

# On rotated backwards self-similar solutions of the incompressible 3D Navier-Stokes equations

Ben Pineau and Vlad Vicol

ABSTRACT. We consider backwards globally self-similar solutions of the 3D incompressible Navier-Stokes equations which are invariant under the *joint action of scaling* (the natural parabolic scaling) *and rotation* about a given axis, at a constant angular speed  $\alpha$  in self-similar time.

For these so-called *rotated self-similar* solutions (RSS), we prove that if they satisfy a Type I upper bound, and if the rotation parameter  $\alpha$  is either too small, or too large, then they must be trivial. This Liouville-type result extends the classical works of Nečas-Růžička-Šverák ('96) and Tsai ('98), which only consider  $\alpha = 0$ , to the case of similarity profiles which experience nontrivial rotation. Our results partially answer a question posed by Perelman.

For backwards globally self-similar solutions which are invariant under the discrete action of scaling and rotation, the so-called *rotated discretely self-similar* solutions (RDSS), we obtain similar Liouville-type results under a Type I upper bound, assuming extreme values of the rotation parameter  $\alpha$ , and if the scaling factor  $\lambda$  is sufficiently close to 1.

The proof of all these results rests on the introduction of a robust weighted- $L^2$  framework. In particular, our method is quantitative and is not sensitive to whether the Bernoulli head pressure satisfies a maximum principle, which was a key obstruction in previous works. July 14, 2026

## 1. Introduction

We consider the Cauchy problem for the 3D incompressible Navier-Stokes equations

$$\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \quad (1.1a)$$

$$\nabla \cdot u = 0, \quad (1.1b)$$

for the unknown velocity field  $u = u(x, t)$  and scalar pressure  $p = p(x, t)$ , with  $x \in \mathbb{R}^3$  and  $t \geq -1$ . We supplement (1.1) with a *smooth, divergence-free*, and sufficiently fast *decaying* initial datum  $u(\cdot, -1) = u_0$ . Every such datum  $u_0$  produces a unique, smooth, decaying solution  $(u, p)$  of (1.1), on a maximal time interval  $[-1, T_*)$ , for some  $T_* = T_*(u_0) \in (-1, \infty]$ , see [18, 39, 58, 2]. Whether  $T_*$  can be finite is a central open problem in the mathematical theory of viscous incompressible fluids [40, 24].

Suppose  $T_*$  is finite, and without loss of generality, let  $T_* = 0$ . The classical candidate for a finite-time singularity is a *self-similar* solution: zooming-in and rescaling the solution at just the right rate in time, one sees the same object as  $t \rightarrow 0^-$ , a *similarity profile*. For many nonlinear equations of mathematical physics this picture is not merely a heuristic, but a theorem.<sup>1</sup>

Short of proving unconditional global-in-time regularity for (1.1), it is thus a natural question to ask whether smooth self-similar solutions of the 3D incompressible Navier-Stokes equations exist.

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<sup>1</sup>Self-similar singularity formation has been rigorously established, for instance in: the semilinear heat equation [25, 30, 7, 45], the incompressible Euler equations [21, 13], shocks for compressible Euler [9], implosions for the compressible Euler and Navier-Stokes [43, 44, 8, 14], Prandtl boundary-layer separation [20, 17], Euler-Poisson gravitational collapse [29], chemotactic aggregation in reaction-diffusion systems [31, 32], the mass-critical nonlinear Schrödinger equation [42], the complex Ginzburg-Landau equation [48, 41], the harmonic map heat flow and wave maps [50, 49], and the gravitational collapse of a self-gravitating scalar field in general relativity [16]. This is a highly non-exhaustive list, which for instance does not cover the many interesting results on self-similar blowup in 1D toy-models arising in fluid dynamics.

**1.1. Backwards self-similar solutions.** In [40] Leray proposed to seek solutions of (1.1) that are invariant under the natural Navier-Stokes scaling:

$$u(x, t) = \lambda u(\lambda x, \lambda^2 t), \quad \text{for all } \lambda > 0.$$

For a solution defined on  $[-1, 0)$  and which blows up as  $t \rightarrow 0^-$  at  $x = 0$ , one may take  $\lambda = \frac{1}{\sqrt{-t}}$ , thus arriving at a *backwards*<sup>2</sup> *globally*<sup>3</sup> *self-similar solution* (SS):

$$u(x, t) = \frac{1}{\sqrt{-t}} U\left(\frac{x}{\sqrt{-t}}\right), \quad t < 0. \quad (1.2)$$

Here  $U: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is a time-independent *similarity profile*. Inserting ansatz (1.2) into the Navier-Stokes system (1.1) shows that the profile  $U = U(y)$  solves the stationary Leray system

$$\frac{1}{2}U + \frac{1}{2}(y \cdot \nabla)U - \Delta U + (U \cdot \nabla)U + \nabla P = 0, \quad (1.3a)$$

$$\nabla \cdot U = 0, \quad (1.3b)$$

on  $\mathbb{R}^3$ . Here we have denoted the self-similar space variable as  $y = \frac{x}{\sqrt{-t}}$ .

Leray's question, whether (1.3) admits a nontrivial finite-energy solution  $U$ , was answered in the negative by Nečas, Růžička, and Šverák [46]: if  $U \in L^3(\mathbb{R}^3)$ , then  $U \equiv 0$ .<sup>4</sup> The proof rests on a maximum principle for the head-pressure, or Bernoulli function, defined by

$$\Pi := P + \frac{1}{2}|U|^2 + \frac{1}{2}(y \cdot U). \quad (1.4)$$

For a solution of (1.3),  $\Pi$  satisfies the elliptic equation

$$-\Delta \Pi + (U + \frac{1}{2}y) \cdot \nabla \Pi = -|\Omega|^2 \leq 0, \quad \Omega = \nabla \times U, \quad (1.5)$$

and hence obeys a maximum principle. The triviality of  $U$  then follows by comparison against exponentially growing radial supersolutions of (1.5).

Tsai [56] established the same conclusion for profiles  $U \in L^p(\mathbb{R}^3)$  with  $p \in (3, \infty]$ , as well as for weak solutions satisfying only local energy estimates. Further Liouville-type theorems for self-similar solutions were obtained by Chae and Wolf [12]; for instance, they prove that a globally self-similar solution with profile  $U \in L^{p,\infty}(\mathbb{R}^3)$  with  $p \in (\frac{3}{2}, \infty)$ , must vanish identically.

**1.2. Rotated self-similar solutions (RSS).** In addition to the scaling invariance  $u(x, t) \mapsto \lambda u(\lambda x, \lambda^2 t)$  for  $\lambda > 0$ , the Navier-Stokes equations are also invariant under rotations  $u(x, t) \mapsto \mathcal{O}u(\mathcal{O}^\top x, t)$ , for  $\mathcal{O} \in SO(3)$ . Thus, a larger class of self-similar solutions may be obtained by composing scaling and rotation. Denote by  $R(s) \in SO(3)$  the matrix representing rotation by angle  $s$  about the  $\vec{e}_3$ -axis,<sup>5</sup> and let  $J$  be the associated antisymmetric generator:

$$R(s) = \begin{pmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad J = R'(0) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (1.6)$$

<sup>2</sup>The adjective *backwards* is used to distinguish these solutions from *forward* self-similar solutions, which satisfy the ansatz  $u(x, t) = \frac{1}{\sqrt{t}}U(\frac{x}{\sqrt{t}})$ , for  $t > 0$ , and which arise from scale-invariant initial data. Forward self-similar and forward discretely self-similar solutions play an important role in the Cauchy theory of the Navier-Stokes equations for scale-invariant data, and in questions related to non-uniqueness; see, for instance, [33, 57, 6, 37, 35, 34, 27, 1]. Since our goal is to rule out possible finite-time blowup profiles for smooth Navier-Stokes solutions, all self-similar, rotated self-similar, and discretely self-similar solutions considered in this paper are backwards in time. See also [28] for related stationary solutions.

<sup>3</sup>We call the solution *globally* self-similar to emphasize that the profile  $U$  is defined on all of  $\mathbb{R}^3$  and does not depend on time.

<sup>4</sup>Results which show that globally defined solutions of an elliptic system must vanish under a mild integrability or growth constraint are sometimes called *Liouville-type* theorems.

<sup>5</sup>The choice of the  $\vec{e}_3$ -axis as the axis of rotation is made here only for convenience.

Letting  $\mathcal{O} = R(2\alpha \log \lambda)$  for some fixed  $\alpha \in \mathbb{R}$  leads to the class of *rotated self-similar* solutions (RSS), as being those for which

$$u(x, t) = \lambda R(2\alpha \log \lambda) u(\lambda R(-2\alpha \log \lambda)x, \lambda^2 t), \quad \text{for all } \lambda > 0.$$

For a solution of (1.1) defined on  $[-1, 0)$  and which blows up as  $t \rightarrow 0^-$  at  $x = 0$ , one may take  $\lambda = \frac{1}{\sqrt{-t}}$  so that  $\mathcal{O} = R(2\alpha \log \lambda) = R(\alpha s)$ , where  $s = -\log(-t)$ ; this corresponds to a similarity profile that is spun at a constant angular speed  $\alpha \in \mathbb{R}$ . The *backwards rotated globally self-similar* ansatz thus reads

$$u(x, t) = \frac{1}{\sqrt{-t}} R(\alpha s) U\left(\frac{R(-\alpha s)x}{\sqrt{-t}}\right), \quad s = -\log(-t), \quad (1.7)$$

for some  $\alpha \in \mathbb{R}$  and a time-independent similarity profile  $U: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ . When  $\alpha = 0$ , (1.7) reduces to the self-similar ansatz (1.2) because  $R(0) = \text{Id}$ .

Substituting (1.7) into (1.1), the profile  $U: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is found to solve

$$\alpha(JU - (Jy \cdot \nabla)U) + \frac{1}{2}U + \frac{1}{2}(y \cdot \nabla)U - \Delta U + (U \cdot \nabla)U + \nabla P = 0, \quad (1.8a)$$

$$\nabla \cdot U = 0, \quad (1.8b)$$

on  $\mathbb{R}^3$ . Here, the self-similar space variable is  $y = R(-\alpha s)\frac{x}{\sqrt{-t}}$ , and  $J$  is defined in (1.6). According to Tsai's book [58, Chapter 8], the backward rotated self-similar ansatz (1.7) was proposed by G. Perelman in a private communication to G. Seregin; see also [5, 35].

In contrast to the non-rotated case ( $\alpha = 0$ ), it is currently an *open problem* to rule out nontrivial rotated backward self-similar singularities of the 3D Navier-Stokes equations. The conjecture (attributed to G. Perelman) stated in [58, Conjecture 8.9], or equivalently in [5, Open Problem 5.2], is the following:

**CONJECTURE 1.1.** *Let  $\alpha \neq 0$ , and let  $U$  be a smooth solution of (1.8) on  $\mathbb{R}^3$ . If there exists a constant  $C_{U,0} > 0$  such that*

$$|U(y)| \leq \frac{C_{U,0}}{1 + |y|}, \quad (1.9)$$

*for all  $y \in \mathbb{R}^3$ , then  $U \equiv 0$ .*

**REMARK 1.2** (Type I blowup rate). Using the ansatz (1.7), it is immediate to show the global bound (1.9) is equivalent (within this class of rotated backwards self-similar solutions) to the borderline Type I blowup rate<sup>6</sup>

$$|u(x, t)| \leq \frac{C_{U,0}}{|x| + \sqrt{-t}} \quad (1.10)$$

for all  $(x, t) \in \mathbb{R}^3 \times [-1, 0)$ . The constants in (1.9) and (1.10) are the same. Thus, an equivalent way to state Conjecture 1.1 is: *Assume that  $u$  is a smooth solution of the 3D Navier-Stokes equations (1.1) on  $\mathbb{R}^3 \times [-1, 0)$  which satisfies the bound (1.10). Furthermore, assume that there exists  $\alpha > 0$  such that  $u$  satisfies the (global) rotated backwards self-similar ansatz (1.7). Then,  $u \equiv 0$ .*

**REMARK 1.3** ( $C_{U,0}$  is independent of  $\alpha$ ). The equivalent formulation Conjecture 1.1 that we have given in Remark 1.2 stresses an important point. The constant  $C_{U,0}$  appearing in (1.10) is just an a-priori Type I upper bound for the magnitude of the Navier-Stokes solution; this constant does not know anything about rotations being present or not. The ansatz (1.7) is separate, and

<sup>6</sup>The Type I blowup rate (1.10) is precisely the rate at which the dissipation balances the material derivative, and it is the scaling-critical threshold underlying partial regularity theory of [10]. In the axisymmetric setting, Type I blowup for 3D Navier-Stokes has been ruled out [15, 54]; consequently, any putative singularity must be of Type II. For recent results on potential Type II blowups, we refer to [51, 52, 53] and references therein. Note that in a Type II blowup scenario for 3D Navier-Stokes, the self-similar scaling is expected to be dictated by the 3D Euler equations; self-similar solutions of 3D Euler can however be shown to not be viable for a “lift” to 3D Navier-Stokes, under a local outgoing property [19].

this is where the rotation parameter  $\alpha$  comes in. Moreover, at the initial time  $t = -1$  we have  $s = -\log(1) = 0$ , and  $R(0) = \text{Id}$ ; the initial data  $u_0$  does not know about the rotation either. As such, *throughout this paper we assume that the constant  $C_{U,0}$  in (1.9) (or (1.10)) depends on  $u_0$ , but not on  $\alpha$ .*

Why is Conjecture 1.1 open for  $\alpha \neq 0$ ? Assumption (1.9) only implies that the profile  $U$  belongs to the weak- $L^3$  class  $L^{3,\infty}(\mathbb{R}^3)$ . Had we assumed that the profile decays a little bit faster, to ensure that  $U \in L^3(\mathbb{R}^3)$ , then we would have  $u \in L^\infty([-1, 0]; L^3(\mathbb{R}^3))$ , and by the theory of Escauriaza, Seregin, and Šverák [22] this would imply regularity (and hence the triviality of  $U$ ). This argument applies to rotated self-similar solutions since (1.7) implies that  $\|u(\cdot, t)\|_{L^3} = \|U\|_{L^3}$  for all  $t < 0$  and  $\alpha \in \mathbb{R}$ . The genuine difficulty therefore lies in the borderline decay rate (1.9), which guarantees  $U \in L^p(\mathbb{R}^3)$  for every  $p > 3$ , but not membership in the scaling-critical space  $L^3$ . When  $\alpha = 0$ , the Liouville theorem established in [56] fundamentally relies on the maximum principle for the head pressure  $\Pi$  (1.4) recalled above. When  $\alpha \neq 0$ , the issue is that the antisymmetric rotation terms  $\alpha(JU - (Jy \cdot \nabla)U)$  present in (1.8) destroy this structure. We are not aware (see also [5, 58]) of any scalar quantity (an  $\alpha$ -dependent modification of (1.4)) playing the role of the Bernoulli function, and which satisfies a maximum principle. This is the central obstruction, and the reason that no Liouville theorem has so far been available for rotated self-similar solutions, when  $\alpha \neq 0$ .

The purpose of this paper is to prove that if  $U$  is a RSS solution of 3D Navier-Stokes and  $|\alpha|$  is either sufficiently small, or sufficiently large when compared to  $C_{U,0}$ , then  $U \equiv 0$ .

