty(A_{r_0,r_1})} \right) \leqslant 2\,, \tag{3.6}$$

$$\lim_{t\to T_*} \partial_\theta\rho(\theta_*, t) = \lim_{t\to T_*} \partial_\theta u_\theta(\theta_*, t) = -\infty\,. \tag{3.7}$$

The vorticity and density satisfy

$$\nu_0 \leqslant \omega(\theta, t) \leqslant 1, \qquad \rho(r, \theta, t) \geqslant r_0 \tfrac{\nu_0}{2}\,, \tag{3.8}$$

for all $\theta \in \mathbb{T}$ and $t \in [0, \varepsilon)$.

Proof of Theorem 3.1. For smooth initial datum w_0, we solve (3.2). Differentiating (3.2) gives the equation $\partial_t(\partial_\theta w) + w\partial_\theta^2 w + (\partial_\theta w)^2 = 0$. Define the flow $\psi(\theta, t)$ by $\partial_t\psi(\theta, t) = w(\psi(\theta, t), t)$ and $\psi(\theta, 0) = \theta$. Then $\partial_t(\partial_\theta w \circ \psi) + (\partial_\theta w \circ \psi)^2 = 0$ so that $(\partial_\theta w) \circ \psi = \frac{\partial_\theta w_0}{t + \partial_\theta w_0}$ and $\psi(\theta, t) = \theta + tw_0(\theta)$. Hence from (3.4), $\partial_\theta w$ forms a shock at time $T_* = \varepsilon$ at the point $\theta_* = \theta_0 + tw_0(\theta_0)$, implying (3.7). By the maximum principle and (3.3) we have

$$\sup_{t\in[0,T_*)} \|w(\cdot, t)\|_{L^\infty} \leqslant 1 \qquad \text{and} \qquad \min_{\theta\in\mathbb{T}} w(\theta, t) \geqslant \nu_0\,.$$

The bounds (3.5)–(3.6) and (3.8) follow directly from the definitions of u_θ, ρ, ω and the above estimate. $\square$

Remark 3.2 (**The Burgers solution continued after the singularity).** In Theorem 3.1 we have considered datum with a global (negative) minimum attained at a single point θ_0, and thus $w_0''(\theta_0) = 0$ and $w_0'''(\theta_0) > 0$. It is shown in [12, Proposition 9] that in the Burgers equation the finite time blowup arising from such initial datum is asymptotically self-similar and that the blowup profile is precisely the stable global-self similar profile $\overline{W}$ defined in (2.14). Moreover, at the blowup time $T_* = \varepsilon$ the solution is Hölder $C^{1/3}$ smooth near the singular point. To simplify the discussion, upon taking into account a Galilean transformation and a rescaling of the initial datum, we have that the blowup occurs at $\theta = 0$ with speed $w(0, T_*) = 0$, and that $w(\theta, T_*) \sim \theta^{1/3}$ to leading order in $|\theta| \ll 1$. The solution of the Burgers equation may be continued in a unique way as an entropy solution also after the blowup time T_*, starting from this Hölder $1/3$ initial datum, and we still denote this solution as $w(\cdot, t)$. We claim that instantaneously, for any $t > T_*$, the entropy solution $w(\cdot, t)$ has a jump discontinuity, with the discontinuity propagating at the correct shock speed, given by the Rankine-Hugoniot condition. This phenomenon is explained in [16, Chapter 11]: for $t > T_*$ one may compute an explicit forward globally self-similar solution, and one notices that this self-similar solution is not single-valued; we thus must have a jump in the solution w at a location and a speed determined by the Rankine-Hugoniot condition. The argument in [16] can be easily made precise by taking advantage of the Lax-Oleinik formula. For simplicity, let us consider initial datum $w(\theta, T_*) = \theta^{1/3}$, which allows us to perform explicit calculations. For $t > T_*$ the Lax-Oleinik formula tells us that the entropy solution equals

$$w(\theta, t) = \frac{\theta - (t - T_*)^{3/2} Y^3 \left(\frac{\theta}{(t-T_*)^{3/2}} \right)}{t - T_*} \tag{3.9}$$

where the function $Y = Y(q)$ is defined implicitly as the *the correct root* of the equation $Y^3 - Y = q$. This root is unique for $|q| > 2/(3\sqrt{3})$ and so the meaning of $Y(q)$ is clear; for $q \in [-2/(3\sqrt{3}), 0]$ we need to define $Y(q)$ as *the smallest root*, which is negative and has the limiting behavior $Y(0) = -1$; while for $q \in (0, 2/(3\sqrt{3})]$ the entropy solution requires us to take the *largest root*, which is positive and has the limiting behavior $Y(0^+) = +1$. Since the formula (3.9) is explicit, it is easy to verify the above claims. We have $w(0^-, t) = w(0, t) = (t - T_*)^{1/2}$ and $w(0^+, t) = -(t - T_*)^{1/2}$. This shows that we have a discontinuity across the shock location $\theta = 0$, the left speed is larger than the right speed at the shock, and their average is 0, which is why the shock location does not move with time.

Remark 3.3 (**The Euler solution continued after the shock).** For all $t \geqslant T_*$, let $\theta_*(t)$ denote the position of the discontinuity of $w(\cdot, t)$. Now for all $\theta \neq \theta_*(t)$, $w(\cdot, t)$ is smooth and hence defines a smooth solution to the Euler equations via the relations $\rho = u_\theta = \frac{1}{2} r w$ and $u_r = 0$. By the Lax-Olienik formula, the shock moves with speed $\frac{d}{dt}\theta_*(t) = \frac{1}{2}(w^- + w^+)$, where $w^- = \lim_{\theta \to \theta_*(t)^-} w(\theta, t)$ and $w^+ = \lim_{\theta \to \theta_*(t)^+} w(\theta, t)$. For $t > T_*$, we denote by $\Gamma(t)$ the line segment given by $\{(r, \theta) \colon \theta = \theta_*(t), r_1 \leqslant r \leqslant r_2\}$. Then for a piecewise smooth function $f(\cdot, t) : A_{r_0, r_1} \to \mathbb{R}$, which is discontinuous across $\Gamma(t)$, we let $[\![f]\!] = f^-(\cdot, t) - f^+(\cdot, t)$. From the discontinuity of $w(\cdot, t)$ we have that $[\![\rho(\cdot, t)]\!] > 0, [\![u_\theta]\!] > 0, [\![u_r]\!] = 0$. Moreover, the Rankine-Hugoniot conditions require that $\frac{d}{dt}\theta_*(t) = \frac{[\![\rho u_\theta]\!]}{[\![\rho]\!]}$. But $\frac{[\![\rho u_\theta]\!]}{[\![\rho]\!]} = \frac{1}{2}(w^- + w^+)$ and so the Rankine-Hugoniot condition is satisfied. This shows that (u_r, u_θ, ρ) is a global entropy solution to the compressible Euler system with $\gamma = 3$, which forms a shock at $T_* = \varepsilon$, becomes discontinuous across the line segment $\Gamma(t)$ for times $t > \varepsilon$, and propagates the shock with the correct shock speed.

4 Formation of shocks for the Euler equations

In this section, we construct a finite-time shock solution to the Euler equations for the general adiabatic constant $\gamma > 1$. We achieve this by studying the system of equations (2.11) on the time interval $-\varepsilon \leqslant t < T_* = \mathcal{O}(\varepsilon^{5/4})$, where T_* is constructed in the proof and $\varepsilon \in (0, 1)$ is a small parameter to be chosen later. We prove that a gradient blowup occurs at time T_* for the variable w, whereas $\partial_\theta z$ and $\partial_\theta a$ remain bounded.

4.1 Assumptions on the initial datum

In this subsection we describe the initial data that is used to construct the shock wave solutions to (2.11). The initial time is given by $-\varepsilon$, and the initial data is denoted as

$$w(\theta,-\varepsilon) = w_0(\theta), \qquad z(\theta,-\varepsilon) = z_0(\theta), \qquad a(\theta,-\varepsilon) = a_0(\theta).$$

We assume that $\partial_\theta w_0$ attains its global minimum at $\theta = 0$, and moreover that

$$w_0(0) = \kappa_0, \qquad \partial_\theta w_0(0) = -\varepsilon^{-1}, \qquad \partial_\theta^2 w_0(0) = 0, \qquad \partial_\theta^3 w_0(0) = 6\varepsilon^{-4}, \tag{4.1}$$

for some $\kappa_0 > 0$ to be determined later and whose main purpose is to ensure that the initial density is bounded from below by a positive constant (cf. (4.7)), and for an $0 < \varepsilon \ll 1$ to be determined. We also assume that w_0 has its first four derivatives bounded as

$$\|\partial_\theta w_0\|_{L^\infty} \leqslant \varepsilon^{-1}, \quad \|\partial_\theta^2 w_0\|_{L^\infty} \leqslant \varepsilon^{-5/2}, \quad \|\partial_\theta^3 w_0\|_{L^\infty} \leqslant 7\varepsilon^{-4}, \quad \|\partial_\theta^4 w_0\|_{L^\infty} \leqslant \varepsilon^{-11/2}, \tag{4.2}$$

which are bounds consistent with (4.1). In order to simplify the proof and to obtain a precise description of the solution's profile at the singular time (cf. (4.29) and (4.83) below), it is convenient to assume a slightly more precise behavior of $\partial_\theta w_0$ near $\theta = 0$. For this purpose we assume

$$\left|\varepsilon(\partial_\theta w_0)(\theta) - (\overline{W}_x)\left(\frac{\theta}{\varepsilon^{3/2}}\right)\right| \leqslant \min\left\{\frac{(\frac{\theta}{\varepsilon^{3/2}})^2}{40(1+(\frac{\theta}{\varepsilon^{3/2}})^2)}, \frac{1}{2(8+(\frac{\theta}{\varepsilon^{3/2}})^{2/3})}\right\} \tag{4.3}$$

for all $\theta \in \mathbb{T}$, where $\overline{W}$ is the stable globally self-similar solution to the Burgers equation defined in (2.14).

For z and a we assume that at the initial time we have

$$\|z_0\|_{C^n} + \|a_0\|_{C^n} \leqslant 1 \tag{4.4}$$

for $0 \leqslant n \leqslant 4$. Furthermore, we assume that w_0, z_0, and a_0 all have compact support such that

$$\operatorname{supp}(w_0(\theta)-\kappa_0) \cup \operatorname{supp}(z_0(\theta)) \cup \operatorname{supp}(a_0(\theta)) \subseteq (-\pi/2, \pi/2), \tag{4.5}$$

and in order to ensure the positivity of the initial density we assume that

$$\|w_0(\cdot) - \kappa_0\|_{L^\infty} \leqslant \frac{\kappa_0}{2}, \tag{4.6}$$

and choose κ_0 suitably. Indeed, in order to ensure that $P_0(\theta) \geqslant \nu_0 > 0$ for all $\theta \in \mathbb{T}$, we simply choose any

$$\kappa_0 \geqslant 4(2 + (2/\alpha)(\nu_0/2)^\alpha). \tag{4.7}$$

With this choice of κ_0, from (2.9), (4.4), and (4.6) we have that $(2/\alpha)P_0^\alpha(\theta) = w_0(\theta) - z_0(\theta) \geqslant \kappa_0/2 - 1 \geqslant (2/\alpha)\nu_0^\alpha$, thereby ensuring the desired strictly positive lower bound on the initial density.

Remark 4.1 (Consistency of the w_0 assumptions). condition (4.3), which may be rewritten in terms of $x = \theta\varepsilon^{-3/2}$ as $\left|\varepsilon(\partial_\theta w_0)(x\varepsilon^{3/2}) - (\overline{W}_x)(x)\right| \leqslant \min\{\frac{x^2}{40(1+x^2)}, \frac{1}{2(8+x^{2/3})}\}$ for all $|x| \leqslant \pi\varepsilon^{-3/2}$, is consistent with (4.1)–(4.2) and with (4.5)–(4.6), meaning that we can find an open set of initial conditions satisfying all of these assumptions. The first bound in the minimum of (4.3) is required in order to ensure that near $\theta = 0$ the deviation from the self-similar profile is parabolic; this is needed in view of (4.1) and the Taylor series of $\overline{W}_x$ near the origin (4.16a). The second condition in the minimum of (4.3) is not required in order to prove a finite-time singularity theorem; rather, this assumption is needed to characterize the blowup profile of $w(\theta,t)$ as $t \to T_*$ as being Hölder $C^{1/3}$ regular. Lastly, we note that (4.3) is consistent with $\partial_\theta w_0$ being the derivative of a periodic function, which implies that it must have zero average and so $\partial_\theta w_0$ cannot have a definite sign. Since $\overline{W}_x(x) < 0$ for all $x \in \mathbb{R}$, it is important that for $|x| \gg 1$, the envelope determined by the second term on the right side of (4.3) allows $\partial_\theta w_0$ to become positive. Indeed, note that in the Taylor series of $\overline{W}_x$ around infinity (4.16b), the coefficient of $x^{-2/3}$ is $-1/3$, while the coefficient of $x^{-2/3}$ in the Taylor series about infinity of the right side of (4.3) is $1/2 > 1/3$, which allows $\partial_\theta w_0$ to take on positive values.

Remark 4.2 (**L^∞ estimates for the solution).** Using assumptions (4.4), (4.6), and the fact that (2.11) is a system of forced transport equations in which the forcing terms show no derivative loss, we deduce via the maximum principle that

$$\|w(t)\|_{L^\infty} + \|z(t)\|_{L^\infty} + \|a(t)\|_{L^\infty} \leqslant M \tag{4.8}$$

holds for any $M \geqslant 4 + 2\kappa_0$, and all times t which are sufficiently small with respect to κ_0. This argument is detailed upon in Proposition 4.10 below, cf. estimate (4.78). In particular, these amplitude bounds hold for all $t \in [0, T_*)$ since $T_* = \mathcal{O}(\varepsilon^{5/4})$, and we take ε to be sufficiently small, in terms of κ_0.

Remark 4.3 (**The spatial support of the solution and an extension from $\mathbb{T}$ to $\mathbb{R}$).** Using (4.8) we obtain that the transport speeds on the left side of (2.11) are bounded solely in terms of M. Therefore, assuming ε to be sufficiently small depending on M and using that the length of $[-\varepsilon, T_*)$ is less than 2ε, by finite speed of propagation the solution (w, z, a) of (2.11) restricted to the region $\mathbb{T}\backslash[-\frac{3\pi}{4}, \frac{3\pi}{4}]$ is uniquely determined by the initial data (w_0, z_0, a_0) on the set $\mathbb{T}\backslash[-\frac{\pi}{2}, \frac{\pi}{2}]$, for all times $t \in [-\varepsilon, T_*]$. In particular, as a consequence of the support assumption (4.5), on the region $\mathbb{T}\backslash[-\frac{3\pi}{4}, \frac{3\pi}{4}]$, the solution (w, z, a) is constant in the angle θ (albeit a time dependent constant), for all times $t \in [-\varepsilon, T_*]$. Hence by abuse of notation we may extend the domain of (w, z, a) to $\theta \in \mathbb{R}$, by setting $w(\theta, t) = w(\pi, t)$, $z(\theta, t) = z(\pi, t)$, and $a(\theta, t) = a(\pi, t)$ for $|\theta| > \pi$. In what follows we adopt this abuse of notation, with the knowledge that the true solution is defined to be the periodization of the restriction to $[-\pi, \pi)$ of the extended solution. Also, we shall use implicitly throughout the proof that $\operatorname{supp}(\partial_\theta w) \cup \operatorname{supp}(\partial_\theta z) \cup \operatorname{supp}(\partial_\theta a) \subseteq [-3\pi/4, 3\pi/4]$.

4.2 Statement of the main result

Theorem 4.4 (**Formation of shocks for Euler).** *Let $\gamma > 1$, $\alpha = \frac{\gamma-1}{2}$, $0 < R_0 < r_0 < r_1 < R_1 < \infty$, and $\nu_0 > 0$. Then, there exist a sufficiently large $\kappa_0 = \kappa_0(\alpha, \nu_0) > 0$, a sufficiently large $M = M(\alpha, \kappa_0, \nu_0) \geqslant 1$, and a sufficiently small $\varepsilon = \varepsilon(\alpha, \kappa_0, \nu_0, M, R_0, R_1, r_0, r_1) \in (0, 1)$ such that the following holds.*

Assumptions on the initial data. Consider initial datum for the Euler equations (2.1), given at initial time $t_0 = -\varepsilon$ given as follows:

$$u_r(r, \theta, t_0) = ra_0(\theta)\,, u_\theta(r, \theta, t_0) = rb_0(\theta)\,, \text{ and } \rho_0(r, \theta, t_0) = r^{1/\alpha} P_0(\theta) \text{ for } (r, \theta) \in A_{R_0, R_1}\,,$$

where $(a_0, b_0, P_0) \in C^4(\mathbb{T})$ and $P_0 \geqslant \nu_0 > 0$. Define $w_0 = b_0 + \frac{1}{\alpha} P_0^\alpha$, $z_0 = b_0 - \frac{1}{\alpha} P_0^\alpha$, and suppose that (w_0, z_0, a_0) satisfy assumptions (4.1)–(4.6).