**THEOREM 1.4** (Main result for RSS solutions). *Let  $u$  be a solution of the 3D incompressible Navier-Stokes equations on  $\mathbb{R}^3 \times [-1, 0)$ , which satisfies the Type I upper bound (1.10) for some  $C_{U,0} > 0$ . Assume that there exists  $U \in C^2(\mathbb{R}^3)$  and  $\alpha \in \mathbb{R}$  such that  $u$  is a backwards rotated globally self-similar solution of (1.1) with profile  $U$  (the RSS ansatz (1.7) holds). There exists  $0 < \underline{\alpha} = \underline{\alpha}(C_{U,0}) \ll 1$  and  $1 \ll \bar{\alpha} = \bar{\alpha}(C_{U,0}) < \infty$ , such that if either  $|\alpha| < \underline{\alpha}$  or  $|\alpha| > \bar{\alpha}$ , then  $U \equiv 0$ .*

Theorem 1.4 resolves Conjecture 1.1 for  $|\alpha| \ll 1$  and  $|\alpha| \gg 1$ , but leaves open the case  $\alpha \approx 1$ .

The main ideas in the proof of Theorem 1.4 are as follows. When  $|\alpha| \ll 1$ , a natural expectation is that *if* one finds a new proof of the results in [46, 56] for  $\alpha = 0$ , which is quantitative and does not rely on the maximum principle, then one should obtain the desired Liouville-type result for small  $|\alpha|$ , with this new proof. This new proof is given in Section 5, and is based on a weighted  $L^2$ -based enstrophy estimate, which relies on two independent facts. First, a naive energy estimate shows that there exists a large enough radius  $\bar{R} > 0$ , such that if the vorticity  $\Omega$  has small  $L^2$  norm when restricted to the ball  $B_{\bar{R}}$ ,<sup>7</sup> then  $U \equiv 0$ ; see Proposition 3.1. Second, we observe that if  $U$  satisfies suitable smoothness and decay properties, then the non-normal operator  $L = -\Delta + (U + \frac{1}{2}y) \cdot \nabla$  present on the left side of (1.5) is such that its  $L^2$ -adjoint  $L^*$  has an element  $w = w(y)$  in its kernel, which is smooth and obeys Gaussian upper and lower bounds; see Proposition 5.1. We may then use this kernel element  $w$  as a weight in a weighted- $L^2$  energy estimate for  $\Omega$  to conclude that  $\int_{\mathbb{R}^3} |\Omega|^2 w dy \lesssim |\alpha|$ , where the implicit constant depends only on  $C_{U,0}$ . Due to the  $\alpha$ -independent Gaussian lower bound for  $w$ , we deduce that  $\int_{B_{\bar{R}}} |\Omega|^2 dy \lesssim |\alpha|$ . Combining the above two facts implies Theorem 1.4 for  $|\alpha| \ll 1$ .

For the case  $|\alpha| \gg 1$  the proof idea is different. For large  $|\alpha|$  one may expect that the term  $\alpha(JU - (Jy \cdot \nabla)U)$  present in (1.8) is dominant, and so a smooth solution  $U$  wants to stay close to the kernel of the operator  $\mathcal{R} = J - (Jy) \cdot \nabla$ . Note that the kernel of  $\mathcal{R}$  consists of axisymmetric vector fields (with respect to rotations around the  $\vec{e}_3$  axis). In Section 6 we are able to quantify this intuition by showing that if  $|\alpha|$  is sufficiently large, then a weighted- $L^2$  norm of  $\mathcal{R}U$  is  $\ll \frac{1}{\alpha}$ , where the implicit constant depends on  $C_{U,0}$ ; see Proposition 6.5. Here it is important that we obtain a weighted- $L^2$  bound on  $\mathcal{R}U$  which is  $\ll \frac{1}{\alpha}$  as opposed to merely  $\lesssim \frac{1}{\alpha}$ , and that the weight is

<sup>7</sup>Here and throughout the paper, we denote by  $B_R$  the ball of radius  $R > 0$  centered at the origin in  $\mathbb{R}^3$ .

independent of  $\alpha$ . With this bound, we return to the  $L^2$ -weighted energy estimate for the vorticity with weight  $w$  (the element in the kernel of  $L^*$  described earlier) and show that if  $|\alpha|$  is sufficiently large, then  $\int_{B_{\bar{R}}} |\Omega|^2 dy \lesssim \alpha \|e^{-|y|^2/8} \mathcal{R}U\|_{L^2(\mathbb{R}^3)} \ll 1$ . The proof of Theorem 1.4 for  $|\alpha| \gg 1$  then follows as before, because there is not enough enstrophy in a large ball around the origin.

**1.3. Discretely self-similar solutions (DSS).** The self-similar solutions of 3D Navier-Stokes discussed in Sections 1.1 and 1.2 are invariant under the scaling symmetry  $u(x, t) \mapsto \lambda u(\lambda x, \lambda^2 t)$  for all  $\lambda > 1$ . One can however consider a more general class of solutions, which are only assumed to be invariant under the Navier-Stokes scaling symmetry for *one particular*<sup>8</sup> value of  $\lambda > 1$ , namely

$$u(x, t) = \lambda u(\lambda x, \lambda^2 t), \quad \text{for a fixed } \lambda > 1.$$

Such solutions are called *discretely self-similar* (DSS). To emphasize the value  $\lambda > 1$  of the similarity factor, this class of solutions is often denoted as  $\lambda$ -DSS; see [58, Section 8].

We emphasize that discretely self-similar solutions do not correspond to a time-independent similarity profile; instead,  $\lambda$ -DSS solutions representing a finite time blowup as  $t \rightarrow 0^-$  at  $x = 0$  are given by the *backwards globally*<sup>9</sup> *discretely self-similar* ansatz

$$u(x, t) = \frac{1}{\sqrt{-t}} U(y, s), \quad y = \frac{x}{\sqrt{-t}}, \quad s = -\log(-t), \quad (1.11a)$$

for a *periodic-in-self-similar-time* profile  $U: \mathbb{R}^3 \times [0, S] \rightarrow \mathbb{R}^3$ , where the time-period is given by

$$S := 2 \log(\lambda) > 0. \quad (1.11b)$$

Note that as  $\lambda \rightarrow 1^+$  the profile  $U$  becomes time-independent since  $S \rightarrow 0^+$  in (1.11b), and thus the ansatz (1.11a) becomes the Leray self-similar ansatz (1.2).

REMARK 1.5 (RSS  $\subseteq$  DSS). Note that because  $R(s + 2\pi) = R(s)$  and  $R(0) = \text{Id}$ , rotated self-similar (RSS) solutions as defined in Section 1.2 for a given  $\alpha \in \mathbb{R}$  are automatically discretely self-similar ( $\lambda$ -DSS) for any factor  $\lambda > 1$  such that  $2\alpha \log(\lambda) \in 2\pi\mathbb{Z}$ . Put differently, if  $u$  is given by ansatz (1.7) with a time-independent profile  $\bar{U}$  and rotation parameter  $\alpha \in \mathbb{R}$ , then  $u$  satisfies the ansatz (1.11a) with the periodic-in-time profile  $U(y, s) = R(\alpha s) \bar{U}(R(-\alpha s)y)$ , and period  $S = \frac{2\pi}{|\alpha|}$ . Due to (1.11b), the discrete similarity factor  $\lambda$  and the rotation strength  $\alpha$  are related via

$$\lambda = e^{\frac{\pi}{|\alpha|}}.$$

Importantly, we note that *the regime*  $|\alpha| \ll 1$  *corresponds to*  $\lambda \gg 1$ , while *the regime*  $|\alpha| \rightarrow \infty$  *corresponds to*  $\lambda \rightarrow 1^+$ .

Substituting (1.11a) into (1.1), we discover that the profile  $U: \mathbb{R}^3 \times [0, S] \rightarrow \mathbb{R}^3$  is a time-periodic solution of

$$\partial_s U + \frac{1}{2} U + \frac{1}{2} (y \cdot \nabla) U - \Delta U + (U \cdot \nabla) U + \nabla P = 0, \quad (1.12a)$$

$$\nabla \cdot U = 0, \quad (1.12b)$$

on  $\mathbb{R}^3$ , with period  $S = 2 \log(\lambda)$  as in (1.11b).

Not much seems to be known about the existence of solutions to (1.12), other than the following Liouville-type theorem of Chae and Wolf [11], which concerns the case of discrete self-similarity factors  $\lambda$  such that  $0 < \lambda - 1 \ll 1$ . Precisely, [11] shows that for every  $C_{U,0} > 0$  there exists  $\lambda_* = \lambda_*(C_{U,0}) > 1$  such that the following holds: if  $u \in C^\infty((-\infty, 0) \times \mathbb{R}^3)$  is a solution of 3D

<sup>8</sup>Clearly if  $u$  is invariant under  $u(x, t) \mapsto \lambda u(\lambda x, \lambda^2 t)$  for one value of  $\lambda > 1$ , then it is also invariant under  $u(x, t) \mapsto \lambda^k u(\lambda^k x, \lambda^{2k} t)$  for all  $k \in \mathbb{N}$ . For consistency, throughout this paper we shall refer to the discrete self-similarity factor  $\lambda > 1$  as being the one which is *smallest possible*.

<sup>9</sup>These solutions are still referred to as *globally* self-similar because (1.11a) is required to hold for all  $y \in \mathbb{R}^3$ .

Navier-Stokes which satisfies the Type I bound (1.10) on  $(-\infty, 0) \times \mathbb{R}^3$  for some  $C_{U,0} > 0$ , and if  $u$  is  $\lambda$ -DSS (meaning that  $u$  is given by the ansatz (1.11)) for some  $\lambda \in (1, \lambda_*)$ , then  $U \equiv 0$ .

The Chae and Wolf result [11] is obtained by a *compactness argument*, which reduces the problem to the known non-existence results for Leray self-similar solutions in [46, 56]. In light of Remark 1.5 which provides a direct analogy between the  $0 < \lambda - 1 \ll 1$  regime in  $\lambda$ -DSS solutions and the  $|\alpha| \gg 1$  regime in RSS solutions, it is worth noting that [11] gives an *alternative proof* of the large  $|\alpha|$  case in Theorem 1.4. The advantage of the proof we present in this paper (see Section 6) is that it is *quantitative* and provides an explicit a-priori bound for the vorticity. In fact, a slight modification of the proof of Theorem 1.4 immediately recovers the main result in [11], which we re-state here for convenience:

**THEOREM 1.6** (Main result for DSS solutions). *Let  $u$  be a solution of the 3D incompressible Navier-Stokes equations on  $\mathbb{R}^3 \times [-1, 0)$ , which satisfies the Type I upper bound (1.10) for some  $C_{U,0} > 0$ . Assume that there exists  $\lambda > 1$  and  $U \in C^2(\mathbb{R}^3 \times [0, S])$  with  $S = 2 \log(\lambda)$ , such that  $u$  is a backwards globally discretely self-similar solution of (1.1) with profile  $U$  (the DSS ansatz (1.11a) holds). There exists  $\underline{\lambda} = \underline{\lambda}(C_{U,0}) > 1$ , such that if  $1 < \lambda < \underline{\lambda}$ , then  $U \equiv 0$ .*

The proof of Theorem 1.6 is given in Section 7 and appears as a particular case ( $\alpha = 0$ ) of the results discussed in the next subsection.

**1.4. Rotated Discretely self-similar solutions (RDSS).** Analogously to how the discretely self-similar (DSS) solutions generalize the Leray self-similar (SS) solutions by requiring that scale invariance holds for a minimal  $\lambda > 1$  instead of for all  $\lambda > 0$ , one may generalize the class of rotated self-similar (RSS) solutions by requiring that invariance under the combined effect of rotation (with self-similar angular speed  $\alpha$ ) and scaling holds for a minimal  $\lambda > 1$ , instead of for all  $\lambda > 0$ . This leads one to the concept of *rotated discretely self-similar* (RDSS) solutions, as being those for which<sup>10</sup>

$$u(x, t) = \lambda R(2\alpha \log \lambda) u(\lambda R(-2\alpha \log \lambda)x, \lambda^2 t), \quad \text{for a fixed } \lambda > 1, \text{ and a fixed } \alpha \in \mathbb{R},$$

for all  $x \in \mathbb{R}^3$  and  $t \in [-1, 0)$ . To emphasize the value  $\lambda > 1$  of the similarity factor and the value  $\alpha \in \mathbb{R}$  of the angular speed, we shall sometimes denote this class of solutions as  $(\alpha, \lambda)$ -RDSS.

As with DSS solutions, we emphasize that RDSS solutions do not correspond to a time-independent rotated similarity profile; instead, one may verify that RDSS solutions representing a finite time blowup as  $t \rightarrow 0^-$  at  $x = 0$  are given by the *backwards globally<sup>11</sup> rotated discretely self-similar* ansatz

$$u(x, t) = \frac{1}{\sqrt{-t}} R(\alpha s) U(y, s), \quad y = \frac{R(-\alpha s)x}{\sqrt{-t}}, \quad s = -\log(-t), \quad (1.13a)$$

for a *periodic-in-self-similar-time* profile  $U: \mathbb{R}^3 \times [0, S] \rightarrow \mathbb{R}^3$ , where the time-period is given by

$$S := 2 \log(\lambda) > 0. \quad (1.13b)$$

Note that as  $\lambda \rightarrow 1^+$  the profile  $U$  becomes time-independent since  $S \rightarrow 0^+$  in (1.13b), and thus the ansatz (1.13a) becomes the rotated self-similar ansatz (1.7), with the same value of  $\alpha$ . Moreover, since  $R(0) = \text{Id}$ , setting  $\alpha = 0$  in (1.13) recovers the discretely self-similar ansatz (1.11a), with the same value of  $\lambda$ . In particular, in light of Remark 1.5 we have the hierarchy (see also [5, 58])

$$\text{SS} \subseteq \text{RSS} \subseteq \text{DSS} \subseteq \text{RDSS},$$

and hence Liouville-type theorems should be hardest to establish in the setting of rotated discretely self-similar solutions.

<sup>10</sup>In [5, Equation (1.9)] and [58, Equation (8.10)] the definition replaces the existence of an  $\alpha \in \mathbb{R}$  with the existence of a phase  $\phi = 2\alpha \log \lambda \in \mathbb{R}$ . Since  $\lambda > 1$ , these are clearly equivalent definitions; we have chosen this definition for consistency with the notation in Section 1.2.

<sup>11</sup>These solutions are still referred to as *globally* self-similar because (1.13a) is required to hold for all  $y \in \mathbb{R}^3$ .

Note that if  $u$  is a  $(\alpha, \lambda)$ -RDSS solution to 3D Navier-Stokes, then by substituting (1.13a) into (1.1), we discover that the profile  $U: \mathbb{R}^3 \times [0, S] \rightarrow \mathbb{R}^3$  is a time-periodic solution of

$$\partial_s U + \alpha(JU - (Jy \cdot \nabla)U) + \frac{1}{2}U + \frac{1}{2}(y \cdot \nabla)U - \Delta U + (U \cdot \nabla)U + \nabla P = 0, \quad (1.14a)$$

$$\nabla \cdot U = 0, \quad (1.14b)$$

on  $\mathbb{R}^3$ , with period  $S = 2 \log(\lambda)$  as in (1.13b).

A small modification of the adjoint weight construction from the proof of Theorem 1.4 immediately leads to the following generalization of Theorem 1.6:

**THEOREM 1.7** (Main result for RDSS solutions). *Let  $u$  be a solution of the 3D incompressible Navier-Stokes equations on  $\mathbb{R}^3 \times [-1, 0)$ , which satisfies the Type I upper bound (1.10) for some  $C_{U,0} > 0$ . Assume that there exist  $\alpha \in \mathbb{R}$ ,  $\lambda > 1$ , and  $U \in C^2(\mathbb{R}^3 \times [0, S])$  with  $S = 2 \log(\lambda)$ , such that  $u$  is a backwards globally rotated discretely self-similar solution of (1.1) with profile  $U$  (the RDSS ansatz (1.13a) holds). There exist  $\underline{\alpha} = \underline{\alpha}(C_{U,0}) > 0$  and  $\underline{\lambda} = \underline{\lambda}(C_{U,0}) > 1$ , such that if  $|\alpha| \leq \underline{\alpha}$  and  $1 < \lambda < \underline{\lambda}$ , then  $U \equiv 0$ . Moreover, there exist  $\bar{\alpha} = \bar{\alpha}(C_{U,0}) > 0$  and  $\bar{\lambda} = \bar{\lambda}(C_{U,0}) > 1$ , such that if  $|\alpha| \geq \bar{\alpha}$  and  $1 < \lambda < \bar{\lambda}^{\frac{1}{1+\alpha^2}}$ , then  $U \equiv 0$ .*

The proof of Theorem 1.7 is given in Sections 7–8. The arguments may be viewed as an extension of the proof of Theorem 1.4, with the additional knowledge that  $\partial_s U$  is not zero, but is instead *small* in a suitable sense, if the period  $S$  is assumed small (equivalently, if  $\lambda$  is close to 1).

**REMARK 1.8** (Navier-Stokes solitons and breathers). We are indebted to V. Šverák for pointing out the following connection. In the terminology of geometric flows, given an evolution  $\partial_t u = F[u]$  with a symmetry group  $G$  acting via  $u \mapsto R(g)u$ , a solution is called a *soliton* if  $u(t) = R(g(t))U$  for a fixed profile  $U$ . RSS solutions are solitons of (1.1) in this sense. The symmetry group of (1.1) is generated by scalings, rotations, translations, and Galilean boosts, and the solitons (those with bounded velocity and decaying pressure) may be classified (up to conjugation by Galilean symmetries) as: steady states; rotating waves  $u(x, t) = R(\omega t)U(R(-\omega t)x)$ ; and the forward and backward rotated self-similar solutions (1.7). In particular, RSS solutions are the general scaling solitons of (1.1). Backward solitons are conjecturally trivial (cf. Conjecture 1.1), while nontrivial forward self-similar solutions exist for every  $(-1)$ -homogeneous initial datum which is locally Hölder continuous away from the origin [33], and play a central role in the non-uniqueness program [34, 27]. Finally, the DSS and RDSS solutions are *periodic* modulo  $G$ , i.e., the Navier-Stokes analogues of *breathers*, and Theorems 1.6–1.7 may be viewed as conditional results of *no shrinking breathers* type, in the spirit of [47].

**1.5. Organization of the paper.** Sections 2–4 gather basic properties of RSS solutions. The Liouville-type results for RSS solutions claimed in Theorem 1.4 are proven in Section 5 ( $|\alpha| \ll 1$ ) and Section 6 ( $|\alpha| \gg 1$ ). The Liouville-type results for RDSS (and hence also DSS) solutions claimed in Theorem 1.7 are analyzed in Section 7 ( $|\alpha| \ll 1$ ) and Section 8 ( $|\alpha| \gg 1$ ).

## 2. Estimates for the pressure and higher derivatives of the velocity

We work within the class of solutions to 3D Navier-Stokes which are rotated backward self-similar, meaning that ansatz (1.7) holds, and which satisfy the Type I bound (1.10), which is equivalent to the scaling-critical upper bound (1.9). By repeating the arguments in [46, 56], we show that (1.10) implies Type I estimates for higher space derivatives of the velocity, and estimates for the pressure. Importantly, *all these bounds are independent of  $\alpha$* . We record:

**LEMMA 2.1.** *Let  $u$  be a solution of the 3D incompressible Navier-Stokes equations (1.1) on  $\mathbb{R}^3 \times [-1, 0)$ , which satisfies the Type I upper bound (1.10) for some  $C_{U,0} > 0$ . Assume that there*

exists  $U \in C^2(\mathbb{R}^3)$  and  $\alpha \in \mathbb{R}$  such that  $u$  is given by the ansatz (1.7). Then  $U \in C^\infty(\mathbb{R}^3)$ , and there exist  $\alpha$ -independent constants  $C_{U,1} = C_{U,1}(C_{U,0}) > 0$  and  $C_{U,2} = C_{U,2}(C_{U,0}) > 0$  such that

$$|\nabla U(y)| \leq \frac{C_{U,1}}{1 + |y|^2}, \quad |\nabla^2 U(y)| \leq \frac{C_{U,2}}{1 + |y|^3}, \quad (2.1)$$

for all  $y \in \mathbb{R}^3$ . Moreover, for any  $\sigma \in (0, 1)$ , there exists an  $\alpha$ -independent constant  $C_{P,0} = C_{P,0}(C_{U,0}, \sigma) > 0$  such that

$$|P(y)| \leq \frac{C_{P,0}}{1 + |y|^{2-\sigma}}, \quad (2.2)$$

for all  $y \in \mathbb{R}^3$ .

This result is essentially known, and follows mutatis mutandis from [46, 56, 39, 58]. We repeat here a sketch of the proof to emphasize the  $\alpha$ -independence of the constants  $C_{U,1}$ ,  $C_{U,2}$ , and  $C_{P,0}$ . Note that at time  $t = -1$  we have  $s = 0$  and  $R(0) = \text{Id}$ ; thus, the initial data  $u_0 = u|_{t=-1}$  does not know about the rotation parameter  $\alpha$ . The Navier-Stokes equations (1.1) also do not contain an  $\alpha$ -dependent term. Thus, it is convenient to prove parts of Lemma 2.1 in original  $(x, t)$  variables.

PROOF OF LEMMA 2.1. The bound (1.9) implies  $U \in L^\infty_{\text{loc}}(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$  for  $q > 3$ . Equation (1.8a) may be written as  $-\Delta U + \nabla P = \nabla \cdot F$ ,  $\nabla \cdot U = 0$  with  $F \in L^\infty_{\text{loc}}$ . The fact that  $U, P \in C^\infty(\mathbb{R}^3)$  follows from standard interior estimates for the Stokes system [58, Lemma 2.12], differentiating the equation, and iterating. The bounds such obtained are qualitative in nature, as they may a-priori depend on  $\alpha$ .

Note that  $\nabla \cdot (JU - (Jy \cdot \nabla)U) = 0$  and  $\nabla \cdot (U + (y \cdot \nabla)U) = 0$  when  $\nabla \cdot U = 0$ , and hence  $-\Delta P = \partial_i \partial_j (U^i U^j)$ . From [58, Lemma 2.1] (see also [46, Lemma 3.1]) it follows that the pressure  $P$  in (1.8a) is given by  $P = R_i R_j (U^i U^j) + \text{a constant}$ , where  $R_i = \partial_i (-\Delta)^{-1/2}$  are Riesz transforms, and we have used the summation convention on repeated indices. Without loss of generality, we take this constant term to vanish, so that  $P = R_i R_j (U^i U^j)$ ; this directly implies  $P \in L^{q/2}(\mathbb{R}^3)$  for  $q > 3$  with an upper bound that only depends on  $q$  and  $C_{U,0}^2$  (and is hence independent of  $\alpha$ ), by the boundedness of Calderon-Zygmund operators on (non-endpoint) Lebesgue spaces.

The bounds (2.1) follow as in [46, 56] by rescaling to an appropriate parabolic cylinder in original  $(x, t)$  variables, using the fact  $(U, P) \in L^q(\mathbb{R}^3) \times L^{q/2}(\mathbb{R}^3)$  implies the smallness of the mean of  $|u|^3 + |p|^{3/2}$  if the center of this cylinder is far enough, and then applying the Caffarelli-Kohn-Nirenberg [10] partial regularity theory (see [38] for a compact summary).

An alternative argument is as follows. Fix  $y \in \mathbb{R}^3$  arbitrary and set  $r = \frac{1}{2} \max\{|y|, 1\}$ . Then the function<sup>12</sup>  $\tilde{u}(\tilde{x}, \tilde{t}) := r u(y + r\tilde{x}, -1 + r^2\tilde{t})$  solves (1.1) on the unit parabolic cylinder  $Q_1 := B_1 \times (-1, 0]$ . Moreover, the bound (1.10) implies that  $\|\tilde{u}\|_{L^\infty(Q_1)} \leq C_{U,0}$ , a bound which is independent of  $y$  and of  $\alpha$ . By interior regularity for the Navier-Stokes system,<sup>13</sup> there exists constants  $C_1, C_2 > 0$  which depend only on  $C_{U,0}$ , such that  $|(\nabla_{\tilde{x}} \tilde{u})(0, 0)| \leq C_1$  and  $|(\nabla_{\tilde{x}}^2 \tilde{u})(0, 0)| \leq C_2$ . We undo the definition of  $\tilde{u}$  to deduce that  $|(\nabla_x u)(y, -1)| \leq C_1 r^{-2}$  and  $|(\nabla_x^2 u)(y, -1)| \leq C_2 r^{-3}$ . Since at  $t = -1$  we have  $s = 0$  and  $R(0) = \text{Id}$ , we may then use (1.7) to equate  $(\nabla U)(y) = (\nabla_x u)(y, -1)$  and  $\nabla^2 U(y) = (\nabla_x^2 u)(y, -1)$ . The bounds (2.1) now follow by the definition of  $r$ , with  $C_{U,1} = 8C_1$  and  $C_{U,2} = 16C_2$ , two constants that depend only on  $C_{U,0}$  and are thus independent of  $\alpha$ .

The bound (2.2) follows from the singular integral kernel representation of  $P = R_i R_j (U^i U^j)$ , and the previously established bounds for  $U$  and  $\nabla U$ . We have that  $C_{P,0}$  depends only on  $\sigma$ ,  $C_{U,0}$ , and  $C_{U,1}$ , and is thus independent of  $\alpha$ .  $\square$

<sup>12</sup>Note that although  $u$  is a priori a solution of Navier-Stokes only on  $\mathbb{R}^3 \times [-1, 0)$ , the ansatz (1.7) allows us to extend it backwards-in-time as a solution of Navier-Stokes on  $\mathbb{R}^3 \times [-T, 0)$  for any  $T \geq 1$ . In particular, setting  $T = 1 + r^2$  makes  $\tilde{u}$  well defined on  $Q_1$ .

<sup>13</sup>One can follow the proof in the classical paper of Serrin [55] and make it quantitative, resulting in a polynomial dependence of  $C_1$  and  $C_2$  on  $C_{U,0}$ . Note that no pressure bounds appear in this interior estimate. Equivalently, one may iterate interior regularity for the non-stationary Stokes system, see e.g. [15, Lemma A.2].

### 3. Small enstrophy implies zero velocity

We reduce the proof of Theorem 1.4 to verifying that there exists a large enough radius  $\bar{R} > 0$ , such that if  $\|\Omega\|_{L^2(B_{\bar{R}}(0))}$  is sufficiently small (independently of  $\alpha$ ), then  $U \equiv 0$ . This observation is summarized as:

**PROPOSITION 3.1.** *Let  $(U, P)$  be a smooth solution of (1.8) with associated vorticity  $\Omega = \nabla \times U$ . Let  $\bar{R} := \sqrt{8C_{U,1}} > 0$ . There exists a universal constant  $C_\Omega > 0$  such that*

$$\|\Omega\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \Omega\|_{L^2(\mathbb{R}^3)}^2 \leq C_\Omega \|\Omega\|_{L^2(B_{\bar{R}})} (\|\Omega\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \Omega\|_{L^2(\mathbb{R}^3)}^2). \quad (3.1)$$

In particular, if

$$\|\Omega\|_{L^2(B_{\bar{R}})} < C_\Omega^{-1}, \quad (3.2)$$

then  $U \equiv 0$ .

Note that both  $C_\Omega$  and  $\bar{R}$  are independent of  $\alpha$ . Also, note that (3.2) automatically holds if  $6C_{U,1} < C_\Omega^{-4/7}$ , in light of (2.1); this rules out the existence of nontrivial *small solutions* of (1.8). Lastly, note that the smallness condition (3.2) may be re-stated in original  $(x, t)$  variables as follows. Undoing the rescaling (1.7) gives  $\omega(x, t) = (-t)^{-1} R(\alpha s) \Omega(y)$  with  $y = R(-\alpha s)x/\sqrt{-t}$ , so that  $|\omega(x, t)| = (-t)^{-1} |\Omega(y)|$ . The change of variables  $x = \sqrt{-t} R(\alpha s) y$  then yields  $\|\Omega\|_{L^2(B_{\bar{R}})}^2 = (-t)^{1/2} \int_{|x| \leq \bar{R}\sqrt{-t}} |\omega(x, t)|^2 dx$ , for all  $t \in (-1, 0)$ . The right-hand side is invariant under the Navier-Stokes scaling  $u \mapsto \lambda u(\lambda \cdot, \lambda^2 \cdot)$ , and its smallness over the parabolic balls  $\{|x| \leq \bar{R}\sqrt{-t}\}$  shrinking to the origin as  $t \rightarrow 0^-$  is *reminiscent of* the scale-invariant hypotheses under which the partial regularity theory of Caffarelli, Kohn and Nirenberg [10] implies that  $(0, 0)$  is a *regular point* of  $u$ .

**PROOF OF PROPOSITION 3.1.** Taking the curl of (1.8a), using the fact that when  $\nabla \cdot U = 0$  we have  $\nabla \times (JU) = -\partial_3 U$ ,  $[\nabla \times, Jy] \cdot \nabla U = -\nabla U_3$ , and  $J\Omega = \nabla U_3 - \partial_3 U$ , we deduce

$$\alpha(J\Omega - Jy \cdot \nabla \Omega) + \Omega + \frac{1}{2}y \cdot \nabla \Omega - \Delta \Omega + U \cdot \nabla \Omega = \Omega \cdot \nabla U. \quad (3.3)$$

By (2.1) we have that  $|\Omega(y)| \leq \frac{2C_{U,1}}{1+|y|^2}$ , and thus  $\Omega \in L^2(\mathbb{R}^3)$ . We may thus take the dot product of (3.3) with  $\Omega$  and integrate over  $\mathbb{R}^3$ , use the fact that  $\nabla \cdot (Jy) = 0$  and  $J\Omega \cdot \Omega = 0$ , to obtain<sup>14</sup>

$$\frac{1}{4} \|\Omega\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \Omega\|_{L^2(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} \Omega^i \partial_i U^j \Omega^j dy.$$

By (2.1), when  $|y| \geq \bar{R} = \sqrt{8C_{U,1}}$ , we have  $|\nabla U(y)| \leq 1/8$ . We obtain from the above identity that

$$\begin{aligned} \frac{1}{8} \|\Omega\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \Omega\|_{L^2(\mathbb{R}^3)}^2 &\leq \int_{|y| < \bar{R}} |\Omega|^2 |\nabla U| dy \\ &\leq \|\Omega\|_{L^2(B_{\bar{R}})} \|\Omega\|_{L^4(\mathbb{R}^3)} \|\nabla U\|_{L^4(\mathbb{R}^3)} \\ &\leq C' \|\Omega\|_{L^2(B_{\bar{R}})} \|\Omega\|_{L^4(\mathbb{R}^3)}^2 \\ &\leq C' \|\Omega\|_{L^2(B_{\bar{R}})} \|\Omega\|_{L^2(\mathbb{R}^3)}^{1/2} \|\Omega\|_{L^6(\mathbb{R}^3)}^{3/2} \\ &\leq C'(C'')^{3/2} \|\Omega\|_{L^2(B_{\bar{R}})} \|\Omega\|_{L^2(\mathbb{R}^3)}^{1/2} \|\nabla \Omega\|_{L^2(\mathbb{R}^3)}^{3/2} \end{aligned} \quad (3.4)$$

where  $C' > 0$  (the operator norm of  $\nabla \nabla \times (-\Delta)^{-1}: L^4(\mathbb{R}^3) \rightarrow L^4(\mathbb{R}^3)$ ) and  $C'' > 0$  (the constant appearing in the Gagliardo-Nirenberg-Sobolev inequality associated to  $\dot{H}^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$ ) are universal constants. The bound (3.1) now directly follows with  $C_\Omega = 6C'(C'')^{3/2}$ .