Shock formation for (a, z, w)-system (2.11). There exists a unique solution $(a, z, w) \in C([-\varepsilon, T_*); C^4(\mathbb{T}))$ to (2.11) which blows up in asymptotically self-similar fashion at time T_* and angle θ_*, such that:

- *the blowup time $T_* = \mathcal{O}(\varepsilon^{5/4})$ and angle $\theta_* = \mathcal{O}(\varepsilon)$ are explicitly computable, with $\theta_* = \lim_{t\to T_*} \xi(t)$,*
- $\sup_{t\in[-\varepsilon, T_*)} \left(\|a\|_{W^{1,\infty}(\mathbb{T})} + \|z\|_{W^{1,\infty}(\mathbb{T})} + \|w\|_{L^\infty(\mathbb{T})}\right) \leqslant C(M)$,
- $\lim_{t\to T_*} \partial_\theta w(\xi(t), t) = -\infty$ *and we have* $\frac{1}{2(T_*-t)} \leqslant \|\partial_\theta w(\cdot, t)\|_{L^\infty} \leqslant \frac{2}{T_*-t}$ *as* $t \to T_*$,
- *$w(\cdot, T_*)$ has a cusp singularity of Hölder $C^{1/3}$ regularity.*

Shock formation for the Euler equations (2.1). Setting $b = \frac{w+z}{2}$ and $P = (\frac{\alpha}{2}(w-z))^{1/\alpha}$, we define (u_r, u_θ, ρ) by (2.6). Consider the time-dependent domain $\Omega(t)$ defined in (2.13) such that $\Omega(t) \subset A_{R_0,R_1}$ for all $t \in [-\varepsilon, T_*]$. Then, $(u_r, u_\theta, \rho) \in C\left([-\varepsilon, T]; C^4(\Omega(t))\right)$ is a unique solution to the Euler equations (1.1) on the domain $\Omega(t)$ for all $-\varepsilon \leqslant t \leqslant T$, for any $T < T_*$, and

$$\lim_{t\to T_*} \partial_\theta u_\theta(r, \xi(t), t) = \lim_{t\to T_*} \partial_\theta \rho(r, \xi(t), t) = -\infty \qquad \text{for all } r \in \Omega(t)\,, \tag{4.9}$$

$$\sup_{t\in[-\varepsilon,T_*)} \sum_{k=0}^{1} \left(\|\partial_r^k \rho(\cdot,t)\|_{L^\infty(\Omega(t))} + \|\partial_r^k u(\cdot,t)\|_{L^\infty(\Omega(t))} \right) + \|\partial_\theta u_r(\cdot,t)\|_{L^\infty(\Omega(t))} \leqslant C(R_1, M)\,. \tag{4.10}$$

The shock occurs along the line segment $\Gamma(T_*) := \{(r,\theta) \in \Omega(T_*) \colon \theta = \theta_*\}$. The graphs of the blowup profiles $u_\theta(r,\theta,T_*)$ and $\rho(r,\theta,T_*)$ are surfaces with cusps along $\Gamma(T_*)$ and are Hölder $C^{1/3}$ smooth.

Non-trivial vorticity and density at the shock. The vorticity and density satisfy

$$\frac{1}{M^2} \leqslant \omega(\theta,t) \leqslant M^2\,, \qquad \rho(r,\theta,t) \geqslant \frac{R_0^{1/\alpha}\nu_0}{2} > 0\,,$$

for all $(r,\theta) \in \Omega(t)$ and $t \in [-\varepsilon, T_*)$.

Remark 4.5. With $u = (u_r, u_\theta)$, the flow η_u solving $\partial_t \eta_u = u \circ \eta_u$ with initial datum $\eta_u(r,\theta,-\varepsilon) = (r,\theta)$ is well defined and smooth on the time interval $[-\varepsilon, T]$ for all $T < T_*$. Moreover, since $\eta_u(r,\theta,t) = (r,\theta) + \int_{-\varepsilon}^t (u\circ\eta_u)(r,\theta,s)ds$, by (4.10), we see that

$$\sup_{[-\varepsilon,T_*)} \|\eta_u(\cdot,t)\|_{L^\infty(A_{r_0,r_1})} \leqslant C$$

Hence, by dominated convergence, we may define $\eta_u(r,\theta,T_*) = \lim_{t\to T_*} \eta_u(r,\theta,t)$. Thus, the set $\Omega(T_*)$ is well defined.

Remark 4.6. We have established that at the initial singularity time $t = T_*$, both u_θ and ρ have cusp singularities with $C^{1/3}$ regularity. For the case that $\gamma = 3$, we have explained how this cusp singularity develops an instantaneous discontinuity and is propagated as a shock wave. In Section 5 we conjecture that the same is true for the more general solution constructed in the previous theorem. We note that Alinhac [1, 2] proved the formation of cusp-type singularities for solutions of a quasilinear wave equation, but the Euler equations do not satisfy the structure of his equations.

Corollary 4.7 (Open set of initial conditions). *The conditions on the initial data (a_0, z_0, w_0) in Theorem 4.4 may be relaxed so that they may be taken to be in an open neighborhood in the C^4 topology.*

Proof of Corollary 4.7. First note that since the system (2.11) has finite speed of propagation, the support properties of the initial data described in (4.5) (see also Remark 4.3) are stable under small perturbations in the C^4 topology. Second, note that κ_0 and ε are free to be taken in an open set (sufficiently large, respectively sufficiently small), and hence the values of $w_0(0)$ and $\partial_\theta w_0(0)$ stated in (4.1) can be taken in an open set of possible values. Next, observe that if $\left\|\partial_\theta^4 w_0\right\|_{L^\infty} \leqslant \varepsilon^{-11/2}$ holds (condition which is stable under small C^4 perturbations) then a Taylor expansion around the origin yields

$$\begin{aligned}\partial_\theta^2 w_0(\theta) &= \partial_\theta^2 w_0(0) + \theta\partial_\theta^3 w_0(0) + \mathcal{O}(\varepsilon^{-11/2}\theta^2) \\ &= \partial_\theta^2 w_0(0) + 6\varepsilon^{-4}\theta + \theta(\partial_\theta^3 w_0(0) - 6\varepsilon^{-4}) + \mathcal{O}(\varepsilon^{-11/2}\theta^2)\,.\end{aligned}$$

Hence by continuity, for any $\bar\varepsilon > 0$ depending on ε, if one assumes $\partial_\theta^2 w_0(0)$ and $\partial_\theta^3 w_0(0) - 6\varepsilon^{-4}$ to be sufficiently small, there exists an θ_0 satisfying $|\theta_0| \leqslant \bar\varepsilon$ such that $\partial_\theta^2 w_0(\theta_0) = 0$. Hence by the change of coordinates $\theta \mapsto \theta + \theta_0$, and taking $\bar{\varepsilon}$ to be sufficiently small, we can relax the condition $\partial_\theta^2 w_0(0) = 0$ to the condition that $\partial_\theta^2 w_0 = 0$ is in a sufficiently small neighborhood of 0 and that $\partial_\theta^3 w_0(0)$ lies in a sufficiently small neighborhood of $6\varepsilon^{-4}$. Next, note that the rescaling $w_0(\theta) \mapsto \mu^{-1} w_0(\mu\theta)$, rescales $\partial_\theta^3 w_0(0)$ and leaves $\partial_\theta w_0(0)$ unchanged. Strictly speaking, such a rescaling would modify the domain; however, since our analysis only concerns a strict subset of the domain (due to (4.5)), and we have finite speed of propagation, as long as μ is sufficiently close to 1 this μ-rescaling does not pose an issue. Setting

$$\widetilde{a}(\theta,t) = \mu^{-1} a(\mu\theta,t), \quad \widetilde{w}(\theta,t) = \mu^{-1} w(\mu\theta,t), \quad \widetilde{z}(\theta,t) = \mu^{-1} z(\mu\theta,t),$$

the equation satisfied by $(\widetilde{a}, \widetilde{w}, \widetilde{z})$ is of the form (2.11), with the right hand side rescaled by a factor of μ. As long as μ is sufficiently close to 1, this rescaling has no effect on the proof of Theorem 4.4. Thus the condition on $\partial_\theta^3 w_0(0)$ may be relaxed to the condition that $\partial_\theta^3 w_0(0)$ lies in a sufficiently small neighborhood of $6\varepsilon^{-4}$. Finally, note that for θ small, (4.3) is implied by (4.1) and (4.2). For θ away from a small neighborhood of 0, the condition (4.3) is an open condition. Thus (4.3) does not pose an impediment to taking the initial data to lie in an open set. $\square$

4.3 Self-similar variables and solution ansatz

For the purpose of satisfying certain normalization constraints on the developing shock, we introduce three dynamic variables $\tau, \xi, \kappa \colon [-\varepsilon, T_*] \to \mathbb{R}$, and fix their initial values as at time $t = -\varepsilon$ as

$$\tau(-\varepsilon) = 0, \qquad \xi(-\varepsilon) = 0, \qquad \kappa(-\varepsilon) = \kappa_0. \tag{4.11}$$

The blowup time T_* and the blowup location θ_* are defined precisely in Remark 4.9. For the moment we only record that $T_* = \mathcal{O}(\varepsilon^{5/4})$, $\tau(T_*) = T_*$, and that by construction we will ensure $\tau(t) > t$ for all $t \in [-\varepsilon, T_*)$ (see Remark 4.9 below).

We introduce the following self-similar variables

$$x(\theta,t) := \frac{\theta - \xi(t)}{(\tau(t) - t)^{\frac{3}{2}}}, \qquad s(t) := -\log(\tau(t) - t). \tag{4.12}$$

The blowup time is defined by the relation $\tau(T_*) = T_*$. In the self-similar time, the blowup time corresponds to $s \to +\infty$. We will use frequently the identities

$$\tau - t = e^{-s}, \qquad \frac{ds}{dt} = \frac{1-\dot{\tau}}{\tau - t} = (1-\dot{\tau})e^s,$$

where we adopt the notation $\dot{f} = \frac{df}{dt}$, and

$$x = e^{\frac{3}{2}s}(\theta - \xi(t)), \qquad \partial_\theta x = e^{\frac{3}{2}s}, \qquad \partial_t x = \frac{-\dot{\xi}}{(\tau - t)^{\frac{3}{2}}} - \frac{3(\dot{\tau}-1)(\theta-\xi)}{2(\tau-t)^{\frac{5}{2}}} = -e^{\frac{3}{2}s}\dot{\xi} + \frac{3}{2}(1-\dot{\tau})xe^s.$$

Notice that at $t = -\varepsilon$, we have $s = -\log\varepsilon$ and hence $e^{-s} = \varepsilon$.

Using the self-similar variables x and s we rewrite w, z and a as

$$w(\theta,t) = e^{-\frac{s}{2}} W(x,s) + \kappa(t), \qquad z(\theta,t) = Z(x,s), \qquad a(\theta,t) = A(x,s). \tag{4.13}$$

As mentioned in Remark 4.3, the functions (W, Z, A) are defined on all of $\mathbb{R}$, but they are constant in x on the complement of the expanding set $\{x \colon -\frac{3\pi}{4}e^{3s/2} \leqslant x \leqslant \frac{3\pi}{4}e^{3s/2}\}$.

Inserting the ansatz (4.13) in the system (2.11), we obtain that W, Z and A satisfy the equations

$$
\begin{aligned}
&(1-\dot\tau)\left(\partial_s - \tfrac12\right)W + \left(e^{\frac s2}\left(\kappa - \dot\xi + \tfrac{1-\alpha}{1+\alpha}Z\right) + \tfrac32(1-\dot\tau)x + W\right)\partial_x W \\
&\qquad = -e^{-\frac s2}\dot\kappa - Ae^{-\frac s2}\left(\tfrac{1-2\alpha}{1+\alpha}Z - \tfrac{3+2\alpha}{1+\alpha}(e^{-\frac s2}W+\kappa)\right) \\
&(1-\dot\tau)\partial_s Z + \left(e^{\frac s2}\left(\tfrac{1-\alpha}{1+\alpha}\kappa - \dot\xi\right) + \tfrac{1-\alpha}{1+\alpha}W + \tfrac32(1-\dot\tau)x + e^{\frac s2}Z\right)\partial_x Z \\
&\qquad = -Ae^{-s}\left(\tfrac{1-2\alpha}{1+\alpha}(e^{-\frac s2}W+\kappa) - \tfrac{3+2\alpha}{1+\alpha}Z\right) \\
&(1-\dot\tau)\partial_s A + \left(e^{\frac s2}\left(\tfrac{1}{1+\alpha}(Z+\kappa) - \dot\xi\right) + \tfrac{1}{1+\alpha}W + \tfrac32(1-\dot\tau)x\right)\partial_x A \\
&\qquad = \tfrac{1}{2(1+\alpha)}e^{-s}\left(-4A^2 + (e^{-\frac s2}W+\kappa+Z)^2 - \alpha(e^{-\frac s2}W+\kappa-Z)^2\right).
\end{aligned}
$$

It is convenient to introduce the transport speeds

$$g_W := \tfrac{1}{1-\dot\tau}e^{\frac s2}\left(\kappa - \dot\xi + \tfrac{1-\alpha}{1+\alpha}Z\right), \tag{4.14a}$$

$$g_Z := \tfrac{1}{1-\dot\tau}\left(e^{\frac s2}\left(\tfrac{1-\alpha}{1+\alpha}\kappa - \dot\xi\right) + \tfrac{1-\alpha}{1+\alpha}W\right), \tag{4.14b}$$

$$g_A := \tfrac{1}{1-\dot\tau}\left(e^{\frac s2}\left(\tfrac{1}{1+\alpha}(Z+\kappa) - \dot\xi\right) + \tfrac{1}{1+\alpha}W\right), \tag{4.14c}$$

and the forcing terms

$$
\begin{aligned}
F_W &:= -\tfrac{e^{-\frac s2}}{(1+\alpha)(1-\dot\tau)}\left((1-2\alpha)AZ - (3+2\alpha)A(e^{-\frac s2}W+\kappa)\right), \\
F_Z &:= -\tfrac{e^{-s}}{(1+\alpha)(1-\dot\tau)}\left((1-2\alpha)A(e^{-\frac s2}W+\kappa) - (3+2\alpha)AZ\right), \\
F_A &:= \tfrac{e^{-s}}{2(1+\alpha)(1-\dot\tau)}\left(-4A^2 + (e^{-\frac s2}W+\kappa+Z)^2 - \alpha(e^{-\frac s2}W+\kappa-Z)^2\right),
\end{aligned}
$$

so that we can rewrite the evolution equations for W, Z and A as

$$\left(\partial_s - \tfrac12\right)W + \left(g_W + \tfrac{3x}{2} + \tfrac{1}{1-\dot\tau}W\right)\partial_x W = -e^{-\frac s2}\tfrac{\dot\kappa}{1-\dot\tau} + F_W, \tag{4.15a}$$

$$\partial_s Z + \left(g_Z + \tfrac{3x}{2} + \tfrac{1}{1-\dot\tau}e^{\frac s2}Z\right)\partial_x Z = F_Z, \tag{4.15b}$$

$$\partial_s A + \left(g_A + \tfrac{3x}{2}\right)\partial_x A = F_A. \tag{4.15c}$$

As long as the solutions remain smooth, the (W,Z,A) system (4.15) is equivalent to the original (w,z,a) formulation in (2.11). In particular, the local well-posedness of (4.15) from C^4-smooth initial datum of compact support follows from the corresponding well-posedness theorem for (2.11). The purpose of this section is to show that the dynamic modulation variables (κ,ξ,τ) remain uniformly bounded in C^1 and that the functions (W,Z,A) remain uniformly bounded in C^4 for all $s \in [-\log\varepsilon, \infty)$. Taking into account the self-similar transformation (4.12)–(4.13), and in view of the continuation criterion (2.12), this means that no singularities occur prior to time $t = T_*$. Additionally, we will ensure that $\partial_x W(0,s) = -1$ for all $s \geqslant -\log\varepsilon$, which in turn implies through the self-similar change of coordinates that $\partial_\theta w$ blows up as $-1/(T_* - t)$ as $t \to T_*$.

Remark 4.8 (The stable globally self-similar solution of the 1D Burgers equation). We view the evolution (4.15a) as a perturbation of the 1D Burgers dynamics. Indeed, if we set $g_W = \dot\tau = \dot\kappa = F_W \equiv 0$ in (4.15a), the resulting steady equation is the globally self-similar version of the 1D Burgers equation as described in (2.15). We recall that this steady globally self-similar solution $\overline{W}$ given explicitly by (2.14), and that its Taylor series expansions of $\partial_x \overline{W}$ at $x = 0$ and $x = \infty$, respectively, are given by

$$\partial_x \overline{W} = -1 + 3x^2 - 15x^4 + \mathcal{O}(x^6) \quad \text{for} \quad |x| \ll 1\,, \tag{4.16a}$$

$$\partial_x \overline{W} = -\tfrac{1}{3}x^{-\frac{2}{3}} - \tfrac{1}{9}x^{-\frac{4}{3}} + \mathcal{O}(x^{-\frac{8}{3}}) \quad \text{for} \quad |x| \gg 1\,. \tag{4.16b}$$

In the proof of our estimates for $\partial_x W$ and $\partial_{xx} W$ we will use a number of properties for $\overline{W}$, which may be checked directly using its explicit formula (2.14).

At this stage it is convenient to record the differentiated version of the system (4.15). For $n \in \mathbb{N}$, after applying ∂_x^n to (4.15) we obtain from the Leibniz rule that

$$\left(\partial_s + \tfrac{3n-1}{2} + \tfrac{n+\mathbf{1}_{n\neq 1}}{1-\dot\tau}\partial_x W + n\partial_x g_W\right)\partial_x^n W + \left(g_W + \tfrac{3x}{2} + \tfrac{1}{1-\dot\tau}W\right)\partial_x^{n+1} W = F_W^{(n)} \tag{4.17a}$$

$$\left(\partial_s + \tfrac{3n}{2} + \tfrac{n+\mathbf{1}_{n\neq 1}}{1-\dot\tau}e^{\frac{s}{2}}\partial_x Z + n\partial_x g_Z\right)\partial_x^n Z + \left(g_Z + \tfrac{3x}{2} + \tfrac{1}{1-\dot\tau}e^{\frac{s}{2}}Z\right)\partial_x^{n+1} Z = F_Z^{(n)} \tag{4.17b}$$

$$\left(\partial_s + \tfrac{3n}{2} + n\partial_x g_A\right)\partial_x^n A + \left(g_A + \tfrac{3x}{2}\right)\partial_x^{n+1} A = F_A^{(n)} \tag{4.17c}$$

where the forcing terms are given by

$$F_W^{(n)} := \partial_x^n F_W - \mathbf{1}_{n\geqslant 2}\partial_x^n g_W \partial_x W - \mathbf{1}_{n\geqslant 3}\sum_{k=2}^{n-1}\binom{n}{k}\left(\tfrac{1}{1-\dot\tau}\partial_x^k W + \partial_x^k g_W\right)\partial_x^{n-k+1} W$$

$$F_Z^{(n)} := \partial_x^n F_Z - \mathbf{1}_{n\geqslant 2}\partial_x^n g_Z \partial_x Z - \mathbf{1}_{n\geqslant 3}\sum_{k=2}^{n-1}\binom{n}{k}\left(\tfrac{1}{1-\dot\tau}e^{\frac{s}{2}}\partial_x^k Z + \partial_x^k g_Z\right)\partial_x^{n-k+1} Z$$

$$F_A^{(n)} := \partial_x^n F_A - \mathbf{1}_{n\geqslant 2}\sum_{k=2}^{n}\binom{n}{k}\partial_x^k g_A\, \partial_x^{n-k+1} A\,.$$

4.4 Constraints on W at $x = 0$ and the definitions of the modulation variables

Inspired by the self-similar analysis of the 1D Burgers equation in [12], we impose the following constraints at $x = 0$, which fully characterize the developing shock:

$$W(0,s) = 0, \qquad \partial_x W(0,s) = -1, \qquad \partial_x^2 W(0,s) = 0\,. \tag{4.18}$$

These constraints will fix our choices of $\tau(t)$, $\xi(t)$, and $\kappa(t)$. In order to compactly write the computations in this section, we shall denote

$$\varphi^0(s) = \varphi(0,s)\,, \qquad \varphi_x(x,s) = \partial_x\varphi(x,s)\,,\ \varphi_{xx}(x,s) = \partial_x^2\varphi(x,s)\,,\ \text{ etc.} \tag{4.19}$$

for any function $\varphi = \varphi(x,s)$.