Finally, if  $\Omega \equiv 0$ , then since  $\nabla \cdot U = 0$  we have that  $U$  is a harmonic function which vanishes at infinity by (1.9); thus  $U \equiv 0$ .  $\square$

<sup>14</sup>In two space dimensions the vortex stretching term is absent from (3.3), and so we immediately deduce  $\frac{1}{2} \|\Omega\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \Omega\|_{L^2(\mathbb{R}^2)}^2 = 0$ , without any further assumption on  $|\alpha|$  as being small or large.

#### 4. The equation satisfied by the Bernoulli function

As in [46], define the head-pressure, or *Bernoulli function*, by

$$\Pi(y) := P(y) + \frac{1}{2}|U(y)|^2 + \frac{1}{2}y \cdot U(y). \quad (4.1)$$

Note that (1.9) and (2.2) imply

$$|\Pi(y)| \leq C_{P,0} + \frac{1}{2}C_{U,0}^2 + \frac{1}{2}C_{U,0}$$

for all  $y \in \mathbb{R}^3$ . Also, define

$$L := -\Delta + (U(y) + \frac{1}{2}y) \cdot \nabla \quad (4.2)$$

which is a non-normal elliptic operator without potential term.

While in the case  $\alpha = 0$  we know that  $L\Pi = -|\Omega|^2$  (see (1.5) above), as soon as  $\alpha \neq 0$  this identity fails. Instead, for general  $\alpha \in \mathbb{R}$ , if  $(U, P)$  solves (1.8) then

$$\begin{aligned} L\Pi + |\Omega|^2 &= -\Delta P - U \cdot \Delta U - |\nabla U|^2 - \frac{1}{2}y \cdot \Delta U + (U + \frac{1}{2}y) \cdot \nabla P + |\Omega|^2 \\ &\quad + ((U + \frac{1}{2}y) \cdot \nabla U) \cdot U + \frac{1}{2}(U + \frac{1}{2}y) \cdot U + \frac{1}{2}((U + \frac{1}{2}y) \cdot \nabla U) \cdot y \\ &= \text{Tr}((\nabla U)^2) - (U + \frac{1}{2}y) \cdot \Delta U - |\nabla U|^2 + |\Omega|^2 \\ &\quad - (U + \frac{1}{2}y) \cdot (\alpha JU - \alpha(Jy \cdot \nabla)U + \frac{1}{2}U - \Delta U + (U + \frac{1}{2}y) \cdot \nabla U) \\ &\quad + ((U + \frac{1}{2}y) \cdot \nabla U) \cdot U + \frac{1}{2}(U + \frac{1}{2}y) \cdot U + \frac{1}{2}((U + \frac{1}{2}y) \cdot \nabla U) \cdot y \\ &= \text{Tr}((\nabla U)^2) - |\nabla U|^2 + |\Omega|^2 - \alpha(U + \frac{1}{2}y) \cdot (JU - (Jy \cdot \nabla)U) \\ &= -\alpha(U + \frac{1}{2}y) \cdot (JU - (Jy \cdot \nabla)U) \\ &= \alpha E, \end{aligned} \quad (4.3)$$

where the *error term*  $E$  is defined as

$$E := \frac{1}{2}(y_1 U_2 - y_2 U_1) + (U(y) + \frac{1}{2}y) \cdot (y_1 \partial_2 U - y_2 \partial_1 U). \quad (4.4)$$

Note that when  $\alpha = 0$ , identity (4.3) recovers (1.5). The main properties of the error term  $E$  are:

LEMMA 4.1. *Let  $U$  be divergence-free, and define  $E$  via (4.4). Then, we have that*

$$|E(y)| \leq C_E := C_{U,0} + C_{U,1} + 2C_{U,0}C_{U,1}, \quad (4.5)$$

for all  $y \in \mathbb{R}^3$ , and also

$$E = \frac{1}{2}\partial_\theta(|U|^2 + U \cdot y), \quad (4.6)$$

where  $\partial_\theta := y_1 \partial_2 - y_2 \partial_1$  denotes the angular derivative about the  $y_3$ -axis.

The observation that  $E$  is a pure  $\partial_\theta$  derivative (4.6) is crucial for dealing with the large  $\alpha$  regime below.

PROOF OF LEMMA 4.1. The bound (1.9) and (2.1) imply

$$|E(y)| \leq C_{U,0} + C_{U,1} + \frac{2}{1+|y|^2}C_{U,0}C_{U,1}.$$

The bound (4.5) then follows. Next, a direct computation gives

$$\frac{1}{2}\partial_\theta(|U|^2) = U \cdot (y_1 \partial_2 U - y_2 \partial_1 U),$$

and

$$\begin{aligned} \frac{1}{2}\partial_\theta(U \cdot y) &= \frac{1}{2}((y_1 \partial_2 U - y_2 \partial_1 U) \cdot y + U \cdot (y_1 e_2 - y_2 e_1)) \\ &= \frac{1}{2}(y_1 U_2 - y_2 U_1) + \frac{1}{2}y \cdot (y_1 \partial_2 U - y_2 \partial_1 U). \end{aligned}$$

Adding these two identities yields

$$\frac{1}{2}\partial_\theta(|U|^2 + U \cdot y) = \frac{1}{2}(y_1 U_2 - y_2 U_1) + (U + \frac{y}{2}) \cdot (y_1 \partial_2 U - y_2 \partial_1 U) = E,$$

which is the identity claimed in (4.6).  $\square$

### 5. RSS: proof of the main result for $\alpha$ small

The goal is to prove that if  $|\alpha|$  is small enough, then (3.2) holds. We note that the arguments given in this section are new even for  $\alpha = 0$ .

The main technical ingredient is the following observation concerning the kernel of the operator

$$L^* := -\Delta - (U(y) + \tfrac{1}{2}y) \cdot \nabla - \tfrac{3}{2} \quad (5.1)$$

which is the  $L^2$  adjoint of the operator  $L$  introduced earlier in (4.2).

**PROPOSITION 5.1.** *Let  $U$  be smooth, divergence-free, which satisfies the bounds (1.9) and (2.1). Given  $\varepsilon \in (0, 1)$  there exist constants  $m = m(\varepsilon, C_{U,0}) > 0$  and  $M = M(\varepsilon, C_{U,0}) > 0$ , and a  $C^2$  smooth strictly positive function  $w: \mathbb{R}^3 \rightarrow \mathbb{R}_+$  satisfying*

$$L^*w = 0 \quad (5.2)$$

and

$$m e^{-\frac{1+\varepsilon}{4}|y|^2} \leq w(y) \leq M e^{-\frac{1-\varepsilon}{4}|y|^2}, \quad \text{for all } y \in \mathbb{R}^3. \quad (5.3)$$

Assuming that Proposition 5.1 holds, we give the proof of Theorem 1.4 for  $|\alpha| \ll 1$ .

**PROOF OF THEOREM 1.4: THE SMALL  $|\alpha|$  CASE.** We multiply (4.3) by  $w$  and integrate over  $\mathbb{R}^3$ , an operation justified by the Gaussian-type decay of the weight function  $w$  established in (5.3),<sup>15</sup> to obtain

$$\begin{aligned} \int_{\mathbb{R}^3} |\Omega|^2 w \, dy &= - \int_{\mathbb{R}^3} L \Pi w \, dy + \alpha \int_{\mathbb{R}^3} E w \, dy \\ &= - \int_{\mathbb{R}^3} \Pi L^* w \, dy + \alpha \int_{\mathbb{R}^3} E w \, dy = \alpha \int_{\mathbb{R}^3} E w \, dy. \end{aligned} \quad (5.4)$$

In the last equality we have appealed to (5.2). Then, using (4.5) and (5.3) with  $\varepsilon = \frac{1}{2}$ , we obtain

$$\begin{aligned} \|\Omega\|_{L^2(B_{\bar{R}})}^2 &\leq \frac{1}{\inf_{B_{\bar{R}}} w} \int_{B_{\bar{R}}} |\Omega|^2 w \, dy \\ &\leq m^{-1} e^{\frac{3}{8}\bar{R}^2} \cdot |\alpha| \|E\|_{L^\infty(\mathbb{R}^3)} \|w\|_{L^1(\mathbb{R}^3)} \leq |\alpha| \cdot (m^{-1} e^{\frac{3}{8}\bar{R}^2} C_E 200 M). \end{aligned}$$

Note that  $m, M, C_E, \bar{R}$  only depend on  $C_{U,0}$  and  $C_{U,1} = C_{U,1}(C_{U,0})$ , these constants are independent of  $\alpha$ . Moreover, the constant  $C_\Omega$  appearing in (3.2) also does not depend on  $\alpha$ . Therefore, letting

$$\underline{\alpha} := \frac{1}{200} m e^{-\frac{3}{8}\bar{R}^2} C_E^{-1} M^{-1} C_\Omega^{-2},$$

we obtain that for  $|\alpha| < \underline{\alpha}$  we have  $\|\Omega\|_{L^2(B_{\bar{R}})} < C_\Omega^{-1}$ . Proposition 3.1 then implies  $U \equiv 0$ .  $\square$

**REMARK 5.2** (A probabilistic interpretation). We are indebted to V. Šverák for the following remark. Since  $\nabla \cdot (U + \frac{1}{2}y) = \frac{3}{2}$ , the equation  $L^*w = 0$  may be rewritten as  $\Delta w + \nabla \cdot ((U + \frac{1}{2}y)w) = 0$ , which is the stationary Fokker-Planck equation for the diffusion  $dX_t = -(U(X_t) + \frac{1}{2}X_t) dt + \sqrt{2} dB_t$ . When  $U \equiv 0$  this is the Ornstein-Uhlenbeck process, whose invariant density is precisely the Gaussian  $\mu$  from (6.7). Proposition 5.1 states that this equilibrium survives, as a deformation, under a divergence-free drift perturbation  $U$  vanishing at infinity. The balance between the drift and the diffusion is altered significantly only on a compact set, and at infinity only through an arbitrarily small loss  $\varepsilon$  in the Gaussian rate. The existence of the invariant probability measure

<sup>15</sup>Technically speaking, to justify this computation we'd also need to have some mild decay information on  $\nabla w$ , such as  $\lim_{R \rightarrow \infty} R^{-1} \int_{R \leq |y| \leq 2R} |\nabla w| dy = 0$ . The weight  $w$  satisfies this property; in fact, we could work a bit harder in the proof of Proposition 5.1 and establish pointwise Gaussian-type upper bounds for  $|\nabla w|$ , analogously to the upper bound in (5.3). The aforementioned mild decay property, may however be obtained via the Caccioppoli inequality for  $L^*w = 0$ , which together with (5.3) yields  $\|\nabla w\|_{L^2(R \leq |y| \leq 2R)} \leq C(C_{U,0}) \|w\|_{L^2(R/2 \leq |y| \leq 4R)} \leq C'(C_{U,0}, \varepsilon) R^{\frac{3}{2}} e^{-\frac{1-\varepsilon}{16} R^2}$ , for all  $R \geq 1$ .

may also be obtained by probabilistic arguments, via the Lyapunov function  $V(y) = |y|^2$ , which satisfies  $\Delta V - (U + \frac{1}{2}y) \cdot \nabla V \leq 6 + 2C_{U,0} - |y|^2 < 0$  outside a compact set, in the sense of Khasminskii [36]; see also [4]. Here we give a self-contained construction via standard elliptic theory, to emphasize the two-sided pointwise Gaussian bounds (5.3), with constants that depend only on  $\varepsilon$  and  $C_{U,0}$ .

The remainder of this section is dedicated to the proof of Proposition 5.1, which clearly did the heavy lifting in the above argument. The argument consists of the following steps. First, along an increasing sequence of radii  $R_k \rightarrow \infty$ , we construct strictly positive smooth eigenfunctions  $w_{R_k}$  for the operator  $L^*$  on the ball  $B_{R_k}$ , with homogeneous Dirichlet boundary conditions. We show that the corresponding eigenvalue  $\lambda_{R_k}$  is non-negative and converges to zero as  $k \rightarrow \infty$ ; see Lemma 5.3. Second, we show by means of a barrier argument that after a suitable normalization, namely  $w_{R_k}(0) = 1$ , the sequence  $w_{R_k}$  satisfies the Gaussian upper bound stated in (5.3) on each  $B_{R_k}$ , uniformly as  $R_k \rightarrow \infty$ ; see Lemma 5.4. Third, we pass to the limit (in say,  $C_{loc}^2$ ) to obtain the existence of a smooth positive function  $w$  which lies in the kernel of  $L^*$  and satisfies the same Gaussian upper bound; see Lemma 5.5. Fourth, using a second barrier argument we show that  $w$  satisfies the Gaussian lower bound stated in (5.3); see Lemma 5.6. Together, these yield the proof of Proposition 5.1, which is given at the end of this section.

LEMMA 5.3 (Existence and basic properties of the principal eigenpair). *For every  $R > 0$  the homogeneous Dirichlet eigenvalue problem*

$$L^* w_R = \lambda_R w_R, \quad \text{in } B_R, \quad (5.5a)$$

$$w_R = 0, \quad \text{on } \partial B_R, \quad (5.5b)$$

*admits a principal eigenpair  $(\lambda_R, w_R)$ , normalized with  $w_R(0) = 1$ , such that:*

- (i)  $w_R \in C^\infty(B_R) \cap C(\overline{B_R})$  and  $w_R > 0$  in  $B_R$ ;
- (ii)  $\lambda_R \in \mathbb{R}$ , and every other eigenvalue  $\sigma \in \text{spec}(L^*)$  satisfies  $\text{Re}(\sigma) \geq \lambda_R$ ;
- (iii)  $\lambda_R \geq 0$ ;
- (iv) there exists  $R_* = R_*(C_{U,0}) > 0$  such that  $\lambda_R \leq (R_*/R)^2$  for every  $R \geq R_*$ .

PROOF OF LEMMA 5.3. We recall a few facts concerning the spectrum of non-symmetric elliptic operators, that sometimes goes under the umbrella name of Krein-Rutman or Donsker-Varadhan theory [3, 26, 23]. Let  $\mathcal{D} \subset \mathbb{R}^3$  be an open bounded connected domain, with  $\partial\mathcal{D}$  that is smooth. Let  $\mathcal{L} := -a^{ij}\partial_{ij} + b^i\partial_i + c$  be a uniformly elliptic operator with smooth symmetric coefficients  $a = \{a^{ij}\}$ , with smooth and bounded drift  $b = \{b^i\}$  and potential  $c$ ; assume furthermore that  $c \geq 0$  in  $\mathcal{D}$ . Consider the spectrum of  $\mathcal{L}$  with homogeneous Dirichlet boundary conditions on  $\partial\mathcal{D}$ . It is well-known (see [23, Theorem 3, § 6.5]) that  $\mathcal{L}$  admits a principal eigenvalue  $\lambda_1 \in \mathbb{R}$ , such that if  $\sigma \in \mathbb{C}$  is any other eigenvalue of  $\mathcal{L}$ , we have  $\text{Re}(\sigma) \geq \lambda_1$ . The corresponding eigenfunction  $w$  is positive within  $\mathcal{D}$ , and the eigenvalue  $\lambda_1$  is simple, so that  $w$  is uniquely selected by normalizing  $w(x_0) = 1$  for some  $x_0 \in \mathcal{D}$ . Moreover, the principal eigenfunction is smooth up to the boundary of the domain [26]. The principal eigenvalue of  $\mathcal{L}$  and its formal adjoint  $\mathcal{L}^*$  (taken with homogeneous Dirichlet boundary conditions) agree, namely  $\lambda_1(\mathcal{L}) = \lambda_1(\mathcal{L}^*)$ . Lastly, if  $\mu \in \mathbb{R}$ , then the operator  $\mathcal{L}$  and the shifted operator  $\mathcal{L}_\mu := \mathcal{L} + \mu\text{Id}$  have the same (normalized) positive principal eigenfunction, and  $\lambda_1(\mathcal{L}_\mu) = \mu + \lambda_1(\mathcal{L})$ .

The operator  $L$  defined in (4.2) and the operator  $L_\mu^*$  for  $\mu \geq 3/2$  (the shift by  $\mu$  ensures that the constant term in (5.1) is non-negative) fall under the class of operators  $\mathcal{L}$  discussed in above paragraph. In particular,  $L_\mu^*$  admits a principal eigenpair which we denote as  $(\mu + \lambda_R, w_R)$ . Note that  $L_\mu^*$  and  $L^*$  have the same *positive* principal and smooth eigenfunction, so that  $w_R > 0$  on  $B_R$ . Moreover  $\mu + \lambda_R = \lambda_1(L_\mu^*) = \mu + \lambda_1(L^*)$ . This proves part (i) of Lemma 5.3. Since  $\lambda_R := \lambda_1(L^*) = \lambda_1(L)$ , we also have that the principal eigenvalue is *real*, thereby proving part (ii) of Lemma 5.3.

Denote the principal eigenpair for the operator  $L$  on  $B_R$  (with homogeneous Dirichlet boundary conditions) by  $(\lambda_R, \varphi_R)$ . Since  $L$  has no zeroth order term, and since  $\varphi_R > 0$  in  $B_R$  is smooth and vanishes on  $\partial B_R$ , it must attain its *nonzero* maximum on  $\overline{B_R}$  at a point  $y_* \in B_R$ . At this point,  $-\Delta\varphi_R(y_*) \geq 0$  and  $\nabla\varphi_R(y_*) = 0$ , so that  $\lambda_R\varphi_R(y_*) = L\varphi_R(y_*) = -\Delta\varphi_R(y_*) \geq 0$ . This shows that  $\lambda_R \geq 0$ , thereby proving part (iii) of Lemma 5.3.

In order to prove part (iv) of Lemma 5.3, we let  $\psi(y) := R^2 - |y|^2$ , so that  $\psi > 0$  in  $B_R$  and  $\psi = 0$  on  $\partial B_R$ . Since  $\nabla\psi = -2y$ ,  $\Delta\psi = -6$ , and (1.9) holds, we have

$$L\psi = -\Delta\psi + (U + \tfrac{1}{2}y) \cdot \nabla\psi = 6 + (U + \tfrac{1}{2}y) \cdot (-2y) = 6 - 2U \cdot y - |y|^2 \leq 6 + 2C_{U,0} - |y|^2.$$

Let  $R_* = 2\sqrt{6 + 2C_{U,0}}$ , and assume that  $R \geq R_*$ . If  $R/2 \leq |y| \leq R$ , then the above inequality shows that  $L\psi(y) \leq 0$ , while  $\psi(y) > 0$  by construction. In the complementary regime  $|y| < R/2$ , we have  $L\psi(y) \leq 6 + 2C_{U,0} = R_*^2/4$ , while  $\psi(y) = R^2 - |y|^2 \geq 3R^2/4$ . Thus, we have that

$$L\psi(y) \leq R_*^2 R^{-2} \psi(y)$$

for all  $y \in B_R$ . Then, since both  $w_R$  and  $\psi$  are positive in  $B_R$  and vanish on  $\partial B_R$ , we obtain

$$\lambda_R \int_{B_R} w_R \psi \, dy = \int_{B_R} L^* w_R \psi \, dy = \int_{B_R} w_R L\psi \, dy \leq R_*^2 R^{-2} \int_{B_R} w_R \psi \, dy.$$

This concludes the proof of  $\lambda_R \leq R_*^2 R^{-2}$  for  $R \geq R_*$ , and thus of the Lemma.  $\square$

LEMMA 5.4 (Gaussian upper bound). *Let  $(\lambda_R, w_R)$  be as in Lemma 5.3, with  $R \geq R_*$ . For any  $\varepsilon \in (0, 1)$  there exists a constant  $M = M(\varepsilon, C_{U,0}) > 0$  such that*

$$w_R(y) \leq M e^{-\frac{1-\varepsilon}{4}|y|^2}$$

for all  $y \in B_R$ . In particular,  $M$  is independent of  $R$  and  $\alpha$ .

PROOF OF LEMMA 5.4. Let  $G_\varepsilon(y) := e^{-\frac{1-\varepsilon}{4}|y|^2}$ , and define  $\phi_R(y) := w_R(y)/G_\varepsilon(y)$ . Our goal is to prove  $\phi_R \leq M$  on  $B_R$ , for some  $M > 0$  independent of  $R$  and  $\alpha$ . Note that  $\phi_R = 0$  on  $\partial B_R$ .

Using the definition of  $L^*$  in (5.1) and the PDE satisfied by  $w_R$  in (5.5a), we obtain

$$\begin{aligned} & -\Delta\phi_R - (U + \tfrac{1}{2}y) \cdot \nabla\phi_R \\ &= G_\varepsilon^{-1}(-\Delta w_R - (U + \tfrac{1}{2}y) \cdot \nabla w_R) - 2\nabla w_R \cdot \nabla G_\varepsilon^{-1} - w_R(\Delta G_\varepsilon^{-1} + (U + \tfrac{1}{2}y) \cdot \nabla G_\varepsilon^{-1}) \\ &= G_\varepsilon^{-1}(L^* w_R + \tfrac{3}{2}w_R) - 2G_\varepsilon \nabla\phi_R \cdot \nabla G_\varepsilon^{-1} - 2\phi_R \nabla G_\varepsilon \cdot \nabla G_\varepsilon^{-1} + \phi_R G_\varepsilon(L^* G_\varepsilon^{-1} + \tfrac{3}{2}G_\varepsilon^{-1}) \\ &= -\nabla\phi_R \cdot \nabla \log G_\varepsilon^{-2} + \phi_R G_\varepsilon(L^* G_\varepsilon^{-1} + (3 + \lambda_R)G_\varepsilon^{-1} - 2G_\varepsilon^{-1} \nabla G_\varepsilon \cdot \nabla G_\varepsilon^{-1}). \end{aligned}$$

Using the fact that  $\nabla \log G_\varepsilon^{-2} = (1 - \varepsilon)y$  and

$$\begin{aligned} & G_\varepsilon(L^* G_\varepsilon^{-1} + (3 + \lambda_R)G_\varepsilon^{-1} - 2G_\varepsilon^{-1} \nabla G_\varepsilon \cdot \nabla G_\varepsilon^{-1}) \\ &= (\tfrac{3}{2} + \lambda_R) - \tfrac{3}{2}(1 - \varepsilon) - \tfrac{1}{4}(1 - \varepsilon)^2 |y|^2 - \tfrac{1}{2}(1 - \varepsilon)(U \cdot y + \tfrac{1}{2}|y|^2) + \tfrac{1}{2}(1 - \varepsilon)^2 |y|^2 \\ &= \tfrac{3}{2}\varepsilon + \lambda_R - \tfrac{1}{2}(1 - \varepsilon)U \cdot y - \tfrac{1}{4}\varepsilon(1 - \varepsilon)|y|^2, \end{aligned}$$

we thus obtain that

$$-\Delta\phi_R - (U + (\varepsilon - \tfrac{1}{2})y) \cdot \nabla\phi_R + \underbrace{\left(\tfrac{\varepsilon(1-\varepsilon)}{4}|y|^2 - \tfrac{3\varepsilon}{2} - \lambda_R + \tfrac{1-\varepsilon}{2}U \cdot y\right)}_{=: V_\varepsilon(y)} \phi_R = 0 \quad (5.6)$$

in  $B_R$ . By (1.9), if  $R \geq R_* = R_*(C_{U,0})$  (as defined in item (iv) of Lemma 5.3), then

$$V_\varepsilon(y) \geq \tfrac{\varepsilon(1-\varepsilon)}{4}|y|^2 - \tfrac{3\varepsilon}{2} - 1 - \tfrac{1-\varepsilon}{2}C_{U,0}.$$

Thus, for all

$$R \geq R_\varepsilon := \max \left\{ R_*, \sqrt{\tfrac{4+6\varepsilon}{\varepsilon(1-\varepsilon)} + \tfrac{2C_{U,0}}{\varepsilon}} \right\}$$

the zeroth order term multiplying  $\phi_R$  in (5.6) is positive, namely  $V_\varepsilon(y) \geq 0$ , for all  $R_\varepsilon < |y| < R$ . Thus, we can apply the weak maximum principle in the annulus  $R_\varepsilon < |y| < R$ , which together with the boundary condition  $\phi_R = 0$  on  $\partial B_R$  implies

$$\max_{R_\varepsilon \leq |y| \leq R} \phi_R = \max_{|y|=R_\varepsilon} \phi_R. \quad (5.7)$$

Therefore, we are left to bound  $\phi_R$  on  $\partial B_{R_\varepsilon}$ , where we recall that  $R_\varepsilon$  is independent of  $R$  and  $\alpha$ .

Since  $\phi_R > 0$  in  $B_{2R_\varepsilon}$ , and both the drift and the potential term of the uniformly elliptic operator acting on  $\phi_R$  in (5.6) are uniformly bounded<sup>16</sup> solely in terms of  $C_{U,0}$  and  $\varepsilon$ , by the Harnack inequality (see e.g. [26, Corollary 9.25]) we obtain that there exists  $M = M(C_{U,0}, \varepsilon) > 0$  such that

$$\sup_{B_{R_\varepsilon}} \phi_R \leq M \inf_{B_{R_\varepsilon}} \phi_R.$$

To conclude, we recall that  $w_R$  was normalized such that  $w_R(0) = 1$ . Since  $G_\varepsilon(0) = 1$ , it thus follows that  $\phi_R(0) = 1$  and so  $\inf_{B_{R_\varepsilon}} \phi_R \leq 1$ . The above Harnack estimate and the continuity of  $\phi_R$  thus imply

$$\max_{|y| \leq R_\varepsilon} \phi_R \leq M. \quad (5.8)$$

Combining (5.7) and (5.8) implies that  $\phi_R \leq M$  on  $B_R$ , with  $M = M(\varepsilon, C_{U,0})$ .  $\square$

LEMMA 5.5 (Passing  $R \rightarrow \infty$ ). *Let  $(\lambda_R, w_R)$  be as in Lemma 5.3 and fix  $\varepsilon \in (0, 1)$  and  $M = M(\varepsilon, C_{U,0})$  as in Lemma 5.4. There exists a sequence  $R_k \rightarrow \infty$  as  $k \rightarrow \infty$  such that  $w_{R_k}$  converges in  $C_{\text{loc}}^2$  to a strictly positive solution  $w$  of*

$$L^*w = 0$$

*in  $\mathbb{R}^3$ . Additionally, we have  $w(0) = 1$  and  $w$  satisfies the Gaussian upper bound*

$$w(y) \leq M e^{-\frac{1-\varepsilon}{4}|y|^2}, \quad \text{for all } y \in \mathbb{R}^3.$$

PROOF OF LEMMA 5.5. This is a standard diagonal limiting argument, which we include for completeness. Fix  $j \in \mathbb{N}$ , and let  $R \geq \max\{j+2, R_*\}$  be arbitrary. Lemma 5.4 shows that the family  $\{w_R\}_{R \geq j+2}$  is uniformly bounded in  $L^\infty(B_{j+2})$ ; the bounds do not depend on  $j$  either. Since  $R \geq R_*$ , by item (iv) of Lemma 5.3, we have  $\lambda_R \leq 1$ , and thus by (5.5a) the family  $\{L^*w_R\}_{R \geq j+2}$  is uniformly bounded in  $L^\infty(B_{j+2})$ ; the bounds do not depend on  $j$  either. The drift  $U + \frac{1}{2}y$  appearing in the definition of the operator uniformly elliptic operator  $L^*$  (recall (5.1)) is independent of  $R$  and lies in  $C^2(B_{j+2})$ ; the bounds do depend on  $j$ . By interior regularity and Schauder estimates, for each fixed  $\mu \in (0, 1)$  we may find a constant  $C_j = C_j(j, \mu, \varepsilon, C_{U,0}, C_{U,1}, C_{U,2}) > 0$  such that  $\|w_R\|_{C^{2,\mu}(B_j)} \leq C_j$ , uniformly for all  $R \geq \max\{j+2, R_*\}$ . By Arzela-Ascoli (as  $R \rightarrow \infty$ , at fixed  $j$ ) and a diagonal argument (as  $j \rightarrow \infty$ ), we may extract a diverging sequence  $\{R_k\}_{k \geq 1}$  and identify a function  $w \in C^2(\mathbb{R}^3)$  such that  $w_{R_k} \rightarrow w$  in  $C_{\text{loc}}^2(\mathbb{R}^3)$  as  $k \rightarrow \infty$ . The normalization  $w(0) = 1$  follows since  $w_{R_k}(0) = 1$  for all  $k$ . By construction, we inherit also the Gaussian upper bound and the lower bound  $w \geq 0$ .

Again, fix  $j \in \mathbb{N}$ , and let  $k$  be large enough so that  $R_k \geq \max\{j+2, R_*\}$ . By the  $C_{\text{loc}}^2$  convergence  $w_{R_k} \rightarrow w$  and the definition of the operator  $L^*$ , we have that  $L^*w_{R_k} \rightarrow L^*w$  uniformly on  $B_j$ , as  $k \rightarrow \infty$ . Moreover, by item (iv) of Lemma 5.3, we have  $\lambda_{R_k} \rightarrow 0$  as  $k \rightarrow \infty$ , and thus  $\lambda_{R_k}w_{R_k} \rightarrow 0$  uniformly on  $B_j$ , as  $k \rightarrow \infty$ . We deduce  $L^*w = 0$  on  $B_j$ ; since  $j$  was arbitrary, we deduce that  $L^*w = 0$  on  $\mathbb{R}^3$ .