In view of (4.18), in addition to (4.15a) we need to record (4.17a) for $n = 1$ and $n = 2$. Using (4.17a) we spell out these two equations

$$\left(\partial_s + 1 + \tfrac{1}{1-\dot\tau}W_x + \tfrac{(1-\alpha)}{(1+\alpha)(1-\dot\tau)}e^{\frac{s}{2}}Z_x\right)W_x + \left(g_W + \tfrac{3x}{2} + \tfrac{1}{1-\dot\tau}W\right)W_{xx} = F_W^{(1)} \tag{4.20a}$$

$$\left(\partial_s + \tfrac{5}{2} + \tfrac{3}{1-\dot\tau}W_x + \tfrac{2(1-\alpha)}{(1+\alpha)(1-\dot\tau)}e^{\frac{s}{2}}Z_x\right)W_{xx} + \left(g_W + \tfrac{3x}{2} + \tfrac{1}{1-\dot\tau}W\right)W_{xxx} = F_W^{(2)} \tag{4.20b}$$

where the forcing terms are given by

$$F_W^{(1)} := \partial_x F_W\,, \qquad \text{and} \qquad F_W^{(2)} := \partial_{xx} F_W - \tfrac{1-\alpha}{(1+\alpha)(1-\dot\tau)}e^{\frac{s}{2}}Z_{xx}W_x\,. \tag{4.21}$$

Using the notation (4.19), and inserting the constraints (4.18) into (4.20a) we arrive at

$$-\dot{\tau} + \tfrac{(1-\alpha)}{1+\alpha} e^{\frac{s}{2}} Z_x^0(s) = -(1-\dot{\tau}) F_W^{0,(1)}(s)\,,$$

which implies that

$$\dot{\tau} = \tfrac{1-\alpha}{1+\alpha} e^{\frac{s}{2}} Z_x^0(s) - e^{-\frac{s}{2}} \left(\tfrac{1-2\alpha}{1+\alpha} (AZ)_x^0(s) - \tfrac{3+2\alpha}{1+\alpha} (\kappa A_x^0(s) - e^{-\frac{s}{2}} A^0(s)) \right). \tag{4.22}$$

Plugging in the constraints (4.18) into (4.15a) and (4.20b), we further obtain that

$$-g_W^0(s) = F_W^0(s) - \tfrac{1}{1-\dot{\tau}} e^{-\frac{s}{2}} \dot{\kappa} \tag{4.23a}$$

$$g_W^0(s) W_{xxx}^0(s) = F_W^{0,(2)}(s)\,. \tag{4.23b}$$

Since we will prove that $W_{xxx}^0(s) \geqslant 5$, we solve the system (4.23a)–(4.23b) as

$$\dot{\xi} - \kappa - \tfrac{1-\alpha}{1+\alpha} Z^0(s) = -(1-\dot{\tau}) e^{-\frac{s}{2}} \frac{F_W^{0,(2)}}{W_{xxx}^0(s)}\,, \tag{4.24a}$$

$$\dot{\kappa} = (1-\dot{\tau}) e^{\frac{s}{2}} \left(F_W^0(s) + \frac{F_W^{0,(2)}}{W_{xxx}^0(s)} \right). \tag{4.24b}$$

The equations (4.22), (4.24a), and (4.24b) are the evolution equations for the dynamic modulation variables which are used in the proof. We also note here that in view of (4.14a) and (4.24a) we may write

$$g_W(x,s) = \frac{F_W^{0,(2)}}{W_{xxx}^0(s)} + \frac{(1-\alpha)}{(1+\alpha)(1-\dot{\tau})} e^{\frac{s}{2}} \left(Z(x,s) - Z^0(s) \right), \tag{4.25}$$

which provides us with a useful bound for g_W for $|x| \lesssim 1$.

4.5 Bootstrap assumptions

For the dynamic modulation variables, we assume that

$$|\kappa(t)| \leqslant 2\kappa_0, \qquad |\tau(t)| \leqslant \varepsilon^{\frac{5}{4}}, \qquad |\xi(t)| \leqslant 6M\varepsilon \tag{4.26a}$$

$$|\dot{\kappa}(t)| \leqslant M^3, \qquad |\dot{\tau}(t)| \leqslant \varepsilon^{\frac{1}{4}}, \qquad \left|\dot{\xi}(t)\right| \leqslant 3M \tag{4.26b}$$

for all $t < T_*$.

Note that from (4.8) and (4.26a) we deduce that (we use $\kappa_0 \leqslant M$)

$$\|W(s)\|_{L^\infty} \leqslant 2Me^{\frac{s}{2}} \qquad \text{and} \qquad \|Z(s)\|_{L^\infty} + \|A(s)\|_{L^\infty} \leqslant M \tag{4.27}$$

for all $s \geqslant -\log \varepsilon$. Therefore, no bootstrap assumptions are needed for the C^0 norms of (W, A, Z).

For the higher order derivatives of W we assume the following estimates for all times $s \geqslant -\log \varepsilon$

$$\left\|\partial_x^3 W\right\|_{L^\infty} \leqslant M^{\frac{3}{4}}, \quad \left\|\partial_x^4 W\right\|_{L^\infty} \leqslant M\,. \tag{4.28}$$

We further assume the more precise bounds

$$\left|W_x(x,s) - \overline{W}_x(x)\right| \leqslant \frac{x^2}{20(1+x^2)}\,, \tag{4.29}$$

$$|W_{xx}(x,s)| \leqslant \frac{12\,|x|}{(1+x^2)^{1/2}}\,, \tag{4.30}$$

$$|W_{xxx}(0,s) - 6| \leqslant 1\,, \tag{4.31}$$

where $\overline{W}$ is the exact self-similar solution of the Burgers equation given by (2.14) (see [3]). A comment is in order concerning (4.29): this inequality and properties of the function $\overline{W}_x$ imply that

$$\|W_x(\cdot,s)\|_{L^\infty} \leqslant 1 \qquad \text{for all} \qquad s \geqslant -\log\varepsilon. \tag{4.32}$$

Moreover, we note that (4.30) implies

$$\|W_{xx}(\cdot,s)\|_{L^\infty} \leqslant 12 \qquad \text{for all} \qquad s \geqslant -\log\varepsilon. \tag{4.33}$$

For the functions Z and A our bootstrap assumptions are

$$\|\partial_x^n Z\|_{L^\infty} + \|\partial_x^n A\|_{L^\infty} \leqslant M e^{-(\frac{1}{2}+\delta)s}\,, \tag{4.34}$$

for $1 \leqslant n \leqslant 4$, where $\delta = \delta(\alpha) > 0$ is defined as

$$\delta = \frac{\min\{\alpha,1\}}{2(1+\alpha)} > 0. \tag{4.35}$$

Note, that by definition, we have $\delta \leqslant \frac{1}{4}$. Moreover, δ is independent of ε or M, and depends only on α. We use essentially that $\gamma > 1$ to ensure that $\delta > 0$.

Remark 4.9 (**Estimating the blowup time and the blowup location).** The blowup time T_* is defined uniquely by the condition $\tau(T_*) = T_*$ which in view of (4.11) is equivalent to

$$\int_{-\varepsilon}^{T_*} (1-\dot\tau(t))dt = \varepsilon\,.$$

We note that in view of the $\dot\tau$ estimate in (4.26b), we have that $|T_*| \leqslant 2\varepsilon^{5/4}$. We also note here that the bootstrap assumption (4.26b) and the definition of T_* ensures that $\tau(t) > t$ for all $t \in [-\varepsilon, T_*)$. Indeed, when $t = -\varepsilon$ we have $\tau(-\varepsilon) = 0 > -\varepsilon$, and the function $t \mapsto \int_{-\varepsilon}^t (1-\dot\tau)dt' - \varepsilon = t - \tau(t)$ is strictly increasing. The blowup location is determined by $\theta_* = \xi(T_*)$, which by (4.11) is the same as

$$\theta_* = \int_{-\varepsilon}^{T_*} \dot\xi(t)dt\,.$$

In view of (4.26b) we deduce that $|\theta_*| \leqslant 6M\varepsilon$, so that the blowup location is $\mathcal{O}(\varepsilon)$ close to the origin.

4.6 Closure of bootstrap

Throughout the proof we shall use the notation $\lesssim$ to denote an inequality which holds up to a sufficiently large multiplicative constant $C > 0$, which may only depend on α (hence on γ), but not on s, M, or ε.

4.6.1 The Z estimates

First we consider the equation obeyed by Z_x, given by (4.17b) with $n = 1$. Recalling (4.14b), and appealing to the bootstrap assumptions (4.26b), (4.29) (in fact, we use its consequence, the bound (4.32)), and (4.34), we see that the damping term in the Z_x evolution may be bounded from below as

$$\begin{aligned}\frac{3}{2} + \frac{e^{\frac{s}{2}} Z_x}{1-\dot\tau} + \partial_x g_Z &= \frac{3}{2} + \frac{e^{\frac{s}{2}} Z_x}{1-\dot\tau} + \frac{(1-\alpha)W_x}{(1-\dot\tau)(1+\alpha)} \\ &\geqslant \frac{3}{2} - (1+2\varepsilon^{\frac{1}{4}})\left(M\varepsilon^\delta + \frac{|1-\alpha|}{1+\alpha}\right) \geqslant \frac{1}{2} + \delta\end{aligned} \tag{4.36}$$

for all $s \geqslant -\log\varepsilon$, where we have used the parameter $\delta = \delta(\alpha)$ defined in (4.35) above. In deriving (4.36), we have used that

$$(1+2\varepsilon^{\frac14})\left(M\varepsilon^{\delta} + \left|\frac{1-\alpha}{1+\alpha}\right|\right) \leqslant (1+2\varepsilon^{\frac14})\left(M\varepsilon^{\delta} + 1 - 2\delta\right) \leqslant 1-\delta$$

which is true as long as ε is taken to be sufficiently small, depending only on α (through δ), and on M.

On the other hand, the forcing term in the Z_x equation, $F_Z^{(1)} = \partial_x F_Z$ may be estimated using (4.8), (4.26a), (4.28), and (4.34) as

$$\begin{aligned}\left\|F_Z^{(1)}\right\|_{L^\infty} &\lesssim \frac{e^{-s}}{1-\dot\tau}\left(\|A_x\|_{L^\infty}\left(\left\|(e^{-\frac{s}{2}}W+\kappa)\right\|_{L^\infty} + \|Z\|_{L^\infty}\right) + \|A\|_{L^\infty}\left(e^{-\frac{s}{2}}\|W_x\|_{L^\infty} + \|Z_x\|_{L^\infty}\right)\right)\\ &\lesssim Me^{-s}\left(Me^{-(\frac12+\delta)s} + e^{-\frac{s}{2}}\right)\\ &\lesssim M^2 e^{-\frac32 s}. \end{aligned} \tag{4.37}$$

With (4.36) and (4.37), from (4.17b) with $n=1$ and a standard maximum principle argument (cf. Lemma A.1, estimate (A.2)), with $\lambda_D = \frac12+\delta$, $\lambda_F = \frac32$, and $s_0 = -\log\varepsilon$), we obtain that

$$\begin{aligned}\|Z_x(s)\|_{L^\infty} &\lesssim \|Z_x(-\log\varepsilon)\|_{L^\infty} e^{-(\frac12+\delta)(s+\log\varepsilon)} + M^2 e^{(1-\delta)\log\varepsilon} e^{-(\frac12+\delta)s}\\ &\lesssim \left(\varepsilon^{1-\delta} + M^2\varepsilon^{1-\delta}\right) e^{-(\frac12+\delta)s}\end{aligned}$$

where we used (4.4) to deduce $\|Z_x(-\log\varepsilon)\|_{L^\infty} = \varepsilon^{\frac32}\|\partial_\theta z_0\|_{L^\infty} \leqslant \varepsilon^{\frac32}$. Then, taking ε sufficiently small in terms of M, and using $\delta \leqslant 1/4$ we obtain

$$\|Z_x(s)\|_{L^\infty} \leqslant \varepsilon^{\frac14} e^{-(\frac12+\delta)s} \leqslant \frac{M}{2} e^{-(\frac12+\delta)s}, \tag{4.38}$$

closing the bootstrap (4.34) for Z_x.

Similarly to the estimate for $\partial_x Z$, we note that for $2 \leqslant n \leqslant 4$, the damping term in (4.17b) may be bounded from below as

$$\begin{aligned}\frac{3n}{2} + \frac{n+1}{1-\dot\tau}e^{\frac{s}{2}}\partial_x Z + n\partial_x g_Z &\geqslant \frac{3n}{2} - n(1+2\varepsilon^{\frac14})\|W_x\|_{L^\infty} - (n+1)(1+2\varepsilon^{\frac14})e^{\frac{s}{2}}\|\partial_x Z\|_{L^\infty}\\ &\geqslant \frac{3n}{2} - n(1+2\varepsilon^{\frac14}) - 5(1+2\varepsilon^{\frac14})M\varepsilon^{\delta} \geqslant \frac34, \end{aligned} \tag{4.39}$$

for all $s \geqslant -\log\varepsilon$, by appealing to our bootstrap assumptions and by assuming ε is sufficiently small in terms of M. On the other hand, using our bootstrap assumptions, and the strong bound established earlier in (4.38), one may show that the forcing term on the right side of (4.17b) may be estimated as

$$\begin{aligned}\left\|F_Z^{(n)}\right\|_{L^\infty} &\lesssim \|\partial_x^n F_Z\|_{L^\infty} + \|\partial_x^n g_Z\|_{L^\infty}\|\partial_x Z\|_{L^\infty} + \mathbf{1}_{n\geqslant 3}\sum_{k=2}^{n-1}\left(e^{\frac{s}{2}}\left\|\partial_x^k Z\right\|_{L^\infty} + \left\|\partial_x^k g_Z\right\|_{L^\infty}\right)\left\|\partial_x^{n-k+1}Z\right\|_{L^\infty}\\ &\lesssim M^2 e^{-s} + M\varepsilon^{\frac14}e^{-(\frac12+\delta)} + \mathbf{1}_{\{n\geqslant 3\}}\sum_{k=2}^{n-1} M\left\|\partial_x^{n-k+1}Z\right\|_{L^\infty}\\ &\lesssim M\left(\varepsilon^{\frac14}e^{-(\frac12+\delta)s} + \mathbf{1}_{\{n\geqslant 3\}}\sum_{k=2}^{n-1}\left\|\partial_x^{n-k+1}Z\right\|_{L^\infty}\right), \end{aligned} \tag{4.40}$$

where we have assumed ε to be sufficiently small, dependent on M in order to bound the first term on the second line in terms of the second term. We also remark that since $\partial_x^n Z(\cdot, -\log\varepsilon) = \varepsilon^{\frac{3n}{2}}\partial_\theta^n z_0(\cdot)$, by (4.4) we have

$$\|\partial_x^n Z(\cdot, -\log\varepsilon)\|_{L^\infty} \leqslant \varepsilon^3 ,$$

for all $n \geqslant 2$.

Let us first treat the case $n = 2$, when the second term on the right side of (4.40) is absent. Therefore, in view of (4.39)–(4.40), and applying Lemma A.1 to the evolution equation for $\partial_x^n Z$ given by (4.17b) (with $\lambda_D = \frac{3}{4}$, $\mathcal{F}_0 = M\varepsilon^{\frac14}$, and $\lambda_F = \frac12 + \delta$), we arrive using (A.1) at

$$\begin{aligned}\left\|\partial_x^2 Z(s)\right\|_{L^\infty} &\lesssim \left\|\partial_x^2 Z(\cdot, -\log\varepsilon)\right\|_{L^\infty} e^{-\frac34(s+\log\varepsilon)} + M\varepsilon^{\frac14}e^{-(\frac12+\delta)s} \\ &\lesssim \varepsilon^{\frac94}e^{-\frac34 s} + M\varepsilon^{\frac14}e^{-(\frac12+\delta)s} \lesssim M\varepsilon^{\frac14}e^{-(\frac12+\delta)s}\end{aligned} \tag{4.41}$$

for all $s \geqslant -\log\varepsilon$.

With (4.41) in hand, we return to treat the case $n = 3$. Then the second term on the right side of (4.40) is estimated by a constant multiple of $M^2\varepsilon^{\frac14}e^{-(\frac12+\delta)s}$. Therefore, the total estimate on the force for $\partial_x^3 Z$ is given by $\left\|F_Z^{(3)}\right\|_{L^\infty} \lesssim M^2\varepsilon^{\frac14}e^{-(\frac12+\delta)s}$. The only modification, as compared to the case $n = 2$, is that M becomes M^2. Therefore, an argument similar to the one yielding (4.41) gives the estimate

$$\left\|\partial_x^3 Z(s)\right\|_{L^\infty} \lesssim M^2\varepsilon^{\frac14}e^{-(\frac12+\delta)s} . \tag{4.42}$$

Using (4.41) and (4.42), we next return to the forcing estimate (4.40) for $n = 4$. Similar arguments yield $\left\|F_Z^{(4)}\right\|_{L^\infty} \lesssim M^3\varepsilon^{\frac14}e^{-(\frac12+\delta)s}$, by taking ε to be sufficiently small, in terms of M. Yet another application of Lemma A.1, similarly to (4.41) implies that

$$\left\|\partial_x^4 Z(s)\right\|_{L^\infty} \lesssim M^3\varepsilon^{\frac14}e^{-(\frac12+\delta)s} . \tag{4.43}$$

In conclusion, assuming that ε is taken to be sufficiently small, dependent on M, then the bounds (4.41), (4.42), and (4.43) close the bootstrap assumptions for $\partial_x^n Z$ (with $2 \leqslant n \leqslant 4$) stated in (4.34).