We are left to show the strict positivity of  $w$ . Note that the operator  $L^* + \frac{3}{2}$  is uniformly elliptic, has smooth locally bounded coefficients, and *contains no zeroth order term*; cf. (5.1). Moreover,  $(L^* + \frac{3}{2})w = \frac{3}{2}w \geq 0$  in  $\mathbb{R}^3$ , since  $w$  is non-negative. By the strong minimum principle (see e.g. [26,

<sup>16</sup>To be precise, we have  $\|U + (\varepsilon - \frac{1}{2})y\|_{L^\infty(B_{2R_\varepsilon})} \leq C_{U,0} + R_\varepsilon$  and  $\|V_\varepsilon\|_{L^\infty(B_{2R_\varepsilon})} \leq R_\varepsilon^2 + \frac{5}{2} + \frac{1}{2}C_{U,0}$ .

Theorem 3.5]), if there would exist  $y_* \in \mathbb{R}^3$  such that  $w(y_*) = 0$ , then  $w$  would be the constant zero function, contradicting  $w(0) = 1$ .  $\square$

LEMMA 5.6 (Gaussian lower bound). *Let  $w$  be as in Lemma 5.5. For any  $\varepsilon \in (0, 1)$  there exists a constant  $m = m(\varepsilon, C_{U,0}) > 0$  such that*

$$w(y) \geq m e^{-\frac{1+\varepsilon}{4}|y|^2}, \quad \text{for all } y \in \mathbb{R}^3.$$

*In particular,  $m$  is independent of  $\alpha$ .*

PROOF OF LEMMA 5.6. For  $m \in (0, 1]$  to be determined, define the function

$$\phi(y) := m e^{-\frac{\varepsilon}{2}|y|^2} - w(y) e^{\frac{1-\varepsilon}{4}|y|^2} = e^{-\frac{\varepsilon}{2}|y|^2} (m - w(y) e^{\frac{1+\varepsilon}{4}|y|^2}).$$

The desired lower bound on  $w$  is thus equivalent to the inequality  $\phi \leq 0$  in  $\mathbb{R}^3$ . Note that the upper bound for  $w$  established in Lemma 5.5, together with  $m \leq 1$  directly implies that

$$-M \leq \phi(y) < e^{-\frac{\varepsilon}{2}|y|^2}$$

for all  $y \in \mathbb{R}^3$ , and thus  $\limsup_{|y| \rightarrow \infty} \phi(y) \leq 0$  since  $\varepsilon > 0$ .

We next verify that  $\phi$  is a sub-solution to an elliptic PDE, on the complement of a large ball. Inspired by the computations performed in the proof of Lemma 5.4 for the function  $\phi_R(y) = w_R(y) e^{\frac{1-\varepsilon}{4}|y|^2}$ , by repeating the same computations that have led to (5.6) we obtain that

$$\begin{aligned} & -\Delta \phi - \left( U + \left( \varepsilon - \frac{1}{2} \right) y \right) \cdot \nabla \phi + \left( \frac{\varepsilon(1-\varepsilon)}{4}|y|^2 - \frac{3\varepsilon}{2} + \frac{1-\varepsilon}{2} U \cdot y \right) \phi \\ &= m \left( -\Delta e^{-\frac{\varepsilon}{2}|y|^2} - \left( U + \left( \varepsilon - \frac{1}{2} \right) y \right) \cdot \nabla e^{-\frac{\varepsilon}{2}|y|^2} + \left( \frac{\varepsilon(1-\varepsilon)}{4}|y|^2 - \frac{3\varepsilon}{2} + \frac{1-\varepsilon}{2} U \cdot y \right) e^{-\frac{\varepsilon}{2}|y|^2} \right) \\ &= -m e^{-\frac{\varepsilon}{2}|y|^2} \left( \frac{\varepsilon(1+\varepsilon)}{4}|y|^2 - \frac{1+\varepsilon}{2} U \cdot y - \frac{3\varepsilon}{2} \right). \end{aligned}$$

We write the above equation as

$$-\Delta \phi - \left( U + \left( \varepsilon - \frac{1}{2} \right) y \right) \cdot \nabla \phi + V \phi = -m e^{-\frac{\varepsilon}{2}|y|^2} F, \quad (5.9)$$

where  $V(y) := \frac{\varepsilon(1-\varepsilon)}{4}|y|^2 - \frac{3\varepsilon}{2} + \frac{1-\varepsilon}{2} U \cdot y$  and  $F(y) := \frac{\varepsilon(1+\varepsilon)}{4}|y|^2 - \frac{1+\varepsilon}{2} U \cdot y - \frac{3\varepsilon}{2}$ . The main observation is that the coefficient of  $|y|^2$  in both  $V$  and  $F$  are positive when  $\varepsilon \in (0, 1)$ . In particular, due to (1.9), we have that

$$V(y) \geq 0 \quad \text{and} \quad F(y) \geq 0, \quad \text{for all } |y| \geq R'_\varepsilon, \quad (5.10)$$

where

$$R'_\varepsilon := \sqrt{\frac{6}{1-\varepsilon} + \frac{2C_{U,0}}{\varepsilon}}.$$

With  $R'_\varepsilon = R'_\varepsilon(\varepsilon, C_{U,0}) > 0$  fixed, we choose the parameter  $m$  as follows. Since  $L^*w = 0$  in  $B_{2R'_\varepsilon}$  and the coefficients of the operator  $L^*$  are uniformly bounded on  $B_{2R'_\varepsilon}$  solely in terms of  $\varepsilon$  and  $C_{U,0}$ , by the Harnack inequality we obtain that there exists  $m = m(\varepsilon, C_{U,0}) \in (0, 1]$  such that

$$\sup_{B_{R'_\varepsilon}} w \leq m^{-1} \inf_{B_{R'_\varepsilon}} w.$$

Since  $w(0) = 1$  and  $w$  is continuous, we deduce that

$$w(y) \geq m \quad \text{for all } y \in \overline{B_{R'_\varepsilon}}.$$

Therefore, by the definition of  $\phi$  we have that

$$\phi(y) = e^{-\frac{\varepsilon}{2}|y|^2} (m - w(y) e^{\frac{1+\varepsilon}{4}|y|^2}) \leq e^{-\frac{\varepsilon}{2}|y|^2} (m - w(y)) \leq 0 \quad \text{for all } y \in \overline{B_{R'_\varepsilon}}.$$

At last, we use (5.9) and (5.10) to prove that  $\phi \leq 0$  also for  $|y| > R'_\varepsilon$ . Assume by contradiction that there exists  $y_*$  with  $|y_*| > R'_\varepsilon$  such that  $\phi(y_*) > 0$ . Pick  $R$  large enough so that  $e^{-\frac{\varepsilon}{2}R^2} < \phi(y_*)$  and  $R > |y_*|$ . On the annulus  $R'_\varepsilon < |y| < R$ , (5.9)–(5.10) shows that  $\phi$  is a sub-solution of the

uniformly elliptic operator  $-\Delta - (U + (\varepsilon - \frac{1}{2})y) \cdot \nabla + V$ , and  $V \geq 0$  on this set. By the weak maximum principle and our a priori global bound for  $\phi$ , we have that

$$\phi(y_*) \leq \max_{R'_\varepsilon \leq |y| \leq R} \phi(y) = \max_{\{|y|=R'_\varepsilon\} \cup \{|y|=R\}} \phi(y) \leq e^{-\frac{\varepsilon}{2}R^2} < \phi(y_*),$$

a contradiction. This proves that  $\phi \leq 0$  on  $B_{R'_\varepsilon}^c$ , and thus concludes the proof of the Lemma.  $\square$

**PROOF OF PROPOSITION 5.1.** The existence of a  $C^2$  smooth function  $w$  lying in the kernel of  $L^*$  is established in Lemma 5.5. The upper bound stated in (5.3) is established as part of Lemma 5.5, while the lower bound stated in (5.3) is established as part of Lemma 5.6.  $\square$

## 6. RSS: proof of the main result for $\alpha$ large

The goal is to prove that if  $|\alpha|$  is large enough,  $U$  is close to the kernel of the operator  $J - Jy \cdot \nabla$ . That is,  $U$  is almost axisymmetric. If  $U$  were truly axisymmetric, then because of (4.6) we'd have  $E \equiv 0$ . Instead, when  $|\alpha| \gg 1$  we show that  $\alpha E$  is small, from which (3.2) follows, and we conclude the proof of  $U \equiv 0$  with Proposition 3.1.

**6.1. Cylindrical coordinates, rotation operator, angular mean.** We write a Cartesian point  $y = (y_1, y_2, y_3) \in \mathbb{R}^3$  in cylindrical coordinates as  $y = (r \cos \theta, r \sin \theta, z)$ , where  $r = \sqrt{y_1^2 + y_2^2} \geq 0$ ,  $z = y_3 \in \mathbb{R}$ , and  $\theta \in [0, 2\pi)$ . As usual, we denote the cylindrical coordinate frame by  $e_r = (\cos \theta, \sin \theta, 0)$ ,  $e_\theta = (-\sin \theta, \cos \theta, 0)$ , and  $e_z = (0, 0, 1)$ .<sup>17</sup> Vector fields  $V: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  are written as

$$V(y) = V^r(r, \theta, z)e_r + V^\theta(r, \theta, z)e_\theta + V^z(r, \theta, z)e_z.$$

We recall that the operator  $J$  defined in (1.6) acts on vector fields  $V = (V_1, V_2, V_3)$  as  $JV = (-V_2, V_1, 0)$ . In cylindrical coordinates, this identity is

$$JV = -V^\theta e_r + V^r e_\theta.$$

Specializing to  $V(y) = y = re_r + ze_z$ , and using that  $\nabla = e_r \partial_r + \frac{1}{r} e_\theta \partial_\theta + e_z \partial_z$ , we deduce that

$$Jy = re_\theta, \quad (Jy) \cdot \nabla = \partial_\theta.$$

In particular, for a vector field  $V: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ , we have

$$((Jy) \cdot \nabla)V = (\partial_\theta V^r - V^\theta)e_r + (\partial_\theta V^\theta + V^r)e_\theta + \partial_\theta V^z e_z.$$

We define

$$\mathcal{R}V := JV - (Jy) \cdot \nabla V, \tag{6.1}$$

to be the operator appearing in (1.8) and (3.3) with amplification coefficient  $\alpha$ . In cylindrical coordinates, the operator  $\mathcal{R}$  acts as

$$\mathcal{R}V = -\partial_\theta V^r e_r - \partial_\theta V^\theta e_\theta - \partial_\theta V^z e_z. \tag{6.2}$$

That is,  $\mathcal{R}$  acts on vector fields by applying  $-\partial_\theta$  to each cylindrical component separately.

Next, for any scalar function  $f: \mathbb{R}^3 \rightarrow \mathbb{R}$  we define

$$\langle f \rangle_\theta := \langle f \rangle_\theta(r, z) = \frac{1}{2\pi} \int_0^{2\pi} f(r, \theta, z) d\theta. \tag{6.3}$$

For a vector field  $V: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ , we accordingly define

$$\langle V \rangle_\theta := \langle V^r \rangle_\theta(r, z)e_r + \langle V^\theta \rangle_\theta(r, z)e_\theta + \langle V^z \rangle_\theta(r, z)e_z. \tag{6.4}$$

Note that a vector field  $V$  is axisymmetric *if and only if*  $V = \langle V \rangle_\theta$ . Thus, in order to measure how far a vector field is from being axisymmetric, we introduce the notation

$$(V)_a := V - \langle V \rangle_\theta = (V^r)_a(r, \theta, z)e_r + (V^\theta)_a(r, \theta, z)e_\theta + (V^z)_a(r, \theta, z)e_z, \tag{6.5}$$

<sup>17</sup>It is convenient to record the identities:  $\partial_\theta e_r = e_\theta$ ,  $\partial_\theta e_\theta = -e_r$ ,  $\partial_\theta e_z = 0$ .

where for any scalar function  $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ , we have denoted

$$(f)_a = f - \langle f \rangle_\theta. \quad (6.6)$$

With this notation  $V$  is axisymmetric *if and only if*  $(V)_a = 0$ . We will say that  $V$  is *nearly axisymmetric* if  $(V)_a \ll 1$  (in a suitable topology). We record a few useful facts.

LEMMA 6.1. *Let  $f, g: \mathbb{R}^3 \rightarrow \mathbb{R}$  be smooth scalar functions and  $V: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a smooth vector field. Then, we have:*

- (i)  $\mathcal{R}(V)_\theta = 0$  and hence  $\mathcal{R}V = \mathcal{R}(V)_a$ ; moreover,  $\langle \mathcal{R}V \rangle_\theta = 0$  and hence  $(\mathcal{R}V)_a = \mathcal{R}(V)_a$ .
- (ii)  $\langle \partial_r f \rangle_\theta = \partial_r \langle f \rangle_\theta$  and  $\langle \partial_z f \rangle_\theta = \partial_z \langle f \rangle_\theta$ ; therefore,  $(\partial_r f)_a = \partial_r (f)_a$  and  $(\partial_z f)_a = \partial_z (f)_a$ .
- (iii)  $(\partial_\theta f)_a = \partial_\theta (f)_a$  and hence  $(\nabla f)_a = \nabla (f)_a$ .
- (iv)  $(\nabla \cdot V)_a = \nabla \cdot (V)_a$  and hence  $(\Delta f)_a = \Delta (f)_a$ .
- (v)  $(fg)_a = (f)_a \langle g \rangle_\theta + \langle f \rangle_\theta (g)_a + ((f)_a (g)_a)_a$ .
- (vi)  $((V \cdot \nabla) f)_a = \langle V \rangle_\theta \cdot \nabla (f)_a + (V^r)_a \partial_r \langle f \rangle_\theta + (V^z)_a \partial_z \langle f \rangle_\theta + ((V)_a \cdot \nabla) (f)_a$ .

PROOF OF LEMMA 6.1. Since  $\mathcal{R}$  acts on vector fields by applying  $-\partial_\theta$  to each of its cylindrical components separately, item (i) follows directly from the definition (6.4).

The claims in item (ii) follow directly from the definitions (6.3) and (6.6).

Note that  $\langle \partial_\theta f \rangle_\theta = 0$ , and thus  $(\partial_\theta f)_a = \partial_\theta f$ . Since  $\partial_\theta \langle f \rangle_\theta = 0$ , it follows that  $\partial_\theta (f)_a = \partial_\theta f$ , proving the first claim in (iii). Since  $\nabla f = e_r \partial_r f + e_\theta \frac{1}{r} \partial_\theta f + e_z \partial_z f$ , from item (ii), (6.5), and the previously established identity, we deduce  $(\nabla f)_a = e_r (\partial_r f)_a + e_\theta (\frac{1}{r} \partial_\theta f)_a + e_z (\partial_z f)_a = e_r \partial_r (f)_a + e_\theta \frac{1}{r} \partial_\theta (f)_a + e_z \partial_z (f)_a = \nabla (f)_a$ .

In cylindrical coordinates we have  $\nabla \cdot V = \partial_r V^r + \frac{1}{r} V^r + \frac{1}{r} \partial_\theta V^\theta + \partial_z V^z$ , and thus  $\langle \nabla \cdot V \rangle_\theta = \langle \partial_r V^r \rangle_\theta + \langle \frac{1}{r} V^r \rangle_\theta + \langle \partial_z V^z \rangle_\theta = \partial_r \langle V^r \rangle_\theta + \frac{1}{r} \langle V^r \rangle_\theta + \partial_z \langle V^z \rangle_\theta = \nabla \cdot \langle V \rangle_\theta$ ; in the last equality we have used that  $\partial_\theta \langle f \rangle_\theta = 0$ . The first claim in item (iv) now follows from (6.5) and (6.6). The second claim follows by letting  $V = \nabla f$ , and using the second claim in item (iii).

The claim in item (v) follows from  $(fg)_a = (\langle f \rangle_\theta \langle g \rangle_\theta + (f)_a \langle g \rangle_\theta + \langle f \rangle_\theta (g)_a + (f)_a (g)_a)_a$  together with the fact that functions independent of  $\theta$  lie in the kernel of  $(\cdot)_a$  and the fact that multiplication by a function independent of  $\theta$  commutes with the operation  $(\cdot)_a$ .

Finally, item (vi) follows from the identity  $V \cdot \nabla f = V^r \partial_r f + \frac{1}{r} V^\theta \partial_\theta f + V^z \partial_z f$ , and item (v).  $\square$

**6.2. Gaussian weighted  $L^2$ .** The velocity (1.8) and vorticity (3.3) equations both contain the elliptic operator  $-\Delta + \frac{1}{2}y \cdot \nabla$ , which is not self-adjoint on  $L^2(\mathbb{R}^3)$ . In order to recover this favorable property (which will be useful in the analysis of this section), we work with the Gaussian weighted space  $L_\mu^2(\mathbb{R}^3)$ , defined as follows. Let

$$\mu = \mu(y) = e^{-\frac{1}{4}|y|^2} = e^{-\frac{1}{4}(r^2+z^2)}.$$

For smooth vector fields  $V, G: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  we define

$$\langle V, G \rangle_{L_\mu^2} := \int_{\mathbb{R}^3} V(y) \cdot G(y) \mu(y) dy = \int_{\mathbb{R}} \int_0^\infty \langle V \cdot G \rangle_\theta(r, z) \mu(r, z) 2\pi r dr dz, \quad \|V\|_{L_\mu^2}^2 = \langle V, V \rangle_{L_\mu^2}, \quad (6.7)$$

while for smooth scalar functions  $f, g: \mathbb{R}^3 \rightarrow \mathbb{R}$  we define

$$\langle f, g \rangle_{L_\mu^2} := \int_{\mathbb{R}^3} f(y) g(y) \mu(y) dy = \int_{\mathbb{R}} \int_0^\infty \langle fg \rangle_\theta(r, z) \mu(r, z) 2\pi r dr dz, \quad \|f\|_{L_\mu^2}^2 = \langle f, f \rangle_{L_\mu^2}. \quad (6.8)$$

With these definitions, we record a few useful facts.

LEMMA 6.2. *Let  $f, g: \mathbb{R}^3 \rightarrow \mathbb{R}$  be smooth scalar functions and  $V, G: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be smooth vector fields. Then we have:*

- (i) The operator  $-\Delta + \frac{1}{2}y \cdot \nabla$  is self-adjoint on  $L_\mu^2(\mathbb{R}^3)$  and  $\langle (-\Delta + \frac{1}{2}y \cdot \nabla)V, V \rangle_{L_\mu^2} = \|\nabla V\|_{L_\mu^2}^2$ .
- (ii)  $\langle \mathcal{R}V, G \rangle_{L_\mu^2} = -\langle V, \mathcal{R}G \rangle_{L_\mu^2}$  and  $\langle \mathcal{R}V, G \rangle_{L_\mu^2} = \langle \mathcal{R}V, (G)_a \rangle_{L_\mu^2}$ .

(iii)  $\langle (f)_a, \langle g \rangle_\theta \rangle_{L_\mu^2} = 0$  and  $\langle (V)_a, \langle G \rangle_\theta \rangle_{L_\mu^2} = 0$ ; hence

$$\|f\|_{L_\mu^2}^2 = \|\langle f \rangle_\theta\|_{L_\mu^2}^2 + \|(f)_a\|_{L_\mu^2}^2, \quad \text{and} \quad \|V\|_{L_\mu^2}^2 = \|\langle V \rangle_\theta\|_{L_\mu^2}^2 + \|(V)_a\|_{L_\mu^2}^2.$$

(iv) The Poincaré inequality in the angular variable  $\theta \in [0, 2\pi)$  implies that

$$\|(V)_a\|_{L_\mu^2} \leq 2\|\mathcal{R}V\|_{L_\mu^2}. \quad (6.9)$$

(v)  $\|fg\|_{L_\mu^2} \leq \|f\|_{L^\infty} \|g\|_{L_\mu^2}.$

PROOF OF LEMMA 6.2. Item (i) follows by the very definition of the Gaussian weight  $\mu$ .

By (6.2), (6.3), and integration by parts, we have that  $\langle \mathcal{R}V \cdot G \rangle_\theta = -\langle \partial_\theta V^r G^r + \partial_\theta V^\theta G^\theta + \partial_\theta V^z G^z \rangle_\theta = \langle V^r \partial_\theta G^r + V^\theta \partial_\theta G^\theta + V^z \partial_\theta G^z \rangle_\theta = -\langle V \cdot \mathcal{R}G \rangle_\theta$ . The first claim in item (ii) now follows by using definition (6.7). The second claim follows by additionally appealing to item (i) of Lemma 6.1.

By (6.8), we have that  $\langle (f)_a \langle g \rangle_\theta \rangle_\theta = \langle (f)_a \rangle_\theta \langle g \rangle_\theta = 0$ , proving the first claim in item (iii). The second claim follows by applying the first one to the  $e_r$ ,  $e_\theta$ , and  $e_z$  components of  $V$  and  $G$  separately. The claims about the orthogonal decomposition follow from (6.3)–(6.6) and the bilinearity of the inner product.

In order to prove (6.9), we appeal to (6.5), (6.6), (6.2), and the Poincaré-Wirtinger inequality in  $\theta$  (applicable since  $(f)_a$  has zero mean on the circle for any scalar function  $f$ ), to conclude that for every fixed  $(r, z) \in \mathbb{R}_+ \times \mathbb{R}$  we have

$$\begin{aligned} \langle (V)_a \cdot (V)_a \rangle_\theta &= \langle (V^r)_a^2 \rangle_\theta + \langle (V^\theta)_a^2 \rangle_\theta + \langle (V^z)_a^2 \rangle_\theta \\ &= \frac{1}{2\pi} \|(V^r)_a\|_{L^2(0, 2\pi)}^2 + \frac{1}{2\pi} \|(V^\theta)_a\|_{L^2(0, 2\pi)}^2 + \frac{1}{2\pi} \|(V^z)_a\|_{L^2(0, 2\pi)}^2 \\ &\leq \frac{2}{\pi} \|\partial_\theta (V^r)_a\|_{L^2(0, 2\pi)}^2 + \frac{2}{\pi} \|\partial_\theta (V^\theta)_a\|_{L^2(0, 2\pi)}^2 + \frac{2}{\pi} \|\partial_\theta (V^z)_a\|_{L^2(0, 2\pi)}^2 \\ &= \frac{2}{\pi} \|\partial_\theta V^r\|_{L^2(0, 2\pi)}^2 + \frac{2}{\pi} \|\partial_\theta V^\theta\|_{L^2(0, 2\pi)}^2 + \frac{2}{\pi} \|\partial_\theta V^z\|_{L^2(0, 2\pi)}^2 \\ &= 4\langle \mathcal{R}V \cdot \mathcal{R}V \rangle_\theta. \end{aligned}$$

The proof of (6.9) now follows from the above estimate and definition (6.7).

The bound in item (v) follows from the definition (6.8) and Hölder's inequality.  $\square$

**6.3. Estimates for the nonlinearity and the pressure gradient.** Inspecting the velocity equation (1.8a) we note that all terms except  $(U \cdot \nabla)U$  and  $\nabla P$  are linear in  $U$ . We estimate these terms as follows:

LEMMA 6.3. Assume that the smooth incompressible vector field  $U: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  satisfies the bounds (1.9) and (2.1). Define

$$\mathcal{N} := -(U \cdot \nabla)U - \nabla P, \quad (6.10)$$

where  $P = R_i R_j (U^i U^j)$  (as in the proof of Lemma 2.1). For any  $\varepsilon \in (0, 1]$ , there exists a constant  $C_\varepsilon = C_\varepsilon(\varepsilon, C_{U,0}) > 0$  such that

$$\|(\mathcal{N})_a\|_{L_\mu^2} \leq \varepsilon + C_\varepsilon \|(U)_a\|_{L_\mu^2} + C_\varepsilon \|\nabla(U)_a\|_{L_\mu^2}. \quad (6.11)$$

PROOF OF LEMMA 6.3. Let us first analyze the nonlinear term  $(U \cdot \nabla)U$ . Since  $U = \langle U \rangle_\theta + (U)_a$ , we may decompose

$$(U \cdot \nabla)U = (\langle U \rangle_\theta \cdot \nabla) \langle U \rangle_\theta + (\langle U \rangle_\theta \cdot \nabla) (U)_a + ((U)_a \cdot \nabla) \langle U \rangle_\theta + ((U)_a \cdot \nabla) (U)_a.$$

Note that the term  $(\langle U \rangle_\theta \cdot \nabla) \langle U \rangle_\theta$  is axisymmetric, meaning that its cylindrical components are independent of  $\theta$ .<sup>18</sup> Therefore, the projection operator  $(\cdot)_a$  annihilates this term:

$$((\langle U \rangle_\theta \cdot \nabla) \langle U \rangle_\theta)_a = 0.$$

<sup>18</sup>To see this note that  $(V \cdot \nabla)V = ((V \cdot \nabla)V^r - \frac{1}{r}(V^\theta)^2)e_r + ((V \cdot \nabla)V^\theta + \frac{1}{r}V^r V^\theta)e_\theta + ((V \cdot \nabla)V^z)e_z$ , and also  $V \cdot \nabla = V^r \partial_r + \frac{1}{r}V^\theta \partial_\theta + V^z \partial_z$ .

To estimate the remaining three terms, note that item (iii) of Lemma 6.2, gives  $\|(V)_a\|_{L_\mu^2} \leq \|V\|_{L_\mu^2}$ . Moreover, using (6.4), (1.9), (2.1), and items (ii) and (iii) of Lemma 6.1 we have that  $\|\langle U \rangle_\theta\|_{L^\infty} \leq C_{U,0}$ ,  $\|(U)_a\|_{L^\infty} \leq 2C_{U,0}$ , and  $\|\nabla \langle U \rangle_\theta\|_{L^\infty} \leq C_{U,1}$ . Using these facts and Hölder's inequality, we deduce

$$\begin{aligned} \|(\langle U \rangle_\theta \cdot \nabla)(U)_a\|_{L_\mu^2} &\leq \|(\langle U \rangle_\theta \cdot \nabla)(U)_a\|_{L_\mu^2} \leq \|\langle U \rangle_\theta\|_{L^\infty} \|\nabla(U)_a\|_{L_\mu^2} \leq C_{U,0} \|\nabla(U)_a\|_{L_\mu^2}, \\ \|((U)_a \cdot \nabla) \langle U \rangle_\theta\|_{L_\mu^2} &\leq \|((U)_a \cdot \nabla) \langle U \rangle_\theta\|_{L_\mu^2} \leq \|(U)_a\|_{L^\infty} \|\nabla \langle U \rangle_\theta\|_{L^\infty} \leq C_{U,1} \|(U)_a\|_{L_\mu^2}, \\ \|((U)_a \cdot \nabla)(U)_a\|_{L_\mu^2} &\leq \|((U)_a \cdot \nabla)(U)_a\|_{L_\mu^2} \leq \|(U)_a\|_{L^\infty} \|\nabla(U)_a\|_{L_\mu^2} \leq 2C_{U,0} \|\nabla(U)_a\|_{L_\mu^2}. \end{aligned}$$

Summing these bounds we obtain

$$\|((U \cdot \nabla)U)_a\|_{L_\mu^2} \leq C_{U,1} \|(U)_a\|_{L_\mu^2} + 3C_{U,0} \|\nabla(U)_a\|_{L_\mu^2}. \quad (6.12)$$

This estimate is clearly consistent with (6.11).

Next, we bound the pressure gradient. Note that we are seeking a bound in the Gaussian weighted  $L_\mu^2$  norm, and so we do not have access to usual Calderon–Zygmund bounds. This is compensated for by the  $\varepsilon$ -factor present on the right side of (6.11). Recall that  $-\Delta P = \partial_i \partial_j (U^i U^j)$ . Using item (iv) of Lemma 6.1, we obtain

$$-\Delta(P)_a = (\partial_i \partial_j (U^i U^j))_a = (\nabla \cdot \partial_j (U U^j))_a = \nabla \cdot (\partial_j (U U^j))_a = \nabla \cdot ((U \cdot \nabla)U)_a.$$

Since  $P$  decays (recall (2.2)) so does  $(P)_a$ , and hence  $(P)_a = (-\Delta)^{-1} \nabla \cdot ((U \cdot \nabla)U)_a$ , where  $(-\Delta)^{-1}$  denotes convolution with the Newtonian potential. Using item (iii) of Lemma 6.1, we arrive at

$$(\nabla P)_a = \nabla(P)_a = \nabla(-\Delta)^{-1} \nabla \cdot ((U \cdot \nabla)U)_a, \quad (6.13)$$

which is to say that

$$(\nabla P)_a^i = R_i R_j V^j, \quad \text{where} \quad V^j = ((U \cdot \nabla)U)_a^j,$$

and  $R_i, R_j$  are Riesz transforms. Note that we have previously (cf. (6.12)) bounded  $\|V\|_{L_\mu^2}$ , and using (6.4), (1.9), (2.1) we obtain that  $|V(y)| \leq 2C_{U,0} C_{U,1} (1 + |y|^3)^{-1}$ . Since  $\mu \leq 1$  and  $R_i R_j$  is bounded on  $L^2(\mathbb{R}^3)$ , we immediately have

$$\|R_i R_j V^j\|_{L_\mu^2} \leq \|R_i R_j V^j\|_{L^2} \leq \|V\|_{L^2}.$$

Then for  $L_\varepsilon \geq 1$ , to be determined, we have

$$\begin{aligned} \|V\|_{L^2} &\leq \|\mathbf{1}_{|y| \leq L_\varepsilon} V\|_{L^2} + \|\mathbf{1}_{|y| > L_\varepsilon} V\|_{L^2} \\ &\leq \|\mathbf{1}_{|y| \leq L_\varepsilon} \mu^{-\frac{1}{2}}\|_{L^\infty} \|\mu^{\frac{1}{2}} V\|_{L^2} + 2C_{U,0} C_{U,1} \|\mathbf{1}_{|y| > L_\varepsilon} |y|^{-3}\|_{L^2} \\ &\leq e^{\frac{1}{8} L_\varepsilon^2} \|V\|_{L_\mu^2} + 8C_{U,0} C_{U,1} L_\varepsilon^{-\frac{3}{2}}. \end{aligned} \quad (6.14)$$

Therefore, if we let

$$L_\varepsilon := \left( \frac{24C_{U,0} C_{U,1}}{\varepsilon} \right)^{\frac{2}{3}},$$

we obtain from (6.13)–(6.14), together with (6.12), that

$$\|(\nabla P)_a\|_{L_\mu^2} \leq \varepsilon + 9e^{\frac{1}{8} L_\varepsilon^2} (C_{U,1} \|(U)_a\|_{L_\mu^2} + C_{U,0} \|\nabla(U)_a\|_{L_\mu^2}). \quad (6.15)$$

Adding (6.12) and (6.15), and recalling that  $C_{U,1} = C_{U,1}(C_{U,0})$ , concludes the proof of (6.11).  $\square$