4.6.2 The A estimates

Next we turn to the $\partial_x^n A$ estimates for $1 \leqslant n \leqslant 4$. These bounds are established very similarly to the Z estimates proven earlier. The damping term in (4.17c) is estimated using (4.32) and (4.34) as

$$\frac{3n}{2} + n\partial_x g_A = \frac{3n}{2} + \frac{n(W_x + e^{\frac{s}{2}}Z_x)}{(1+\alpha)(1-\dot\tau)} \geqslant \frac{3n}{2} - \frac{n(1+2\varepsilon^{\frac14})(1+M\varepsilon^\delta)}{1+\alpha} \geqslant \frac{n}{2} + \delta , \tag{4.44}$$

upon taking ε small enough in terms of δ (as defined in (4.35) above) and in terms of $\alpha > 0$ and M. The forcing term on the right side of (4.17c) may be bounded from above using our bootstrap assumptions as

$$\left\|F_A^{(n)}\right\|_{L^\infty} \lesssim M^2 e^{-s} + M\mathbf{1}_{\{n\geqslant 2\}}\sum_{k=2}^{n}\left\|\partial_x^{n-k+1}A\right\|_{L^\infty} . \tag{4.45}$$

Moreover, note that by (4.4) we have

$$\|\partial_x^n A(\cdot, -\log\varepsilon)\|_{L^\infty} \leqslant \varepsilon^{\frac32} ,$$

for all $n \geqslant 1$. At this stage one may employ a similar scheme to the one employed in the Z estimates. First, we treat the case $n = 1$ since in that case the second forcing term on the right side of (4.45) is absent. With (4.44) in mind we apply Lemma A.1, and deduce (similarly to (4.41)) that

$$\|\partial_x A(s)\|_{L^\infty} \leqslant \varepsilon^{\frac{1}{4}} e^{-(\frac{1}{2}+\delta)s}, \tag{4.46}$$

where again we absorbed M^2 and the implicit constants by assuming ε to be sufficiently small. Using the bound (4.46) we may return the case $n = 2$, and use that the extra forcing term present on the right side of (4.45) is bounded a constant multiple of $M \|\partial_x A\|_{L^\infty} \lesssim M\varepsilon^{\frac{1}{4}} e^{-(\frac{1}{2}+\delta)s}$, upon taking ε sufficiently small. This argument may be then iterated essentially because in the sum on the right side of (4.45) we always have $n-k+1 \leqslant n-1$, so that only norms of A that are already known to be small arise. Using Lemma A.1 one may then show iteratively that

$$\|\partial_x^n A(s)\|_{L^\infty} \lesssim M^{n-1} \varepsilon^{\frac{1}{4}} e^{-(\frac{1}{2}+\delta)s} \tag{4.47}$$

for all $2 \leqslant n \leqslant 4$. Taking ε sufficiently small, dependent on M, then (4.46) and (4.47) close the bootstrap assumptions on $\partial_x^n A$ stated in (4.34).

4.6.3 Bounds on the modulation variables τ, κ, and ξ

From (4.22), using the bounds (4.8), (4.26a), (4.38), and (4.46), we obtain

$$\begin{aligned}
|\dot\tau| &\lesssim e^{\frac{s}{2}} \|Z_x\|_{L^\infty} + e^{-\frac{s}{2}} \|A\|_{L^\infty} \left(\|Z_x\|_{L^\infty} + e^{-\frac{s}{2}}\right) + e^{-\frac{s}{2}} \|A_x\|_{L^\infty} \left(\|Z\|_{L^\infty} + \kappa_0\right) \\
&\lesssim \varepsilon^{\frac{1}{4}} e^{-\delta s} + M e^{-s} \left(\varepsilon^{\frac{1}{4}} e^{-\delta s} + 1\right) + \varepsilon^{\frac{1}{4}} e^{-(1+\delta)s} (M + \kappa_0)
\end{aligned}$$

The implicit constant is universal. Hence for $s \geqslant -\log \varepsilon$, upon taking ε small to be sufficiently small solely in terms of M and δ, we obtain from the above that

$$|\dot\tau| \leqslant C\varepsilon^{\frac{1}{4}} e^{-\delta s} \leqslant C\varepsilon^{\frac{1}{4}+\delta} \leqslant \tfrac{1}{2}\varepsilon^{\frac{1}{4}}. \tag{4.48}$$

Integrating in t for $t \leqslant T_*$, and using that $\tau(-\varepsilon) = 0$, we obtain

$$|\tau| \leqslant \tfrac{1}{2}\varepsilon^{\frac{5}{4}},$$

proving the τ bounds in (4.26a)–(4.26b).

Aa consequence of (4.24b), (4.8) and the bootstrap assumptions, by inspection we obtain

$$|\dot\kappa| \leqslant 2e^{\frac{s}{2}} \left(\left|F_W^0(s)\right| + \left|F_W^{0,(2)}(s)\right| \right) \leqslant M^3$$

assuming that M is taken to be sufficiently large (in terms of just universal constants). Integrating in t from $-\varepsilon$ to T_*, and assuming that ε is sufficiently small (in terms of M and κ_0), yields

$$|\kappa(t)| \leqslant \tfrac{3}{2}\kappa_0.$$

This establishes the κ bounds in (4.26a)–(4.26b).

Similarly, from(4.24a), (4.8) and the bootstrap assumptions, by inspection we obtain

$$\left|\dot\xi\right| \leqslant |\kappa| + \left|Z^0(s)\right| + e^{-\frac{s}{2}} \left|F_W^{0,(2)}\right| \leqslant \tfrac{3}{2}(\kappa_0 + M) \leqslant \tfrac{5}{2}M$$

upon taking ε to be sufficiently small, in terms of M, and recalling cf. Remark 4.2 that $2\kappa_0 \leqslant M$. Integrating in t from $-\varepsilon$ to T_*, which obeys $|T_*| \leqslant 2\varepsilon^{5/4}$, and using that $\xi(-\varepsilon) = 0$, we arrive at

$$|\xi(t)| \leqslant 5M\varepsilon,$$

which proves the ξ estimates in (4.26a)–(4.26b).

4.6.4 Estimates for W

The third derivative at $x = 0$. Our first goal is to establish (4.31). The evolution of $\partial_x^3 W^0(s)$ is obtained by restricting (4.17a) with $n = 3$ to $x = 0$, using the constraints (4.18), and the definition of $\dot{\xi}$ in (4.24a). We obtain (noting that $\partial_x^3 F_W$ also contains the term $\partial_x^3 W$):

$$\left(\partial_s + 4\left(1 - \frac{1}{1-\dot{\tau}}\right) + 3\frac{e^{\frac{s}{2}}}{1-\dot{\tau}}\frac{1-\alpha}{1+\alpha}Z_x^0(s) - \frac{e^{-s}(3+2\alpha)}{(1+\alpha)(1-\dot{\tau})}A^0(s)\right)W_{xxx}^0(s)$$
$$= \frac{F_W^{0,(2)}(s)}{W_{xxx}^0(s)}W_{xxxx}^0(s) + \frac{(1-\alpha)e^{\frac{s}{2}}Z_{xxx}^0(s)}{(1+\alpha)(1-\dot{\tau})} - \frac{e^{-\frac{s}{2}}(1-2\alpha)}{(1+\alpha)(1-\dot{\tau})}(AZ)_{xxx}^0(s)$$
$$+ \frac{e^{-\frac{s}{2}}(3+2\alpha)}{(1+\alpha)(1-\dot{\tau})}\left(\kappa A_{xxx}^0(s) - 3e^{-\frac{s}{2}}A_{xx}^0(s)\right). \tag{4.49}$$

We bound the terms of the above evolution using (4.8), (4.26b), (4.28), (4.31), (4.32), and (4.34). After a calculation, we obtain that the right side of (4.49) is bounded by

$$\lesssim M\left(M^2e^{-(\frac{1}{2}+\delta)s} + Me^{-(1+\delta)s}\right) + Me^{-\delta s} + M^2e^{-(1+\delta)s} + e^{-\frac{s}{2}}\left(\kappa_0 Me^{-(\frac{1}{2}+\delta)s} + Me^{-(1+\delta)s}\right)$$
$$\lesssim Me^{-\delta s}$$

where we have assumed ε to be sufficiently small such that the second term dominates all other terms. On the other hand, the damping term on the left side of (4.49) may be estimated in absolute value, upon appealing to the first inequality in (4.48), by

$$\lesssim \varepsilon^{\frac{1}{4}}e^{-\delta s} + Me^{-\delta s} + Me^{-s} \lesssim Me^{-\delta s}$$

for $s \geqslant -\log\varepsilon$. Therefore, by also appealing to the bootstrap assumption (4.31), we have proven that

$$\left|\partial_s W_{xxx}^0(s)\right| \lesssim Me^{-\delta s}(\left|W_{xxx}^0(s)\right| + 1) \lesssim Me^{-\delta s}.$$

Recalling that $W_{xxx}^0(0) = 6$, and using the fundamental theorem of calculus in time, we obtain

$$\left|W_{xxx}^0(s) - 6\right| \lesssim M\int_{-\log\varepsilon}^{s} e^{-\delta s'}ds' \lesssim \frac{M}{\delta}\varepsilon^{\delta} \leqslant \varepsilon^{\frac{\delta}{2}} \tag{4.50}$$

upon taking ε to be sufficiently small, in terms of M and δ. Since $\varepsilon < 1$, we close the bootstrap (4.31).

The first derivative. We prove (4.29) in two steps, first for $|x| \leqslant \ell$ for some $\ell > 0$ to be determined below (cf. (4.51)), and then for $|x| \geqslant \ell$. Using a Taylor expansion around $x = 0$ together with the constraints (4.18), we obtain

$$W_x(x,s) + 1 - 3x^2 = x^2\left(\frac{1}{2}W_{xxx}^0(s) - 3\right) + \frac{x^3}{6}W_{xxxx}(x',s)$$

for some x' with $|x'| < |x|$. Using (4.50) and (4.28) we arrive at

$$\left|W_x(x,s) + 1 - 3x^2\right| \leqslant x^2\left(\varepsilon^{\frac{\delta}{2}} + \frac{M|x|}{6}\right) \leqslant x^2\left(\varepsilon^{\frac{\delta}{2}} + \frac{M\ell}{6}\right)$$

for all $|x| \leqslant \ell$. Then, recalling (4.16a), we see that the above estimate implies

$$\left|W_x(x,s) - \overline{W}_x\right| \leqslant x^2\left(\varepsilon^{\frac{\delta}{2}} + \frac{M\ell}{6} + 15\ell^2\right) \leqslant \frac{x^2}{40(1+x^2)}$$

for all $|x| \leqslant \ell$, as soon as we choose

$$\ell \leqslant \frac{1}{40M}\,, \tag{4.51}$$

M sufficiently large, and ε sufficiently small in terms of M and δ. Thus, we improve upon the bootstrap assumption (4.29) for $|x| \leqslant \ell$, as desired.

It remains to establish (4.29) for $|x| \geqslant \ell$. For this purpose it is convenient to define

$$\widetilde{W} = W - \overline{W}\,,$$

so that from (4.20a) and the differentiated form of (2.15), $\widetilde{W}_x$ is the solution of

$$\begin{aligned}\left(\partial_s + 1 + \frac{\widetilde{W}_x + 2\overline{W}_x}{1-\dot\tau} + \frac{(1-\alpha)e^{\frac{s}{2}}Z_x}{(1+\alpha)(1-\dot\tau)}\right)\widetilde{W}_x + \left(g_W + \frac{3x}{2} + \frac{W}{1-\dot\tau}\right)\widetilde{W}_{xx}\\ = \partial_x F_W - \left(g_W + \frac{\widetilde{W} + \dot\tau\overline{W}}{1-\dot\tau}\right)\overline{W}_{xx} - \left(\frac{\dot\tau\overline{W}_x}{1-\dot\tau} + \frac{(1-\alpha)e^{\frac{s}{2}}Z_x}{(1+\alpha)(1-\dot\tau)}\right)\overline{W}_x\,.\end{aligned} \tag{4.52}$$

Note that by (4.18) and (4.16a), we have $\widetilde{W}(0,s) = \widetilde{W}_x(0,s) = \widetilde{W}_{xx}(0,s) = 0$. Next, we define

$$V(x,s) = \frac{\widetilde{W}_x(1+x^2)}{x^2}$$

so that establishing (4.29) is equivalent to proving that $|V| \leqslant \frac{1}{20}$ for all $s > -\log\varepsilon$ and all $|x| \geqslant \ell$. It is important here that we are avoiding $x = 0$ (since we concerned with $|x| > \ell$), in view of the division by x^2. It follows from (4.52) and a short computation that

$$\begin{aligned}&\partial_s V + \left(1 + \frac{\widetilde{W}_x + 2\overline{W}_x}{1-\dot\tau} + \frac{(1-\alpha)e^{\frac{s}{2}}Z_x}{(1+\alpha)(1-\dot\tau)} + \frac{2}{x(1+x^2)}\left(g_W + \frac{3x}{2} + \frac{W}{1-\dot\tau}\right)\right)V\\ &\quad + \left(g_W + \frac{3x}{2} + \frac{W}{1-\dot\tau}\right)V_x\\ &= \frac{(1+x^2)\partial_x F_W}{x^2} - \left(g_W + \frac{\dot\tau\overline{W}}{1-\dot\tau}\right)\frac{(1+x^2)\overline{W}_{xx}}{x^2} - \left(\frac{\dot\tau\overline{W}_x}{1-\dot\tau} + \frac{(1-\alpha)e^{\frac{s}{2}}Z_x}{(1+\alpha)(1-\dot\tau)}\right)\frac{(1+x^2)\overline{W}_x}{x^2}\\ &\quad - \frac{1}{1-\dot\tau}\frac{(1+x^2)\overline{W}_{xx}}{x^2}\int_0^x V(x')\frac{(x')^2}{1+(x')^2}dx'\,.\end{aligned} \tag{4.53}$$

The evolution equation for V takes the form of a damped and non-locally forced transport equation, of the general form given in (A.3) below. Our goal is to apply Lemma A.2 to (4.53).

The main observation which allows us to bound the solution V of (4.53) is that the *explicit formula* for $\overline{W}$ in (2.14) implies the lower bound

$$1 + 2\overline{W}_x + \frac{2}{x(1+x^2)}\left(\frac{3x}{2} + \overline{W}\right) \geqslant \frac{6x^2}{1+8x^2} \tag{4.54}$$

for all $x \in \mathbb{R}$. Since we are analyzing $|x| \geqslant \ell$, the above estimate yields a strictly positive damping term in the V equation. In order to see this, let us estimate the remaining terms in the damping factor for V on the left side of (4.53). We claim that for all $|x| \geqslant \ell$, we have that

$$\left|\frac{\widetilde{W}_x + 2\dot\tau\overline{W}_x}{1-\dot\tau} + \frac{(1-\alpha)e^{\frac{s}{2}}Z_x}{(1+\alpha)(1-\dot\tau)} + \frac{2}{x(1+x^2)}\left(g_W + \widetilde{W} + \frac{\dot\tau W}{1-\dot\tau}\right)\right| \leqslant \frac{5x^2}{4(1+8x^2)} + \varepsilon^{\frac{\delta}{2}}\,. \tag{4.55}$$

Indeed, using the $\dot\tau$ estimate (4.26b), the fact that $\left|\overline{W}_x\right| \leqslant 1$, and the bootstrap assumptions, we deduce that

$$\begin{aligned}\left|\frac{\widetilde{W}_x + 2\dot\tau\overline{W}_x}{1-\dot\tau} + \frac{(1-\alpha)e^{\frac{s}{2}}Z_x}{(1+\alpha)(1-\dot\tau)} + \frac{2\widetilde{W}}{x(1+x^2)}\right| &\leqslant (1+2\varepsilon^{\frac14})\left(\frac{3x^2}{20(1+x^2)} + 2\varepsilon^{\frac14} + M\varepsilon^{\delta}\right)\\ &\leqslant \frac{5x^2}{4(1+8x^2)} + 2M\varepsilon^{\delta}\end{aligned} \tag{4.56}$$

since ε is sufficiently small. Here we have used that $\widetilde{W}(0,s)=0$, and thus that

$$\left|\frac{2\widetilde{W}(x,s)}{x(1+x^2)}\right| \leqslant \frac{2}{|x|\,(1+x^2)}\int_0^{|x|}\left|\widetilde{W}_x(x',s)\right|dx' \leqslant \frac{1}{10\,|x|\,(1+x^2)}\int_0^{|x|}\frac{(x')^2}{1+(x')^2}dx' \leqslant \frac{x^2}{10(1+x^2)}\,.$$

Similarly, using the constraint (4.18) and the bound (4.32), we may directly estimate

$$\frac{2\,|\dot\tau W(x,s)|}{x(1+x^2)(1-\dot\tau)} \leqslant \frac{4\varepsilon^{\frac14}}{x}\int_0^x \left|W_x(x',s)\right|dx' \leqslant 4\varepsilon^{\frac14}\,. \tag{4.57}$$

Recall the identities (4.25) and (4.21). Note that by (4.34) we have $\left|Z(x,s)-Z^0(s)\right| \leqslant M|x|e^{-(\frac12+\delta)s}$. Then, by appealing to (4.8), (4.34) and the constraints (4.18), we may deduce that

$$\begin{aligned}|g_W(x,s)| &\leqslant \frac{(1-\alpha)e^{\frac{s}{2}}}{(1+\alpha)(1-\dot\tau)}\left|Z(x,s)-Z^0(s)\right| + \frac{\left|F_W^{0,(2)}\right|}{W^0_{xxx}(s)}\\ &\lesssim |x|\,Me^{-\delta s} + \left\|\partial_{xx}F_W^0\right\|_{L^\infty} + e^{\frac{s}{2}}\left\|Z^0_{xx}\right\|_{L^\infty}\\ &\lesssim |x|\,Me^{-\delta s} + M^2e^{-s} + Me^{-\delta s}\\ &\lesssim \ell^{-1}M\,|x|\,e^{-\delta s}\end{aligned} \tag{4.58}$$

for any $\ell \leqslant |x|$. Choosing ε sufficiently small in terms of δ and M, and combining (4.56)–(4.58) yields the proof of (4.55). In turn, combining (4.54) and (4.55) we obtain that the total damping term in (4.53) may be bounded from below as

$$1 + \frac{\widetilde{W}_x + 2\overline{W}_x}{1-\dot\tau} + \frac{(1-\alpha)e^{\frac{s}{2}}Z_x}{(1+\alpha)(1-\dot\tau)} + \frac{2}{x(1+x^2)}\left(g_W + \frac{3x}{2} + \frac{W}{1-\dot\tau}\right) \geqslant \frac{9x^2}{2(1+8x^2)} \tag{4.59}$$

pointwise for all $|x| \geqslant \ell$. Here we have implicitly used that $\varepsilon^{\frac{\delta}{2}} \leqslant \frac{\ell^2}{16} \leqslant \frac{x^2}{4(1+8x^2)}$ for $|x| \geqslant \ell$ since by (4.51), ℓ is small enough when M is large. From (4.59) and the fact that the function $\frac{9x^2}{2(1+8x^2)}$ is monotone increasing in $|x|$, we obtain that the damping term in (4.53) is bounded from below by $\lambda_D := \frac{9\ell^2}{2(1+8\ell^2)}$ for all $|x| \geqslant \ell$, as required by (A.4).

Our next observation concerns the last term on the right side of (4.53), which is nonlocal in V. We may write this term as the integral of $V(x',s)$ against the kernel

$$\mathcal{K}(x,x',s) = -\frac{1}{1-\dot\tau}\frac{(1+x^2)\overline{W}_{xx}(x)}{x^2}\mathbf{1}_{[0,x]}(x')\frac{(x')^2}{1+(x')^2}\,.$$

Since we know $\overline{W}_{xx}$ exactly, we may show that pointwise in x and s we have the bound

$$\int_{\mathbb{R}}\left|\mathcal{K}(x,x',s)\right|dx' \leqslant \frac{\left|\overline{W}_{xx}\right|(1+x^2)}{(1-\dot\tau)x^2}\int_0^{|x|}\frac{(x')^2}{1+(x')^2}dx' \leqslant \frac{3(1+2\varepsilon^{1/4})x^2}{1+8x^2}\,. \tag{4.60}$$

In view of (4.59), (4.60), and the bound $3(1+2\varepsilon^{1/4}) \leqslant 9/2 \cdot 3/4$, which holds since ε is sufficiently small, the kernel $\mathcal{K}$ obeys the assumption (A.6) of Lemma A.2.

Next, we estimate the forcing term in (4.53) for $|x| \geqslant \ell$ in order to identify the constant $\mathcal{F}_0$ from Lemma A.2. Indeed, using the explicit properties of $\overline{W}$, the first line on the right side of (4.53) is bounded from above by

$$\begin{aligned}
&\left\|\frac{1+x^2}{x^2}\partial_x F_W\right\|_{L^\infty(|x|\geqslant\ell)} + \left(\left\|\frac{g_W}{x}\right\|_{L^\infty(|x|\geqslant\ell)} + 2\,|\dot\tau|\left\|\frac{\overline{W}}{x}\right\|_{L^\infty(|x|\geqslant\ell)}\right)\left\|\frac{(1+x^2)\overline{W}_{xx}}{x}\right\|_{L^\infty(|x|\geqslant\ell)}\\
&\quad + 2\left(|\dot\tau|\left\|\overline{W}_x\right\|_{L^\infty} + e^{\frac{s}{2}}\left\|Z_x\right\|_{L^\infty}\right)\left\|\frac{(1+x^2)\overline{W}_x}{x^2}\right\|_{|x|\geqslant\ell}\\
&\lesssim \ell^{-2}\left\|\partial_x F_W\right\|_{L^\infty(|x|\geqslant\ell)} + \left\|\frac{g_W}{x}\right\|_{L^\infty(|x|\geqslant\ell)} + |\dot\tau| + \ell^{-2}(|\dot\tau| + e^{\frac{s}{2}}\left\|Z_x\right\|_{L^\infty})\\
&\lesssim \ell^{-2}M^2e^{-s} + \ell^{-1}Me^{-\delta s} + \varepsilon^{\frac14} + \ell^{-2}(\varepsilon^{\frac14} + Me^{-\delta s})\\
&\lesssim \ell^{-2}M\varepsilon^{\delta}
\end{aligned}$$

where we have employed (4.8), (4.26b) (4.34), (4.58), and assumed ε to be sufficiently small, dependent on M. Therefore, taking ε smaller if need be, the estimate on the force required by (A.5) in Lemma A.2 holds, with $\mathcal{F}_0 = \varepsilon^{\frac{\delta}{2}}$.

Lastly, we verify the bounds (A.7). We already know that for $|x| \leqslant \ell$, and for $s \geqslant -\log\varepsilon$, we have the inequality $|V(x,s)| \leqslant 1/40$. Moreover, in view of the assumption (4.3), at the initial time $s = -\log\varepsilon$ we have that $x\varepsilon^{\frac32} = \theta$ and thus

$$|V(x,-\log\varepsilon)| = \frac{1+x^2}{x^2}\left|W_x(x,-\log\varepsilon) - \overline{W}_x(x)\right| = \frac{\varepsilon^3+\theta^2}{\theta^2}\left|\varepsilon(\partial_\theta w_0)(\theta) - \overline{W}_x\left(\frac{\theta}{\varepsilon^{\frac32}}\right)\right| \leqslant \frac{1}{40}\,.$$

Thus, (A.7) holds with $m = 1/20$.

In order to apply Lemma A.2 we finally need to verify the condition (A.8). In view of our determined values for $\lambda_D, \mathcal{F}_0$ and m, we have

$$m\lambda_D = \frac{1}{20}\frac{9\ell^2}{2(1+8\ell^2)} \geqslant 4\varepsilon^{\frac{\delta}{2}} = 4\mathcal{F}_0$$

once ε is chosen to be sufficiently small, in terms of $\ell \leqslant 1$ (and thus of M). Also, note that by Remark 4.3 we have that W_x is compactly supported, while from (4.16b) we have that $\overline{W}_x$ decays as $|x| \to \infty$. Therefore, we have $|V(x,\cdot)| \to 0$ as $|x| \to \infty$. We may thus apply Lemma A.2 and conclude from (A.9) that

$$\|V(\cdot,s)\|_{L^\infty(\mathbb{R})} \leqslant \frac{3}{80}$$

which proves the bootstrap assumption (4.29).

The second derivative. We note that from (4.28), the constraint $W_{xx}(0,s) = 0$ in (4.18), and the bound (4.50), we obtain that

$$\begin{aligned}
|W_{xx}(x,s)| &\leqslant |x|\,W_{xxx}(0,s) + \frac{x^2}{2}\left\|\partial_x^4 W\right\|_{L^\infty}\\
&\leqslant (6+\varepsilon^{\frac{\delta}{2}})\,|x| + \frac{M}{2}x^2 \leqslant \frac{7\,|x|}{(1+x^2)^{\frac12}}\,, \qquad \text{for all} \qquad |x| \leqslant \frac{1}{M}\,,
\end{aligned} \tag{4.61}$$

and all $s \geqslant -\log\varepsilon$. Here we have assumed that M is sufficiently large. This shows that (4.30) automatically holds for $|x| \leqslant 1/M$, with an even better constant.

Next, we observe that (4.2) implies $\left\|\partial_x^2 W(\cdot,-\log\varepsilon)\right\|_{L^\infty} \leqslant 1$ and $\left\|\partial_x^4 W(\cdot,-\log\varepsilon)\right\| \leqslant 1$. Using (4.1), and a Taylor expansion, together with the uniform bound (4.2), we conclude that

$$\|W_{xx}(x,-\log\varepsilon)\| \leqslant \min\left\{6\,|x| + \frac{x^2}{2}, 1\right\} \leqslant \frac{7\,|x|}{(1+x^2)^{1/2}} \tag{4.62}$$

for all $x \in \mathbb{R}$.

Similarly to the above subsection, in order to prove (4.30) for $|x|$ large, we introduce a new variable which is a weighted version of W_{xx}; we define

$$\widetilde{V}(x,s) = \frac{(1+x^2)^{\frac{1}{2}} W_{xx}(x,s)}{x}\,. \tag{4.63}$$

From (4.20b), we see that $\widetilde{V}(x,s)$ is a solution of

$$\begin{aligned}
&\partial_s \widetilde{V} + \left(\frac{5}{2} + \frac{3W_x}{1-\dot{\tau}} + \frac{2(1-\alpha)e^{\frac{s}{2}} Z_x}{(1+\alpha)(1-\dot{\tau})} + \frac{1}{x(1+x^2)}\left(g_W + \frac{3x}{2} + \frac{W}{1-\dot{\tau}}\right) - \frac{(3+2\alpha)e^{-s}A}{(1+\alpha)(1-\dot{\tau})}\right)\widetilde{V} \\
&\qquad + \left(g_W + \frac{3x}{2} + \frac{W}{1-\dot{\tau}}\right)\partial_x \widetilde{V} \\
&= -\frac{e^{-\frac{s}{2}}(1+x^2)^{\frac{1}{2}}}{x(1+\alpha)(1-\dot{\tau})}\left((1-2\alpha)(AZ)_{xx} - (3+2\alpha)\left(A_{xx}(e^{-\frac{s}{2}}W + \kappa) + 2e^{-\frac{s}{2}}A_x W_x\right)\right) \\
&\qquad - \frac{(1-\alpha)e^{\frac{s}{2}}(1+x^2)^{\frac{1}{2}} Z_{xx} W_x}{x(1+\alpha)(1-\dot{\tau})}\,.
\end{aligned} \tag{4.64}$$

Here we have used that $\partial_x^2 F_W$ contains a term with a factor of W_{xx}; the corresponding weighted term has been grouped with the other damping terms on the left of (4.64). The idea is simple: the damping term in (4.64) is larger than the forcing term, for all $|x| \geqslant 1/M$, once ε is chosen sufficiently small.

In order to make this precise, we first estimate the damping term from below. The main observation is that for the exact self-similar profile $\overline{W}$, we have

$$\frac{5}{2} + 3\overline{W}_x + \frac{1}{x(1+x^2)}\left(\frac{3x}{2} + \overline{W}\right) \geqslant \frac{x^2}{1+x^2} \tag{4.65}$$

for all $x \in \mathbb{R}$. This bound is similar to (4.54), and it holds because we know $\overline{W}$ precisely. Using the estimates (4.8), (4.26b), (4.32), (4.34), (4.58) and (4.65), we thus may bound from below

$$\begin{aligned}
&\frac{5}{2} + \frac{3W_x}{1-\dot{\tau}} + \frac{2(1-\alpha)e^{\frac{s}{2}} Z_x}{(1+\alpha)(1-\dot{\tau})} + \frac{1}{1+x^2}\left(\frac{g_W}{x} + \frac{3}{2} + \frac{W}{x(1-\dot{\tau})}\right) - \frac{(3+2\alpha)e^{-s}A}{(1+\alpha)(1-\dot{\tau})} \\
&\geqslant \frac{x^2}{1+x^2} - (3+6\varepsilon^{\frac{1}{4}})\left|\widetilde{W}_x\right| - CMe^{-\delta s} - \frac{1}{1+x^2}\left(C\ell^{-1}Me^{-\delta s} + \left|\frac{\widetilde{W}}{x}\right|\right) - CMe^{-s}
\end{aligned} \tag{4.66}$$

where $C > 0$ only depends on α. Using (4.29) and the fundamental theorem of calculus, we have

$$\begin{aligned}
(3+6\varepsilon^{\frac{1}{4}})\left|\widetilde{W}_x\right| + \frac{1}{1+x^2}\left|\frac{\widetilde{W}}{x}\right| &\leqslant \frac{3x^2}{20(1+x^2)} + \varepsilon^{\frac{1}{4}} + \frac{1}{|x|\,(1+x^2)}\left|\int_0^x \frac{y^2}{20(1+y^2)}dy\right| \\
&\leqslant \frac{x^2}{5(1+x^2)} + \varepsilon^{\frac{1}{4}}
\end{aligned}$$

where we used that $\left|\frac{1+x^2}{x^3}\int_0^x \frac{y^2 dy}{1+y^2}\right| \leqslant 1$ for all $x \in \mathbb{R}$. Taking ε sufficiently small, depending on M, α, δ, we may thus bound the right hand side of (4.66), and thus the total damping terms on the left side of (4.64), from below by

$$\geqslant \frac{4x^2}{5(1+x^2)} - \varepsilon^{\frac{\delta}{2}} \geqslant \frac{1}{2M^2} \qquad \text{for all} \qquad |x| \geqslant \frac{1}{M}\,, \tag{4.67}$$

upon taking ε to be small enough in terms of δ and M.

Similarly, for $|x| \geqslant 1/M$ the forcing term on the right hand side of (4.64) may be bounded by

$$\begin{aligned}
&\lesssim\; e^{-\frac{s}{2}}\frac{(1+x^2)^{\frac{1}{2}}}{|x|}\Big(|(AZ)_{xx}| + \Big(|A_{xx}|\,(e^{-\frac{s}{2}}\,|W| + \kappa) + e^{-\frac{s}{2}}\,|A_x W_x|\Big)\Big) + \frac{e^{\frac{s}{2}}(1+x^2)^{\frac{1}{2}}\,|Z_{xx}W_x|}{|x|}\\
&\lesssim (M^2 e^{-s} + M e^{-\delta s})\frac{(1+x^2)^{\frac{1}{2}}}{|x|} \lesssim M^2 e^{-\delta s}
\end{aligned} \tag{4.68}$$

where we assumed ε to be sufficiently small dependent on M.

To close the bootstrap, we wish to apply Lemma A.2 (with $\mathcal{K} \equiv 0$) to the evolution equation (4.64). Using (4.61) and (4.62), the condition (A.7) is satisfied with $m = 14$ and $\Omega = \{x \,:\, |x| \leqslant 1/M\}$. From (4.68) we verify that (A.5) holds with $\mathcal{F}_0 = \varepsilon^{\frac{\delta}{2}}$, after talking ε to be small enough to absorb the implicit constant and the M^2 factor. Owing to (4.67), the condition (A.8) then amounts to checking

$$14\frac{1}{2M^2} \geqslant e^{-\frac{\delta}{2}}$$

which is easily seen to be satisfied by taking ε to be sufficiently small, dependent on M. Applying Lemma A.2 we obtain

$$\left\|\widetilde{V}\right\|_{L^\infty} \leqslant \frac{21}{2} < 12$$

which closes the bootstrap (4.30) upon recalling the definition of $\widetilde{V}$ in (4.63).

The fourth derivative. The evolution of the fourth derivative of W is governed by (4.17a) with $n = 4$. The damping term in this equation may be bounded from below as

$$\begin{aligned}
\frac{11}{2} + \frac{5}{1-\dot{\tau}}\partial_x W + 4\partial_x g_W &\geqslant \frac{11}{2} - 5(1+2\varepsilon^{\frac{1}{4}})\left(1 + 4e^{\frac{s}{2}}\left\|\partial_x^4 Z\right\|_{L^\infty}\right)\\
&\geqslant \frac{11}{2} - 5(1+2\varepsilon^{\frac{1}{4}})\left(1 + 4M\varepsilon^{\delta s}\right) \geqslant \frac{1}{4}
\end{aligned} \tag{4.69}$$

where we have used that $|W_x| \leqslant 1$, ε is sufficiently small, and (4.34) holds. On the other hand, the forcing term $F_W^{(4)}$ may be estimated using (4.26b), (4.27), (4.28), and (4.32)–(4.34) as

$$\begin{aligned}
\left\|F_W^{(4)}\right\|_{L^\infty} &\lesssim e^{-\frac{s}{2}}\left\|AZ\right\|_{\dot{C}^4} + e^{-\frac{s}{2}}\left\|A(e^{-\frac{s}{2}}W + \kappa)\right\|_{\dot{C}^4} + \|W\|_{\dot{C}^3}\|W\|_{\dot{C}^2} + \sum_{k=1}^{3}\|W\|_{\dot{C}^k}\, e^{\frac{s}{2}}\,\|Z\|_{\dot{C}^{5-k}}\\
&\lesssim M^2 e^{-s} + M^{\frac{3}{4}} + M^{\frac{7}{4}} e^{-\delta s} \lesssim M^{\frac{3}{4}}
\end{aligned} \tag{4.70}$$

assuming ε to be sufficiently small, dependent on M. Appealing to Lemma A.1, estimate (A.1), with $\lambda_F = 0$, $\lambda_D = 1/4$, and $\mathcal{F}_0 = CM^{\frac{3}{4}}$ where C is the (universal) implicit constant in (4.70), we arrive at

$$\left\|\partial_x^4 W(\cdot, s)\right\|_{L^\infty} \leqslant \left\|\partial_x^4 W(\cdot, -\log\varepsilon)\right\|_{L^\infty} e^{-\frac{1}{4}(s+\log\varepsilon)} + 4CM^{\frac{3}{4}} \leqslant 1 + 4CM^{\frac{3}{4}} \leqslant \frac{M}{2} \tag{4.71}$$

for any $s \geqslant -\log\varepsilon$. In the second inequality above, we have used the initial datum assumption (4.2) on the fourth derivative of the initial datum, while in the third inequality we have used that M is sufficiently large, in terms of the universal constant C. This estimate proves the fourth derivative bound in (4.28).

Global bound for the third derivative. Using the mean value theorem and the bound (4.71) we have

$$|W_{xxx}(x,s) - W_{xxx}(0,s)| \leqslant |x|\, M$$

which may be combined with (4.50) to arrive at

$$|W_{xxx}(x,s)| \leqslant 6 + \varepsilon^{\frac{\delta}{2}} + |x|\, M \leqslant \frac{M^{\frac34}}{2} \qquad \text{for} \qquad |x| \leqslant \frac{1}{4M^{\frac14}} \tag{4.72}$$

and all $s \geqslant -\log\varepsilon$, assuming M is sufficiently large. At the initial time, in view of (4.2), the estimate

$$|W_{xxx}(x,-\log\varepsilon)| \leqslant 7 \leqslant \frac{M^{\frac34}}{2} \tag{4.73}$$

holds for all $x \in \mathbb{R}$. We next claim that

$$|W_{xxx}(x,s)| \leqslant \frac{3M^{\frac34}}{4} \tag{4.74}$$

holds for all $s > -\log\varepsilon$ and all $|x| \geqslant 1/(4M^{1/4})$. The estimate (4.74) would then immediately imply the bootstrap assumption for the third derivative in (4.28). The proof of (4.74) is based on Lemma A.2 (with $\mathcal{K} \equiv 0$), and a lower bound on the damping term for the $\partial_x^3 W$ evolution.

We recall from (4.17a) with $n = 3$, and carefully computing the forcing term $F_W^{(3)}$, that

$$\begin{aligned}
&\left(\partial_s + 4\left(1+\partial_x W\right) + \frac{\dot\tau W_x}{1-\dot\tau} + \frac{4(1-\alpha)e^{\frac{s}{2}}Z_x}{(1+\alpha)(1-\dot\tau)} - \frac{(3+2\alpha)e^{-s}A}{(1+\alpha)(1-\dot\tau)}\right)\partial_x^3 W + \left(g_W + \frac{3x}{2} + \frac{W}{1-\dot\tau}\right)\partial_x^4 W \\
&= \frac{e^{-\frac{s}{2}}}{(1+\alpha)(1-\dot\tau)}\left((3+2\alpha)\left(\partial_x^3 A(e^{-\frac{s}{2}}W+\kappa) + 3e^{-\frac{s}{2}}\partial_x^2 A\partial_x W + 3e^{-\frac{s}{2}}\partial_x A\partial_x^2 W\right) - (1-2\alpha)\partial_x^3(AZ)\right) \\
&\quad - \frac{(1-\alpha)e^{\frac{s}{2}}}{(1+\alpha)(1-\dot\tau)}\left(\partial_x^3 Z\partial_x W + 3\partial_x^2 Z\partial_x^2 W\right) - \frac{3}{1-\dot\tau}\left(\partial_x^2 W\right)^2
\end{aligned} \tag{4.75}$$

holds. In order to prove (4.74), we first estimate the right side of (4.75). From (4.8), (4.26b), (4.28), (4.32), and (4.34), we may directly estimate the error term on the right side of (4.75) in absolute value by

$$\lesssim M^2 e^{-\delta s} + 1 \tag{4.76}$$

assuming M is sufficiently large, and ε is sufficiently small, dependent on M and δ. Returning to the damping term in the evolution for $\partial_x^3 W$, for any x and any $s \geqslant -\log\varepsilon$, we have that

$$\begin{aligned}
&4\left(1+\partial_x W\right) + \frac{\dot\tau W_x}{1-\dot\tau} + \frac{4(1-\alpha)e^{\frac{s}{2}}Z_x}{(1+\alpha)(1-\dot\tau)} - \frac{(3+2\alpha)e^{-s}A}{(1+\alpha)(1-\dot\tau)} \\
&\quad\geqslant 4\left(1 + \overline{W}_x - \frac{x^2}{20(1+x^2)}\right) - 2\varepsilon^{\frac14} - 8M\varepsilon^{\delta} - 6M\varepsilon \geqslant \frac{x^2}{1+x^2} - \varepsilon^{\frac{\delta}{2}}\,.
\end{aligned}$$

Above we have appealed to (4.8), (4.26b), (4.29), (4.32), and (4.34), and have taken ε to be sufficiently small, in terms of M and δ. In the second inequality above we have also appealed to the pointwise estimate $1 + \overline{W}_x - \frac{3x^2}{4(1+x^2)} \geqslant 0$ holds for all $x \in \mathbb{R}$. Now, for $|x| > 1/(4M^{\frac14})$ we obtain that

$$4\left(1+\partial_x W\right) + \frac{\dot\tau W_x}{1-\dot\tau} + \frac{4(1-\alpha)e^{\frac{s}{2}}Z_x}{(1+\alpha)(1-\dot\tau)} - \frac{(3+2\alpha)e^{-s}A}{(1+\alpha)(1-\dot\tau)} \geqslant \frac{1}{1+16M^{\frac12}} - \varepsilon^{\frac{\delta}{2}} \geqslant \frac{1}{32M^{\frac12}}\,, \tag{4.77}$$

upon taking ε sufficiently small, solely in terms of M and δ.

We return to (4.75) with the information (4.76) and (4.77) in hand. In view of (4.73), we know that at the initial time and on the compact set $\Omega = \{x \colon |x| \leqslant 1/(4M^{\frac{1}{4}})\}$, the inequality (4.74) holds, with the constant $3/4$ being replaced by the constant $1/2$, i.e. condition (A.7) is satisfied with $m = M^{\frac{3}{4}}$. Moreover, from (4.76) and (4.77), condition (A.8) amounts to checking

$$M^{\frac{3}{4}} \frac{1}{32M^{\frac{1}{2}}} \geqslant 4(CM^2\varepsilon^{\delta} + 1)$$

where C is the implicit constant in (4.76). This condition is true so long as M is sufficiently large and ε is chosen sufficiently small, dependent on M. Hence we may apply Lemma A.2 to deduce that (4.74) holds for all $s \geqslant -\log \varepsilon$.

4.7 Proof of Theorem 4.10

In this section we show that the already established bootstrap bounds (4.26a)–(4.34), together with a number of *a-posteriori estimates* give the proof of Theorem 4.10. First, we note that from (4.12)–(4.13), the definition of T_* in Remark 4.9, and (4.26a)–(4.34), we obtain that the solutions (w, z, a) remain C^4 smooth at all times prior to T_*. Second, we remark that (4.18) implies $\partial_\theta w(\xi(t), t) = e^s W_x(0, s) = -e^s$, while (4.32) yields $\|\partial_\theta w(\cdot, t)\|_{L^\infty} \leqslant e^s$. These bounds prove the claimed blowup behavior of $\partial_\theta w$ as $t \to T_*$, upon recalling that e^s and ${}^1\!/\!(T_* - t)$ only differ by a factor $\leqslant 2$. Third, we notice that the claimed ε dependent bounds on T_* and θ_* were established in Remark 4.9, while Remark 4.2 (see also estimate (4.78) below) give the claimed amplitude bounds for (w, z, a).

It remains for us to prove that $\|\partial_\theta a(\cdot, t)\|_{L^\infty}$, $[w(\cdot, t)]_{C^{1/3}}$, and $\|\partial_\theta z(\cdot, t)\|_{L^\infty}$ remain uniformly bounded on $[-\varepsilon, T_*)$, that the claimed upper and lower bounds for the vorticity hold, and that the lower bound for the density also holds. In Proposition 4.10 below, we prove the desired vorticity, density and $\partial_\theta a$ bounds. The uniform-in-time Hölder $C^{1/3}$ bound is more delicate and it does not directly follow from the proven bootstrap estimates. Rather, to establish this $C^{1/3}$ bound, we use the second estimate on the right side of (4.91) and prove that it can be propagated forward in time, in self-similar variables. This is achieved in Section 4.11. As explained in Remark 4.11 below, these improved bounds on the blowup profile W as $|x| \to \infty$, imply the desired Hölder estimate. Using this information, we prove in Section 4.7.3 that the distance between the Lagrangian flow of the transport velocity in the z equation and $\xi(t)$ remains too large as $t \to T_*$ for a blowup to occur; namely, this distance is $\mathcal{O}(T_* - t)$ instead of $\mathcal{O}((T_* - t)^{3/2})$, which in turn implies that $\partial_\theta z$ remains uniformly bounded all the way up to the blowup time T_*.

Finally, once these a posteriori estimates for (w, z, a) as well as for ω and P are established, the estimates for solutions (u_r, u_θ, ρ) of the Euler equations (2.1) immediately follow from the the definition of the Riemann variables (2.9) together with our homogeneity assumption (2.6) on the solutions. We note that the blowup segment $\Gamma(T_*)$ is the natural extension of the blowup point θ_* in the radial direction.

4.7.1 Density, vorticity, and $\partial_\theta a$ bounds

Proposition 4.10. *Let $\nu_0, \kappa_0, M, \varepsilon$, and T_* be as in the statement of Theorem 4.4, and assume that (w_0, z_0, a_0) satisfy the bounds (4.1)–(4.6). Then, we have that the L^∞ bound (4.8) holds, and additionally that the bounds*

$$\frac{\nu_0}{2} \leqslant P(\theta, t) \leqslant M, \qquad \frac{1}{M^2} \leqslant \omega(\theta, t) \leqslant M^2, \qquad |\partial_\theta a(\theta, t)| \leqslant 3M^2\,,$$

hold for all $\theta \in \mathbb{T}$ and for all $t \in [0, T_*)$.

Proof of Proposition 4.10. From (2.11) we see that any $\varphi \in \{w, z, a\}$ satisfies an equation of the type $\partial_t \varphi + \lambda(w, z)\varphi' = Q(w, z, a)$ where Q is an explicit quadratic polynomial which obeys $|Q(w, z, a)| \leqslant$

$C_\alpha(\max\{|w|, |z|, |a|\})^2$ for some constant C_α that only depends on α, and λ is a speed that is explicitly computable in terms of w, z and α. Recall that our initial datum assumptions imply $\kappa_0/2 \leqslant w_0 \leqslant 3\kappa_0/2$ on $\mathbb{T}$, and that $\|z_0\|_{L^\infty} + \|a_0\|_{L^\infty} \leqslant 1$. From the maximum principle for forced transport equations, upon recalling that $|T_*| \leqslant \varepsilon$, and upon taking ε to be sufficiently small, we deduce that

$$\frac{\kappa_0}{4} \leqslant w(\cdot,t) \leqslant 2\kappa_0 \qquad \|z(\cdot,t)\|_{L^\infty} \leqslant 2 \qquad \|a(\cdot,t)\|_{L^\infty} \leqslant 2 \tag{4.78}$$

for any $t \in [-\varepsilon, T_*)$. The above estimate shows that (4.8) holds as soon as $M \geqslant 4 + 2\kappa_0$, as claimed in Remark 4.2.

Since $P = (\frac{\alpha}{2}(w-z))^{1/\alpha}$, from (4.78) we deduce that

$$\sup_{t\in[-\varepsilon,T_*)} \|P(\cdot,t)\|_{L^\infty(\mathbb{T})} \leqslant (\alpha(\kappa_0+1))^{1/\alpha} \leqslant M\,, \tag{4.79}$$

upon taking M to be sufficiently large (in terms of α and κ_0), and moreover that

$$P(\theta,t) \geqslant \left(\tfrac{\alpha}{2}\left(\tfrac{\kappa_0}{4} - 2\right)\right)^{1/\alpha} \geqslant \tfrac{\nu_0}{2} > 0 \tag{4.80}$$

for all $\theta \in \mathbb{T}$ and $t \in [-\varepsilon, T_*)$, by appealing to the lower bound (4.7) on κ_0. The above two bounds give the desired density estimates.

Next, we consider estimates related to the vorticity. Since $\omega_0 = 2b_0 - \partial_\theta a_0 = w_0 + z_0 - \partial_\theta a_0$, from (4.4), (4.7), and (4.7) we deduce that

$$\tfrac{\kappa_0}{4} \leqslant \tfrac{\kappa_0}{2} - 2 \leqslant \omega_0 \leqslant \tfrac{3\kappa_0}{2} + 2 \leqslant 2\kappa_0\,,$$

and since $\varpi_0 = \frac{\omega_0}{P_0}$, from (4.79)–(4.80) we obtain

$$\tfrac{\kappa_0}{4M} \leqslant \varpi_0 \leqslant \tfrac{4\kappa_0}{\nu_0}\,.$$

Furthermore, from equation (2.8), we have that ϖ obeys a forced transport equation, and upon composing this equation with the flow of b, and exponentiating, the standard Grönwall inequality and the previously established bound (4.78) imply that

$$\frac{\kappa_0}{8M} \leqslant \frac{\kappa_0}{4M}e^{-\frac{2t}{\alpha}} \leqslant \varpi(\cdot,t) \leqslant \frac{4\kappa_0}{\nu_0}e^{\frac{2t}{\alpha}} \leqslant \frac{8\kappa_0}{\nu_0}\,, \qquad \text{for all} \qquad t \in [-\varepsilon, T_*)\,.$$

Here we have used that ε is taken sufficiently small in terms of α, κ_0, M and ν_0. Combining the above bound with (4.79)–(4.80) and the identity $\omega = \varpi P$, we deduce that

$$\frac{1}{M^2} \leqslant \frac{\kappa_0\nu_0}{16M} \leqslant \omega(\cdot,t) \leqslant \frac{8\kappa_0 M}{\nu_0} \leqslant M^2\,, \qquad \text{for all} \qquad t \in [-\varepsilon, T_*)\,,$$

which is the desired vorticity upper and lower bound. Here we have assumed that M may be taken to be sufficiently large, in terms of κ_0 and ν_0. Finally, since $\partial_\theta a = w + z - \omega$ we deduce from the above bound and (4.78) that

$$\|\partial_\theta a(\cdot,t)\|_{L^\infty} \leqslant 2\kappa_0 + 2\frac{8\kappa_0 M}{\nu_0} \leqslant 3M^2\,, \qquad \text{for all} \qquad t \in [-\varepsilon, T_*)\,, \tag{4.81}$$

upon taking M sufficiently large, in terms of κ_0 and ν_0. $\square$

4.7.2 Sharp bounds for W and W_x as $|x| \to \infty$ and Hölder $1/3$ estimates

From the bootstrap assumption (4.29) we know that as $|x| \to \infty$ we have $\left|\widetilde{W}_x\right| = \left|W_x - \overline{W}_x\right| \leqslant {}^1\!/_{20}$. Note, however, that (4.16b) implies the asymptotic behavior $\left|x^{2/3}\overline{W}_x\right| \to {}^1\!/_3$ as $|x| \to \infty$. Our goal is to show that in fact W_x itself also has a $|x|^{-2/3}$ decay rate as $|x| \to \infty$, uniformly in s. To prove this, we show that this asymptotic behavior is valid for $\widetilde{W}_x$, which we recall satisfies the evolution equation (4.52). In order to normalize the behavior at infinity, we consider the function $\mathcal{V}$ defined as

$$\mathcal{V}(x,s) = (x^{2/3} + 8)\widetilde{W}_x(x,s)\,, \tag{4.82}$$

where the translation of $x^{2/3}$ by 8 will be explained below in the course of the argument. Our objective is to show that

$$\|\mathcal{V}(\cdot,s)\|_{L^\infty} \leqslant 1 \tag{4.83}$$

for all $s \geqslant -\log\varepsilon$. We remark that at the initial time $s = -\log\varepsilon$, we have

$$|\mathcal{V}(x,-\log\varepsilon)| = \left(\frac{\theta^{2/3}}{\varepsilon} + 8\right)\left|\varepsilon(\partial_\theta w_0)(\theta) - (\overline{W}_x)\left(\frac{\theta}{\varepsilon^{3/2}}\right)\right| \leqslant \frac{1}{2}\,, \qquad \text{for all} \qquad x \in \mathbb{R}\,,$$

in view of assumption (4.3) on the initial datum. Additionally, note that by (4.29), we have

$$|\mathcal{V}(x,s)| \leqslant \frac{x^2(x^{2/3}+8)}{20(1+x^2)} \leqslant \frac{1}{2}\,, \qquad \text{for all} \qquad |x| \leqslant 2\,, \tag{4.84}$$

and thus (4.83) is automatically satisfied with a better constant (${}^1\!/_2$ instead of 1) for $|x| \leqslant 2$.

Similarly to (4.53), a simple computation shows that $\mathcal{V}$ satisfies

$$\begin{aligned}
&\partial_s\mathcal{V} + \left(1 + \widetilde{W}_x + 2\overline{W}_x - \frac{2x^{2/3}}{3(x^{2/3}+8)}\left(\frac{3}{2} + \frac{\overline{W}+\widetilde{W}}{x}\right)\right)\mathcal{V} + \left(g_W + \frac{3x}{2} + \frac{W}{1-\dot\tau}\right)\mathcal{V}_x \\
&= (x^{2/3}+8)\partial_x F_W - \left(g_W + \frac{\dot\tau\overline{W}}{1-\dot\tau}\right)(x^{2/3}+8)\overline{W}_{xx} - \left(\frac{\dot\tau\overline{W}_x}{1-\dot\tau} + \frac{(1-\alpha)e^{\frac{s}{2}}Z_x}{(1+\alpha)(1-\dot\tau)}\right)(x^{2/3}+8)\overline{W}_x \\
&\quad - \left(\frac{\dot\tau(\widetilde{W}_x + 2\overline{W}_x) + \frac{1-\alpha}{1+\alpha}e^{\frac{s}{2}}Z_x}{1-\dot\tau} - \frac{2x^{2/3}}{3(x^{2/3}+8)}\left(\frac{\dot\tau W}{(1-\dot\tau)x} + \frac{g_W}{x}\right)\right)\mathcal{V} \\
&\quad - \frac{1}{1-\dot\tau}(x^{2/3}+8)\overline{W}_{xx}\int_0^x \mathcal{V}(x')\frac{1}{(x')^{2/3}+8}dx'\,.
\end{aligned} \tag{4.85}$$

It is convenient to rewrite (4.85) schematically as

$$\partial_s\mathcal{V} + \mathcal{D}(x,s)\mathcal{V} + \mathcal{U}(x,s)\mathcal{V}_x = \mathcal{F}_1(x,s) + \mathcal{F}_2(x,s) + \int_0^\infty \mathcal{V}(x',s)\mathcal{K}(x,x',s)dx' \tag{4.86}$$

where $\mathcal{D}$ and $\mathcal{U}$ are determined by the first line on the left side of (4.85), the forcing term $\mathcal{F}_1$ is given by the first line on the right side of (4.85), the forcing term $\mathcal{F}_2$ is given by the second line on the right side of (4.85), and $\mathcal{K}$ is defined by the last line of the $\mathcal{V}$ evolution as $\mathcal{K}(x,x',s) = -\frac{1}{1-\dot\tau}(x^{2/3}+8)\overline{W}_{xx}(x)\frac{\mathbf{1}_{[0,x]}(x')}{(x')^{2/3}+8}$. The argument fundamentally consists of a comparison between the damping term $\mathcal{D}$ with the $L^1_{x'}$-norm of the kernel $\mathcal{K}$, similar in spirit to the one used to prove Lemma A.2.

Using the fundamental theorem of calculus, the fact that $\widetilde{W}(0,s)=0$, and the bootstrap assumption (4.83), we obtain the following lower bound on the damping term:

$$\mathcal{D}(x,s) \geqslant 1-\frac{1}{x^{2/3}+8}+2\overline{W}_x-\frac{2x^{2/3}}{3(x^{2/3}+8)}\left(\frac{3}{2}+\frac{\overline{W}}{x}+\frac{1}{x}\int_0^x\frac{dx'}{(x')^{2/3}+8}\right)=:\mathcal{D}_{\text{upper}}(x)$$

On the other hand, using our bound for $\dot{\tau}$ (4.48), we have that

$$\int_{\mathbb{R}}\left|\mathcal{K}(x,x',s)\right|dx' \leqslant (1+2\varepsilon^{1/4})(x^{2/3}+8)\left|\overline{W}_{xx}(x)\right|\int_0^{|x|}\frac{dx'}{(x')^{2/3}+8}=:\mathcal{D}_{\text{lower}}(x)\,.$$

The choice of the translation constant 8 in the weight appearing in (4.82) was chosen so that by letting ε be sufficiently small, we ensure that

$$0<\mathcal{D}_{\text{lower}}(x)\leqslant\mathcal{D}_{\text{upper}}(x), \qquad \text{for all} \qquad |x|\geqslant 2. \tag{4.87}$$

While, in fact, $\mathcal{D}_{\text{lower}}(x)\leqslant\frac{3}{4}\mathcal{D}_{\text{upper}}(x)$ for $|x|\geqslant 2$ as required by (A.6), the reason we cannot apply Lemma A.2 is that for $|x|\gg 1$ we have $\mathcal{D}_{\text{upper}}(x)=5x^{-2/3}+\mathcal{O}(|x|^{-1})$, and so we cannot obtain a uniform in x lower bound on the damping, as required by (A.4). Nonetheless, we will still apply an argument similar to the one used to prove Lemma A.2.

Next, we estimate the forcing term $\mathcal{F}_1$. The most delicate term is the one due to $\partial_x F_W$, which is bounded using (4.83) and the support property discussed in Remark 4.3, as

$$\begin{aligned}\left\|(x^{2/3}+8)\partial_x F_W\right\|_{L^\infty} &\lesssim e^{-s/2}\left\|(x^{2/3}+8)\partial_x(AZ)\right\|_{L^\infty}+e^{-s/2}\left\|(x^{2/3}+8)\partial_x A\right\|_{L^\infty}\left\|e^{-s/2}W+\kappa\right\|_{L^\infty}\\ &\quad+e^{-s}\left\|A\right\|_{L^\infty}\left(1+\left\|(x^{2/3}+8)\overline{W}_x\right\|_{L^\infty}\right)\\ &\lesssim M^2e^{-\delta s}+Me^{-s}\end{aligned}$$

where the implicit constant depends only on α. The remaining forcing terms are easier to estimate since we already know the decay rates $\overline{W}_x=\mathcal{O}(|x|^{-2/3})$ and $\overline{W}_{xx}=\mathcal{O}(|x|^{-5/3})$ as $|x|\to\infty$. Using the available estimate (4.48) for $\dot{\tau}$, the bound (4.38) for $\partial_x Z$, and the third line of (4.58) to bound g_W, after a computation we deduce that the total forcing term may be estimated as

$$\|\mathcal{F}_1(\cdot,s)\|_{L^\infty}\leqslant CM^2e^{-\delta s}+CMe^{-s}\leqslant e^{-\delta s/2} \tag{4.88}$$

by choosing ε to be sufficiently small in terms of M and the constant C which only depends on α. Similarly, we have that

$$\|\mathcal{F}_2(\cdot,s)\|_{L^\infty(|x|\geqslant 2)}\leqslant e^{-\delta s/2}\,, \tag{4.89}$$

which follows from the previously established properties of $\dot{\tau}$, W_x, $\overline{W}_x$, Z_x, and g_W, after choosing ε to be sufficiently small in terms of α,δ,M.

In order to conclude the proof of (4.83), we claim that

$$\|\mathcal{V}(\cdot,s)\|_{L^\infty}\leqslant\frac{3}{4} \tag{4.90}$$

which would show that the bootstrap assumption (4.83) holds with an even better constant ($3/4$ instead of 1), thereby closing it. If (4.90) were to fail at some time $s_1>-\log\varepsilon$, by continuity in time there exists a time $s_0\in(-\log\varepsilon,s_1)$ such that $\|\mathcal{V}(\cdot,s)\|_{L^\infty}\geqslant\|\mathcal{V}(\cdot,s_0)\|_{L^\infty}=5/8$ for all $s\in[s_0,s_1]$. Then, for $s\in[s_0,s_1)$ we may evaluate (4.86) at the global maximum of $|\mathcal{V}|$, which is ensured to be attained at a point $x_*=x_*(s)$

with $|x_*| \geqslant 2$, since W_x is compactly supported, $(x^{2/3}+8)\left|\overline{W}_x\right| \to 1/3 < 5/8$ as $|x| \to \infty$, and (4.84) holds. Without loss of generality, let us consider the case when $\mathcal{V}(x_*(s), s)$ is the global maximum for $\mathcal{V}$ (the case of a global minimum is treated similarly). At this maximum point $\mathcal{V}_x$ vanishes, and using (4.87) we obtain

$$\mathcal{D}(x_*(s), s)\mathcal{V}(x_*(s), s) \geqslant \mathcal{D}_{\text{upper}}(x_*(s)) \left\|\mathcal{V}(\cdot, s)\right\|_{L^\infty} \\ \geqslant \mathcal{D}_{\text{lower}}(x_*(s)) \left\|\mathcal{V}(\cdot, s)\right\|_{L^\infty} \geqslant \left|\int_{\mathbb{R}} \mathcal{K}(x_*(s), x', s)\mathcal{V}(x', s)dx'\right| .$$

Therefore, at $x_*(s)$ the second term on the left side of (4.86) dominates the third term on the right side of (4.86). Next, via a standard Rademacher argument (applicable since $\mathcal{V}$ is smooth), and using the bounds (4.88)–(4.89) we obtain that a.e. in s

$$\frac{d}{ds} \left\|\mathcal{V}(\cdot, s)\right\|_{L^\infty} \leqslant 2e^{-\delta s/2} .$$

Using that by assumption $\left\|\mathcal{V}(\cdot, s_0)\right\|_{L^\infty} = 5/8$, we integrate the above inequality for $s \geqslant s_0$ and deduce that

$$\left\|\mathcal{V}(\cdot, s)\right\|_{L^\infty} \leqslant (5/8 + 1)e^{\frac{4}{\delta}\varepsilon^{\delta/2}} - 1 < 3/4$$

for all $s > s_0 > -\log\varepsilon$, upon taking ε to be sufficiently small. This provides the desired contradiction and thus (4.90) holds, concluding the proof.

Remark 4.11 (Uniform Hölder bounds). Estimate (4.83) and properties of the function $\overline{W}_x$ imply that

$$|W_x(x, s)| \leqslant \frac{1}{x^{2/3} + 8} + \left|\overline{W}_x(x)\right| \leqslant \frac{2}{x^{2/3}} \tag{4.91}$$

for all $x \in \mathbb{R}$ and $s \geqslant -\log\varepsilon$. Since $W(0, s) = 0$ for all s, integrating the above estimate in x we arrive at

$$|W(x, s)| \leqslant 6\,|x|^{1/3} \tag{4.92}$$

for all $x \in \mathbb{R}$ and $s \geqslant -\log\varepsilon$. The bounds (4.83)–(4.92) imply that $w \in L^\infty([-\varepsilon, T_*); C^{1/3}(\mathbb{T}))$. To see this, consider any two points $\theta \neq \theta' \in \mathbb{T}$. Accordingly, define the points $x = \frac{\theta - \xi(t)}{(\tau - t)^{3/2}} \neq x' = \frac{\theta' - \xi(t)}{(\tau - t)^{3/2}}$ by the scaling (4.12). Due to the description (4.13) of w we have that

$$\frac{|w(\theta, t) - w(\theta', t)|}{|\theta - \theta'|^{1/3}} = \frac{|W(x, s) - W(x', s)|}{|x - x'|^{1/3}} . \tag{4.93}$$

At this stage we remark that when $x' = 0$, and x is taken to be arbitrary, the bound (4.92) implies that the right side of (4.93) is bounded by 6 uniformly in s. To consider the general case of $x \neq x'$, we combine (4.91) with (4.32) to deduce that $|W_x(x, s)| \lesssim (1 + x^2)^{-1/3}$ where the implicit constant is universal. Then, using the fundamental theorem of calculus we estimate

$$\sup_{x > x'} \frac{|W(x, s) - W(x', s)|}{|x - x'|^{1/3}} \lesssim \sup_{x > x'} \frac{\int_{x'}^{x} (1 + y^2)^{-1/3} dy}{(x - x')^{1/3}} \lesssim 1$$

where the implicit constant is universal, and is in particular independent of s. This concludes the proof of the uniformy in time Hölder $1/3$ estimate for w. It is not hard to see that C^α Hölder norms of w, with $\alpha > 1/3$ blow up as $t \to T_*$ with a rate proportional to $(T_* - t)^{(1-3\alpha)/2}$.

4.7.3 Bounds for $\partial_\theta z$ as $t \to T_*$

In view of the relation $\partial_\theta z = e^{3s/2}\partial_x Z$, and the already established bound (4.34), we have that $\|\partial_\theta z(\cdot,t)\|_{L^\infty} \leqslant 2M(T_*-t)^{-1+\delta}$, for $t \in [-\varepsilon, T_*)$. Here we have used that

$$(1-\varepsilon^{1/4})(T_*-t) \leqslant \tau(t) - t \leqslant (1+\varepsilon^{1/4})(T_*-t)\,, \tag{4.94}$$

which is a consequence of Remark 4.9 and the identity $\tau(t) - t = \varepsilon - \int_{-\varepsilon}^{t}(1-\dot\tau) = \int_t^{T_*}(1-\dot\tau)$, and the fact that $\tau(t) - t = e^{-s}$. We may, however, show that $\partial_\theta z$ remains in fact bounded as $t \to T_*$.

Upon differentiating (2.11b) with respect to θ, we obtain

$$\begin{aligned}\left(\partial_t + \left(z + \tfrac{1-\alpha}{1+\alpha}w\right)\partial_\theta\right)(\partial_\theta z) = -\left(\partial_\theta z + \tfrac{1-\alpha}{1+\alpha}\partial_\theta w\right)(\partial_\theta z) - \tfrac{1-2\alpha}{1+\alpha}a(\partial_\theta w) - \tfrac{3+2\alpha}{1+\alpha}a(\partial_\theta z)\\ - \tfrac{1}{1+\alpha}\partial_\theta a\big((1-2\alpha)w + (3+2\alpha)z\big)\,.\end{aligned} \tag{4.95}$$

Note that by (4.78) and (4.81), we know that a, z, w, and $\partial_\theta a$ remain uniformly bounded in $L^\infty(\mathbb{T})$ over $[-\varepsilon, T_*)$, and so we may think of these terms as constants in (4.95). Moreover, since $\|\partial_\theta z_0\|_{L^\infty} \leqslant 1$, the term $-(\partial_\theta z)^2$ on the right side of (4.95) cannot by itself cause a finite time singularity in time $\mathcal{O}(\varepsilon)$. The blowup of $\partial_\theta z$ could only be caused by the terms involving $\partial_\theta w$ on the right side of (4.95); specifically the $-\frac{1-\alpha}{1+\alpha}(\partial_\theta z)(\partial_\theta w)$ term is dominant near a putative singularity of $\partial_\theta z$. Indeed, $\|\partial_\theta w\|_{L^\infty} = e^s\|W_x\|_{L^\infty} = e^s \geqslant (1/2)(T_*-t)^{-1}$, and so $\int_{-\varepsilon}^{T_*}\|\partial_\theta w(\cdot,t)\|_{L^\infty} = +\infty$, which could be sufficient to cause a singularity.

Our main observation is that if we compose (4.95) with its natural Lagrangian flow $\zeta_{x_0}(t)$, defined as

$$\tfrac{d}{dt}\zeta_{\theta_0}(t) = z(\zeta_{\theta_0}(t),t) + \tfrac{1-\alpha}{1+\alpha}w(\zeta_{\theta_0}(t),t)\,, \qquad \zeta_{\theta_0}(-\varepsilon) = \theta_0\,, \tag{4.96}$$

then the quantity $\int_{-\varepsilon}^{T_*}\partial_\theta w(\zeta_{\theta_0}(t),t)dt$ is the relevant one to study for bounding $\|\partial_\theta z\|_{L^\infty}$. We claim that as $t \to T_*$ the quantity $|\partial_\theta w(\zeta_{\theta_0}(t),t)|$ does not blow up at a non-integrable rate. Once the claim is proven, standard ODE arguments imply that the solution $\partial_\theta z$ of (4.95) remains bounded in L^∞ as $t \to T_*$.

For the remainder of this proof we drop the subindex θ_0 of ζ_{θ_0} (it is frozen) and we use e^{-s} and $T_* - t$ interchangeably as they are comparable up to a factor of $1 \pm \varepsilon^{1/4}$ by (4.94). By the definition of W in (4.13) and the previously established bound (4.91), we have that

$$|\partial_\theta w(\zeta(t),t)| = e^s\left|W_x\left((\zeta(t)-\xi(t))e^{\frac{3s}{2}}, s\right)\right| \lesssim \frac{1}{T_*-t}\left(1 + \frac{|\zeta(t)-\xi(t)|}{(T_*-t)^{3/2}}\right)^{-2/3}. \tag{4.97}$$

Consider the case that $\zeta(T_*) \neq \xi(T_*)$. Then, by continuity, $|\zeta(t)-\xi(t)| \geqslant c$ for t sufficiently close to T_*. Therefore, from (4.97), $|\partial_\theta w(\zeta(t),t)|$ is bounded. Otherwise, $\zeta(t)-\xi(t) \to 0$ as $t \to T_*$. Our goal is to show that there exists a constant c_* such that for all t sufficiently close to T_* we have $|\zeta(t)-\xi(t)| \geqslant c_*(T_*-t)$. Once this claim is established, it follows from (4.97) that $\int_{-\varepsilon}^{T_*}|\partial_\theta w(\zeta(t),t)|\,dt < \infty$, as desired.

It remains to prove the claimed lower bound for $\zeta - \xi$. Using the definition of $\zeta(t)$ in (4.96) and the definition of $\dot\xi$ in (4.24a), we derive that

$$\begin{aligned}\zeta(t) - \xi(t) &= \int_t^{T_*}\dot\xi(t') - z(\zeta(t'),t') - \tfrac{1-\alpha}{1+\alpha}w(\zeta(t'),t')dt'\\ &= \tfrac{2\alpha}{1+\alpha}\int_t^{T_*}\kappa(t')dt' + \int_t^{T_*}\tfrac{1-\alpha}{1+\alpha}Z^0(s') - Z\left((\zeta(t')-\xi(t'))e^{\frac{3s'}{2}}, s'\right)dt'\\ &\quad - \int_t^{T_*}\tfrac{1-\alpha}{1+\alpha}e^{-\frac{s'}{2}}W\left((\zeta(t')-\xi(t'))e^{\frac{3s'}{2}}, s'\right) + (1-\dot\tau)e^{-\frac{s'}{2}}\frac{F_W^{0,(2)}(s')}{W_{xxx}^0(s')}dt'\\ &= I_1(t) + I_2(t) - I_3(t)\end{aligned} \tag{4.98}$$

where $e^{-s'} = \tau(t') - t'$. From (4.26b) we deduce that $I_1(t) \geqslant \frac{\alpha\kappa_0}{1+\alpha}(T_* - t)$, upon taking ε sufficiently small in terms of M and κ_0. It is essential here that $\alpha > 0$, i.e. $\gamma > 1$. Using (4.78) we immediately obtain that $|I_2(t)| \leqslant \frac{4}{1+\alpha}(T_* - t)$. Lastly, using our bootstrap assumptions and the estimate (4.92), after a tedious computation we deduce that the integrand of I_3 may be bounded in absolute value as $\lesssim e^{-\delta s'} \lesssim (T_* - t')^\delta$, and therefore $|I_3(t)| \lesssim (T_* - t)^{1+\delta} \lesssim \varepsilon^\delta (T_* - t)$. We collect the above estimates and insert them in (4.98), to deduce that $|\zeta(t) - \xi(t)| \geqslant \frac{\alpha\kappa_0}{1+\alpha}(T_* - t) - \frac{5}{1+\alpha}(T_* - t) \geqslant \frac{1}{1+\alpha}(T_* - t)$, by taking κ_0 sufficiently large, in terms of α. As discussed above, this lower bound concludes our proof for the boundedness of $\partial_\theta z$.

5 Concluding remarks

By considering homogeneous solutions to the isentropic 2D compressible Euler equations, and using a transformation to self-similar coordinates with dynamic modulation variables, we have proven that for an open set of smooth initial data with $\mathcal{O}(1)$ amplitude, $\mathcal{O}(1)$ vorticity, and with minimum initial slope $-1/\varepsilon$, there exist smooth solutions of the Euler equations which form an asymptotically self-similar shock within $\mathcal{O}(\varepsilon)$ time. Our method is based on perturbing purely azimuthal waves which inherently possess nontrivial vorticity, and thus, our constructed solutions have $\mathcal{O}(1)$ vorticity at the shock, as well as a lower-bound on the density, so that no vacuum regions can form during the formation of the shock singularity.

A key feature of our method is that the purely azimuthal wave is governed exactly by the Burgers equations (as demonstrated for the special case that $\gamma = 3$), and thus our construction uses precise information on the stable self-similar solution $\overline{W}$ of the Burgers equation. This allows us to provide detailed information about the blowup: by using the ODEs solved by $\tau(t)$ and $\xi(t)$, it is possible to compute the exact blowup time and location for our solutions to the 2D Euler equations. Moreover, we have shown that the blowup profiles have cusp singularities with Hölder $C^{1/3}$ regularity.

We have shown in Remark 3.3 that in the case that $\gamma = 3$, the first singularity can be continued as a discontinuous propagating shock wave for all time.[3] In fact, we believe that the solutions we have constructed have this type of continuation property for general $\gamma > 1$.

Conjecture 5.1. Given that the asymptotically self-similar shock solutions constructed in Theorem 4.4 form a $C^{1/3}$ cusp at the initial blowup time $t = T_*$, these solutions can be continued for short time as propagating piecewise smooth discontinuous (possibly non-unique) shock profiles which solve the Euler equations on either side of the time-dependent *curve of discontinuity*, and the evolution of this shock (or discontinuity) is governed by the Rankine-Hugoniot conditions.

The solution we have constructed consists of a sound wave which steepens and shocks in the azimuthal direction as well as the azimuthal velocity which also steepens and shocks in the azimuthal direction. The radial component of velocity can steepen in the azimuthal direction but does not shock.

Conjecture 5.2. Suppose that (ρ, u_r, u_θ) denotes the solution to the Euler equations given in Theorem 4.4. Then at the first blowup time $t = T_*$, the variable $\partial_\theta u_r$ is Lipschitz and no better. In turn, let $\Omega(t)$ denote the material curve defined in (2.13). Then $\partial\Omega(T_*)$ forms a corner singularity.

A Toolshed

Lemma A.1. *Assume that the function $f = f(x, s)$ obeys the forced and damped transport equation*

$$\partial_s f + \mathcal{D} f + \mathcal{U} \partial_x f = \mathcal{F}$$

[3]Note that even the purely azimuthal shock solution has vorticity, and this is extremely important for the shock continuation problem as initially irrotational flows can generate vorticity after the shock [7].

for $s \in [s_0, \infty)$ and $x \in \mathbb{R}$. Assume that $\mathcal{U}$, $\mathcal{D}$ and $\mathcal{F}$ are smooth, that

$$\inf_{(x,s)\in\mathbb{R}\times[s_0,\infty)} \mathcal{D}(x,s) \geqslant \lambda_D$$

for some $\lambda_D \in \mathbb{R}$, and that

$$\|\mathcal{F}(\cdot,s)\|_{L^\infty(\mathbb{R})} \leqslant \mathcal{F}_0 e^{-s\lambda_F}$$

for all $s \geqslant s_0$, for some $\mathcal{F}_0 \in [0,\infty)$ and $\lambda_F \in \mathbb{R}$. For $\lambda_F < \lambda_D$ the function f obeys the estimate

$$\|f(\cdot,s)\|_{L^\infty} \leqslant \|f(\cdot,s_0)\|_{L^\infty} e^{-\lambda_D(s-s_0)} + \frac{\mathcal{F}_0}{\lambda_D-\lambda_F} e^{-s\lambda_F}\,. \tag{A.1}$$

for all $s \geqslant s_0$. On the other hand, for $\lambda_F > \lambda_D$, we have

$$\|f(\cdot,s)\|_{L^\infty} \leqslant \|f(\cdot,s_0)\|_{L^\infty} e^{-\lambda_D(s-s_0)} + \frac{\mathcal{F}_0 e^{-s_0\lambda_F}}{\lambda_F-\lambda_D} e^{-\lambda_D(s-s_0)}\,. \tag{A.2}$$

for all $s \geqslant s_0$.

Proof of Lemma A.1. Let $\partial_s\psi = \mathcal{U}\circ\psi$ for $s > s_0$ and $\psi(x,s_0) = x$. Then $\frac{d}{ds}\left(e^{\int_{s_0}^s(\mathcal{D}\circ\psi)ds'}(f\circ\psi)\right) = e^{\int_{s_0}^s(\mathcal{D}\circ\psi)ds'}(\mathcal{F}\circ\psi)$, from which it follows by integration that

$$f(x,s) = f(x,s_0)e^{-\int_{s_0}^s(\mathcal{D}\circ\psi)dr} + \int_{s_0}^s e^{-\int_{s'}^s(\mathcal{D}\circ\psi)ds''}(\mathcal{F}\circ\psi)ds'\,.$$

From this identity, the inequalities (A.1) and (A.2) immediately follow. □

The following lemma is a version of the maximum principle which is tailored to the needs of this paper.

Lemma A.2. *Assume that the function f obeys the damped and non-locally forced transport equation*

$$\partial_s f(x,s) + \mathcal{D}(x,s)f(x,s) + \mathcal{U}(x,s)\partial_x f(x,s) = \mathcal{F}(x,s) + \int_{\mathbb{R}} f(x',s)\mathcal{K}(x,x',s)dx' \tag{A.3}$$

for $s \in [s_0,\infty)$ and $x \in \mathbb{R}$. Assume that the drift $\mathcal{D}$, the transport velocity $\mathcal{U}$, the forcing $\mathcal{F}$ and the kernel $\mathcal{K}$ are smooth functions, and assume we are given that the solution f decays at spatial infinity: $\lim_{|x|\to\infty}|f(x,s)| = 0$. Let $\Omega \subset \mathbb{R}$ be a compact set, and assume that on its complement the damping obeys

$$\inf_{(x,s)\in\Omega^c\times[s_0,\infty)} \mathcal{D}(x,s) \geqslant \lambda_D > 0 \tag{A.4}$$

and that the forcing is bounded as

$$\|\mathcal{F}(\cdot,s)\|_{L^\infty(\Omega^c)} \leqslant \mathcal{F}_0 < \infty \tag{A.5}$$

for all $s \geqslant s_0$. For the kernel $\mathcal{K}$ we assume the estimate

$$\int_{\mathbb{R}} \left|\mathcal{K}(x,x',s)\right| dx' \leqslant \tfrac{3}{4}\mathcal{D}(x,s) \qquad \textit{for all} \qquad (x,s) \in \Omega^c\times[s_0,\infty)\,. \tag{A.6}$$

Then, if for some $m > 0$ we have

$$\|f(\cdot,s_0)\|_{L^\infty(\mathbb{R})} \leqslant \tfrac{1}{2}m\,, \qquad \textit{and} \qquad \|f(\cdot,s)\|_{L^\infty(\Omega)} \leqslant \tfrac{1}{2}m\,, \tag{A.7}$$

and the the forcing-to-damping relation

$$m\lambda_D \geqslant 4\mathcal{F}_0 \tag{A.8}$$

holds, then the solution f obeys

$$\|f(\cdot,s)\|_{L^\infty(\mathbb{R})} \leqslant \tfrac{3}{4}m \tag{A.9}$$

for all $s \geqslant s_0$.

Proof of Lemma A.2. Assume for the sake of contradiction that (A.9) fails. Then, by the smoothness of solutions to (A.3) and the assumption that the solution f vanishes as $|x| \to \infty$, there exists a *first time* s_* and a *location* x_* such that $|f(s_*, x_*)| = 3m/4$. In view of (A.7) we must have $x_* \in \Omega^c$. We may first assume that f attains its *global maximum* at this point, i.e. that $f(s_*, x_*) = 3m/4$. By the minimality of s_*, we must have $(\partial_s f)(x_*, s_*) \geqslant 0$. We will prove that the opposite inequality holds, thereby contradicting the existence of the breakthrough point (x_*, s_*). For this purpose, evaluate the forced and damped transport equation at (x_*, s_*), and note that because f attains its global maximum at this point, we have $\partial_x f(x_*, s_*) = 0$. Additionally, from the assumption on the kernel, we have

$$\left|\int_{\mathbb{R}} f(x', s_*)\mathcal{K}(x_*, x', s_*)dx'\right| \leqslant \tfrac{3}{4} \|f(\cdot, s_*)\|_{L^\infty(\mathbb{R})} \mathcal{D}(x_*, s_*) = \tfrac{3}{4} f(x_*, s_*)\mathcal{D}(x_*, s_*)$$

and therefore, using (A.8) we obtain

$$(\partial_s f)(x_*, s_*) \leqslant |\mathcal{F}(x_*, s_*)| - \tfrac{1}{4}\mathcal{D}(x_*, s_*)f(x_*, s_*) \leqslant \mathcal{F}_0 - \tfrac{3}{16} m\lambda_D \leqslant -\tfrac{\mathcal{F}_0}{4} < 0$$

which yields the desired contradiction.

If on the other hand f attains its *global minimum at this point*, i.e. $f(s_*, x_*) = -3m/4$, then by the minimality of s_*, we must have $(\partial_s f)(x_*, s_*) \leqslant 0$. We prove that the opposite inequality holds, yielding the contradiction. For this purpose, evaluate the forced and damped transport equation at (x_*, s_*), and note that because f attains its global minimum at this point, we have $\partial_x f(x_*, s_*) = 0$. Also, we have

$$\left|\int_{\mathbb{R}} f(x', s_*)\mathcal{K}(x_*, x', s_*)dx'\right| \leqslant \tfrac{3}{4} \|f(\cdot, s_*)\|_{L^\infty(\mathbb{R})} \mathcal{D}(x_*, s_*) = -\tfrac{3}{4} f(x_*, s_*)\mathcal{D}(x_*, s_*)$$

so that

$$(\partial_s f)(x_*, s_*) \geqslant \mathcal{F}(x_*, s_*) - \tfrac{1}{4}\mathcal{D}(x_*, s_*)f(x_*, s_*) \geqslant -\mathcal{F}_0 + \tfrac{3}{16} m\lambda_D \geqslant \tfrac{1}{4}\mathcal{F}_0 > 0\,.$$

Therefore, the breakthrough point (x_*, s_*) does not exist, concluding the proof of (A.9). □

Acknowledgments

T.B. was supported by the NSF grant DMS-1600868. S.S. was supported by the Department of Energy Advanced Simulation and Computing (ASC) Program. V.V. was supported by the NSF grant DMS-1652134. The authors are grateful to Tarek Elgindi for numerous conversations on the topic of self-similar blowup in fluids and for stimulating discussions in the early phase of this project. We are also grateful to Jared Speck who explained to us the results of the paper [28], as well as some of the history of the subject.

References

[1] S. Alinhac, *Blowup of small data solutions for a class of quasilinear wave equations in two space dimensions. II*, Acta Math. **182** (1999), no. 1, 1–23. MR1687180

[2] S. Alinhac, *Blowup of small data solutions for a quasilinear wave equation in two space dimensions*, Ann. of Math. (2) **149** (1999), no. 1, 97–127. MR1680539

[3] K. W. Cassel, F. T. Smith, and J. D. A. Walker, *The onset of instability in unsteady boundary-layer separation*, Journal of Fluid Mechanics **315** (1996), 223–256.

[4] G.-Q. Chen and M. Feldman, *Global solutions of shock reflection by large-angle wedges for potential flow*, Ann. of Math. (2) **171** (2010), no. 2, 1067–1182. MR2630061

[5] G.-Q. G. Chen and M. Feldman, *The mathematics of shock reflection-diffraction and von Neumann's conjectures*, Annals of Mathematics Studies, vol. 197, Princeton University Press, Princeton, NJ, 2018. MR3791458

[6] J. Chen, T. Y. Hou, and D. Huang, *On the finite time blowup of the De Gregorio model for the 3d Euler equation*, arXiv:1905.06387 (2019).

[7] D. Christodoulou, *The formation of shocks in 3-dimensional fluids*, EMS Monographs in Mathematics, European Mathematical Society (EMS), Zürich, 2007. MR2284927

[8] D. Christodoulou, *The shock development problem*, EMS Monographs in Mathematics, European Mathematical Society (EMS), Zürich, 2019. MR3890062

[9] D. Christodoulou and S. Klainerman, *The global nonlinear stability of the Minkowski space*, Princeton Mathematical Series, vol. 41, Princeton University Press, Princeton, NJ, 1993. MR1316662

[10] D. Christodoulou and S. Miao, *Compressible flow and Euler's equations*, Surveys of Modern Mathematics, vol. 9, International Press, Somerville, MA; Higher Education Press, Beijing, 2014. MR3288725

[11] C. Collot, T.-E. Ghoul, S. Ibrahim, and N. Masmoudi, *On singularity formation for the two dimensional unsteady Prandtl's system*, arXiv:1808.05967 (2018).

[12] C. Collot, T.-E. Ghoul, and N. Masmoudi, *Singularity formation for Burgers equation with transverse viscosity*, arXiv:1803.07826 (2018).

[13] C. Collot, T.-E. Ghoul, and N. Masmoudi, *Unsteady separation for the inviscid two-dimensional prandtl's system*, arXiv:1903.08244 (2019).

[14] C. M. Dafermos, *Hyperbolic conservation laws in continuum physics*, Third, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 325, Springer-Verlag, Berlin, 2010. MR2574377

[15] A.-L. Dalibard and N. Masmoudi, *Separation for the stationary Prandtl equation*, arXiv:1802.04039 (2018).

[16] J. Eggers and M. A. Fontelos, *The role of self-similarity in singularities of partial differential equations*, Nonlinearity **22** (2009), no. 1, R1–R44. MR2470260

[17] T. M. Elgindi, *Finite-time singularity formation for $C^{1,\alpha}$ solutions to the incompressible Euler equations on R^3*, arXiv:1904.04795 (2019).

[18] T. M. Elgindi and I.-J. Jeong, *Symmetries and critical phenomena in fluids*, arXiv preprint arXiv:1610.09701 (2016).

[19] Y. Giga and R. V. Kohn, *Asymptotically self-similar blow-up of semilinear heat equations*, Communications on Pure and Applied Mathematics **38** (1985), no. 3, 297–319.

[20] O. Guès, G. Métivier, M. Williams, and K. Zumbrun, *Existence and stability of multidimensional shock fronts in the vanishing viscosity limit*, Arch. Ration. Mech. Anal. **175** (2005), no. 2, 151–244. MR2118476

[21] G. Holzegel, S. Klainerman, J. Speck, and W. W.-Y. Wong, *Small-data shock formation in solutions to 3D quasilinear wave equations: an overview*, J. Hyperbolic Differ. Equ. **13** (2016), no. 1, 1–105. MR3474069

[22] F. John, *Formation of singularities in one-dimensional nonlinear wave propagation*, Comm. Pure Appl. Math. **27** (1974), 377–405. MR0369934

[23] S. Klainerman and I. Rodnianski, *Improved local well-posedness for quasilinear wave equations in dimension three*, Duke Math. J. **117** (2003), no. 1, 1–124. MR1962783

[24] S. Klainerman, *Long time behaviour of solutions to nonlinear wave equations*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), 1984, pp. 1209–1215. MR804771

[25] S. Klainerman and A. Majda, *Formation of singularities for wave equations including the nonlinear vibrating string*, Comm. Pure Appl. Math. **33** (1980), no. 3, 241–263. MR562736

[26] P. D. Lax, *Development of singularities of solutions of nonlinear hyperbolic partial differential equations*, J. Mathematical Phys. **5** (1964), 611–613. MR0165243

[27] T. P. Liu, *Development of singularities in the nonlinear waves for quasilinear hyperbolic partial differential equations*, J. Differential Equations **33** (1979), no. 1, 92–111. MR540819

[28] J. Luk and J. Speck, *Shock formation in solutions to the 2D compressible Euler equations in the presence of non-zero vorticity*, Invent. Math. **214** (2018), no. 1, 1–169. MR3858399

[29] A. Majda, *The existence of multidimensional shock fronts*, Mem. Amer. Math. Soc. **43** (1983), no. 281, v+93. MR699241

[30] A. Majda, *The stability of multidimensional shock fronts*, Mem. Amer. Math. Soc. **41** (1983), no. 275, iv+95. MR683422

[31] A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables*, Applied Mathematical Sciences, vol. 53, Springer-Verlag, New York, 1984. MR748308

[32] Y. Martel, F. Merle, and P. Raphaël, *Blow up for the critical generalized Korteweg–de Vries equation. I: Dynamics near the soliton*, Acta Math. **212** (2014), no. 1, 59–140. MR3179608

[33] F. Merle, *Asymptotics for L^2 minimal blow-up solutions of critical nonlinear Schrödinger equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **13** (1996), no. 5, 553–565. MR1409662

[34] F. Merle and P. Raphael, *The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation*, Ann. of Math. (2) **161** (2005), no. 1, 157–222. MR2150386

[35] F. Merle, P. Raphaël, and I. Rodnianski, *Blowup dynamics for smooth data equivariant solutions to the critical Schrödinger map problem*, Invent. Math. **193** (2013), no. 2, 249–365. MR3090180

[36] F. Merle and H. Zaag, *Stability of the blow-up profile for equations of the type $u_t = \Delta u + |u|^{p-1}u$*, Duke Math. J. **86** (1997), no. 1, 143–195. MR1427848

[37] F. Merle and H. Zaag, *On the stability of the notion of non-characteristic point and blow-up profile for semilinear wave equations*, Comm. Math. Phys. **333** (2015), no. 3, 1529–1562. MR3302641

[38] G. Métivier, *Stability of multidimensional shocks*, Advances in the theory of shock waves, 2001, pp. 25–103. MR1842775

[39] S. Miao and P. Yu, *On the formation of shocks for quasilinear wave equations*, Invent. Math. **207** (2017), no. 2, 697–831. MR3595936

[40] Y. Pomeau, M. Le Berre, P. Guyenne, and S. Grilli, *Wave-breaking and generic singularities of nonlinear hyperbolic equations*, Nonlinearity **21** (2008), no. 5, T61–T79. MR2412317

[41] B. Riemann, *Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite*, Abhandlungen der Königlichen Gesellschaft der Wissenschaften in Göttingen **8** (1860), 43–66.

[42] T. C. Sideris, *Formation of singularities in three-dimensional compressible fluids*, Comm. Math. Phys. **101** (1985), no. 4, 475–485. MR815196

[43] J. Speck, *Shock formation in small-data solutions to 3D quasilinear wave equations*, Mathematical Surveys and Monographs, vol. 214, American Mathematical Society, Providence, RI, 2016. MR3561670