**6.4. Estimate for  $\mathcal{R}U$ .** Recall that  $U$  solves the rotated backward self-similar Navier-Stokes equations (1.8). Using the operator  $\mathcal{R}$  defined in (6.1) and the nonlinear terms  $\mathcal{N}$  defined in (6.10), the  $U$  equation may be rewritten as

$$\alpha \mathcal{R}U + \frac{1}{2}U + (-\Delta + \frac{1}{2}(y \cdot \nabla))U = \mathcal{N}, \quad (6.16a)$$

$$\nabla \cdot U = 0. \quad (6.16b)$$

The main result of this subsection is:

LEMMA 6.4. *Let  $\alpha \in \mathbb{R}$  and let  $U$  be a smooth solution of (6.16) which satisfies the bounds (1.9) and (2.1). Then, we have*

$$|\alpha| \|\mathcal{R}U\|_{L_\mu^2}^2 + \frac{1}{2} \|(U)_a\|_{L_\mu^2}^2 + \|\nabla(U)_a\|_{L_\mu^2}^2 \leq 3 \|\mathcal{R}U\|_{L_\mu^2} \|(\mathcal{N})_a\|_{L_\mu^2}. \quad (6.17)$$

PROOF OF LEMMA 6.4. We begin by noting that the operators  $\mathcal{R}$  and  $-\Delta + \frac{1}{2}y \cdot \nabla$  commute; this holds because  $\mathcal{R}$  acts as  $-\partial_\theta$  on each cylindrical component of the vector fields it acts on, and both  $-\Delta$  and  $y \cdot \nabla$  are differential operators with  $\theta$ -independent coefficients, when written in cylindrical coordinates.<sup>19</sup> Item (ii) of Lemma 6.2 shows that  $\mathcal{R}$  is skew-adjoint on  $L_\mu^2$ , whereas item (i) shows that  $-\Delta + \frac{1}{2}y \cdot \nabla$  is self-adjoint. Combined, these facts yield

$$\langle \mathcal{R}U, (-\Delta + \frac{1}{2}y \cdot \nabla)U \rangle_{L_\mu^2} = 0.$$

Since we also have  $\langle \mathcal{R}U, U \rangle_{L_\mu^2} = 0$ , taking the  $L_\mu^2$  inner product of (6.16a) with  $\text{sgn}(\alpha)\mathcal{R}U$ , and using the second identity in item (ii) of Lemma 6.2, we obtain

$$|\alpha| \|\mathcal{R}U\|_{L_\mu^2}^2 = \text{sgn}(\alpha) \langle \mathcal{R}U, \mathcal{N} \rangle_{L_\mu^2} = \text{sgn}(\alpha) \langle \mathcal{R}U, (\mathcal{N})_a \rangle_{L_\mu^2} \leq \|\mathcal{R}U\|_{L_\mu^2} \|(\mathcal{N})_a\|_{L_\mu^2}. \quad (6.18)$$

Note however that bounding  $(\mathcal{N})_a$  requires knowledge of  $\nabla(U)_a$ , which motivates our next computation.

We first apply the projection  $(\cdot)_a$  to equation (6.16a). We use Lemma 6.1 as follows: item (i) gives us  $(\mathcal{R}U)_a = \mathcal{R}(U)_a$ ; the definition of  $\Delta U$  in cylindrical coordinates (see Footnote 19) together with items (iii) and (iv) gives  $(\Delta U)_a = \Delta(U)_a$ ; and item (v) with  $V = y = re_r + ze_z$  so that  $(V)_a = (y)_a = 0$ , we obtain that  $(y \cdot \nabla U)_a = y \cdot \nabla(U)_a$ . With these identities, (6.16a) gives

$$\alpha \mathcal{R}(U)_a + \frac{1}{2}(U)_a + (-\Delta + \frac{1}{2}(y \cdot \nabla))(U)_a = (\mathcal{N})_a.$$

We take the  $L_\mu^2$  inner product of this identity with  $(U)_a$ , appeal to items (i) and (ii) in Lemma 6.2, and deduce

$$\frac{1}{2} \|(U)_a\|_{L_\mu^2}^2 + \|\nabla(U)_a\|_{L_\mu^2}^2 = \langle (\mathcal{N})_a, (U)_a \rangle_{L_\mu^2}.$$

By appealing to the angular Poincaré inequality (6.9), we thus obtain

$$\frac{1}{2} \|(U)_a\|_{L_\mu^2}^2 + \|\nabla(U)_a\|_{L_\mu^2}^2 \leq 2 \|(\mathcal{N})_a\|_{L_\mu^2} \|\mathcal{R}U\|_{L_\mu^2}. \quad (6.19)$$

Adding (6.18) and (6.19) concludes the proof.  $\square$

By combining Lemmas 6.3 and 6.4 we may show that  $\mathcal{R}U$  is small when  $|\alpha|$  is large, in the following sense:

PROPOSITION 6.5. *Let  $\alpha \in \mathbb{R}$  and let  $U$  be a smooth solution of (6.16) which satisfies the bounds (1.9) and (2.1). Let  $\varepsilon \in (0, 1]$  be arbitrary. Then, there exists a constant  $A_\varepsilon = A(\varepsilon, C_{U,0}) \geq 1$ , such that for all  $|\alpha| \geq A_\varepsilon$  we have*

$$|\alpha| \|\mathcal{R}U\|_{L_\mu^2} \leq \varepsilon. \quad (6.20)$$

<sup>19</sup>Recall that for a scalar function  $f$  we have  $\Delta f = (\partial_{rr} + \frac{1}{r}\partial_r)f + \frac{1}{r^2}\partial_{\theta\theta}f + \partial_{zz}f$ , while for a vector field  $V$  we have  $\Delta V = (\Delta V^r - \frac{1}{r^2}V^r - \frac{2}{r^2}\partial_\theta V^\theta)e_r + (\Delta V^\theta - \frac{1}{r^2}V^\theta + \frac{2}{r^2}\partial_\theta V^r)e_\theta + (\Delta V^z)e_z$ . Moreover,  $y \cdot \nabla = r\partial_r + z\partial_z$ .

PROOF OF PROPOSITION 6.5. The bounds (6.11) (with  $\varepsilon$  replaced by  $\frac{1}{6}\varepsilon$ ), (6.17), and Young's inequality show that there exists  $C_\varepsilon = C_\varepsilon(\varepsilon, C_{U,0}) > 0$ , which is in particular independent of  $\alpha$ , such that

$$\begin{aligned} & |\alpha| \|\mathcal{R}U\|_{L_\mu^2}^2 + \frac{1}{2} \|(U)_a\|_{L_\mu^2}^2 + \|\nabla(U)_a\|_{L_\mu^2}^2 \\ & \leq 3\|\mathcal{R}U\|_{L_\mu^2} \left( \frac{1}{6}\varepsilon + C_\varepsilon \|(U)_a\|_{L_\mu^2} + C_\varepsilon \|\nabla(U)_a\|_{L_\mu^2} \right) \\ & \leq \frac{1}{2}\varepsilon \|\mathcal{R}U\|_{L_\mu^2}^2 + \frac{1}{2} \|(U)_a\|_{L_\mu^2}^2 + \frac{9}{2}C_\varepsilon^2 \|\mathcal{R}U\|_{L_\mu^2}^2 + \|\nabla(U)_a\|_{L_\mu^2}^2 + \frac{9}{4}C_\varepsilon^2 \|\mathcal{R}U\|_{L_\mu^2}^2. \end{aligned}$$

Absorbing the appropriate terms into the left side of the above inequality gives

$$|\alpha| \|\mathcal{R}U\|_{L_\mu^2}^2 \leq \frac{1}{2}\varepsilon \|\mathcal{R}U\|_{L_\mu^2}^2 + \frac{27}{4}C_\varepsilon^2 \|\mathcal{R}U\|_{L_\mu^2}^2.$$

Letting  $A_\varepsilon := \frac{27}{2}C_\varepsilon^2$ , the above inequality shows that if  $|\alpha| \geq A_\varepsilon$ , then  $\frac{1}{2}|\alpha| \|\mathcal{R}U\|_{L_\mu^2}^2 \leq \frac{1}{2}\varepsilon \|\mathcal{R}U\|_{L_\mu^2}^2$ , concluding the proof.  $\square$

**6.5. Proof of the main result for RSS solutions.** We conclude this section by giving the proof of our main result, when  $|\alpha| \gg 1$ .

PROOF OF THEOREM 1.4: THE LARGE  $|\alpha|$  CASE. From (4.6) and identity (5.4), we have that

$$\int_{\mathbb{R}^3} |\Omega|^2 w \, dy = \frac{1}{2}\alpha \int_{\mathbb{R}^3} \partial_\theta(|U|^2 + U \cdot y) w \, dy,$$

where  $w$  is the positive weight constructed in Proposition 5.1, with parameter  $\varepsilon \mapsto \frac{1}{4}$ .

We first note that  $\partial_\theta|U|^2 = 2U^r \partial_\theta U^r + 2U^\theta \partial_\theta U^\theta + 2U^z \partial_\theta U^z$ . Recalling the definition of  $\mathcal{R}$  in (6.1) we deduce the pointwise-in- $y$  bound

$$|\partial_\theta|U|^2| \leq 6|U| |\mathcal{R}U|.$$

Similarly, since  $\partial_\theta(U \cdot y) = \partial_\theta(rU^r + zU^z) = r\partial_\theta U^r + z\partial_\theta U^z$ , we have the pointwise-in- $y$  bound

$$|\partial_\theta(U \cdot y)| \leq 2|y| |\mathcal{R}U|.$$

Combining the inequalities above with the  $U$  bound in (1.9), the  $w$  bound in (5.3) (since we fixed the small parameter at  $\frac{1}{4}$ , we have  $M = M(C_{U,0})$ ), and the definition of  $L_\mu^2$  in (6.7), we arrive at

$$\begin{aligned} \int_{\mathbb{R}^3} |\Omega|^2 w \, dy & \leq |\alpha| \int_{\mathbb{R}^3} (3|U| + |y|) |\mathcal{R}U| w \, dy \\ & \leq |\alpha| \int_{\mathbb{R}^3} w(3C_{U,0} + |y|) \mu^{-\frac{1}{2}} |\mathcal{R}U| \mu^{\frac{1}{2}} \, dy \\ & \leq |\alpha| \|\mathcal{R}U\|_{L_\mu^2} M \left( \int_{\mathbb{R}^3} e^{-\frac{3}{8}|y|^2} (3C_{U,0} + |y|)^2 e^{\frac{1}{4}|y|^2} \, dy \right)^{\frac{1}{2}} \\ & \leq |\alpha| \|\mathcal{R}U\|_{L_\mu^2} M(1 + 3C_{U,0}) \left( \int_{\mathbb{R}^3} e^{-\frac{1}{8}|y|^2} (1 + |y|)^2 \, dy \right)^{\frac{1}{2}}. \end{aligned} \quad (6.21)$$

One may verify that the integral term appearing in the last line of (6.21) is  $\leq 100$ . Thus, we deduce that there exists a constant  $M' = M'(C_{U,0}) > 0$  (which is in particular independent of  $\alpha$ ) such that

$$\|\Omega\sqrt{w}\|_{L^2}^2 \leq M' |\alpha| \|\mathcal{R}U\|_{L_\mu^2}^2.$$

Combining this bound with Proposition 6.5 shows that for any  $\varepsilon \in (0, 1]$ , there exists  $A_\varepsilon = A_\varepsilon(\varepsilon, C_{U,0})$  such that

$$\|\Omega\sqrt{w}\|_{L^2}^2 \leq \varepsilon M', \quad \text{whenever} \quad |\alpha| \geq A_\varepsilon.$$

The proof is now concluded in analogy to the small  $|\alpha|$  case. From (5.3), and with  $\bar{R} = \bar{R}(C_{U,0}) > 0$  as in Proposition 3.1 we deduce that there exists  $m = m(C_{U,0}) > 0$  such that

$$\|\Omega\|_{L^2(B_{\bar{R}})}^2 \leq m^{-1} e^{\frac{5}{16}\bar{R}^2} \cdot \varepsilon M'$$

whenever  $|\alpha| \geq A_\varepsilon$ . Setting

$$\varepsilon := \frac{1}{2} \min\{(M')^{-1} \cdot m e^{-\frac{5}{16}\bar{R}^2} \cdot C_\Omega^{-2}, 1\},$$

so that  $\varepsilon$  is a parameter that only depends on  $C_{U,0}$ , we obtain that (3.2) holds for all  $|\alpha| \geq A_\varepsilon$ , and thus  $U \equiv 0$  by Proposition 3.1.  $\square$

## 7. DSS and RDSS: proof of the main result for $\alpha$ small and $\lambda$ close to 1

Setting  $\alpha = 0$  in Theorem 1.7 recovers the statement of Theorem 1.6, and hence we only discuss the case of rotated discretely self-similar solutions, with small period  $S$  and small (absolute) angular speed  $|\alpha|$ . The arguments presented here closely follow those in Sections 2–5, and thus we omit details which are identical.

Without loss of generality, we let  $\underline{\alpha} \leq 1$  and  $\underline{\lambda} \leq e^{\frac{1}{2}}$ , so that  $|\alpha| \leq 1$  and  $S \leq 1$ , due to (1.13b).

**7.1. Uniform-in-time Type I bounds for higher derivatives.** We seek a statement analogous to Lemma 2.1, with bounds for  $\nabla U, \nabla^2 U$ , and  $P$ , which are *uniform-in-time* for  $s \in [0, S]$  and only depend on  $C_{U,0}$ . In the process, we also obtain bounds for  $\partial_s U$  and  $\partial_s \nabla U$ .

**LEMMA 7.1.** *Under the assumptions of Theorem 1.7 (in particular  $|\alpha| \leq 1$ ), the profile  $U(\cdot, s)$  satisfies the pointwise bound (1.9) for all  $s \in [0, S]$ , with the same constant  $C_{U,0} > 0$  from the Type I upper bound (1.10). The space-gradients of  $U$  satisfy (2.1) and the pressure  $P(\cdot, s)$  obeys estimate (2.2) for every  $s \in [0, S]$ , with the same  $(\alpha, S)$ -independent constants  $C_{U,1}, C_{U,2}, C_{P,0}$  as in Lemma 2.1. Moreover, there exists an  $(\alpha, S)$ -independent constant  $C_{U,s} = C_{U,s}(C_{U,0}) > 0$  such that*

$$|\partial_s U(y, s)| \leq \frac{C_{U,s}}{1 + |y|}, \quad |\nabla \partial_s U(y, s)| \leq \frac{C_{U,s}}{1 + |y|^2}, \quad (7.1)$$

for all  $(y, s) \in \mathbb{R}^3 \times [0, S]$ .

**PROOF OF LEMMA 7.1.** The proof is nearly identical to that of Lemma 2.1, so we only emphasize the differences. Since  $R(\alpha s) \in SO(3)$ , using (1.13a) we obtain that  $|U(y, s)| = \sqrt{-t} |u(x, t)|$  for all  $(y, s) \in \mathbb{R}^3 \times [0, S]$ . The Type I bound (1.10) then gives  $|U(y, s)| \leq C_{U,0}(1 + |y|)^{-1}$  for all  $(y, s)$ , thereby proving (1.9).

The qualitative smoothness of  $(U, P)$  and the pressure representation formula  $P = R_i R_j (U^i U^j)$  are established exactly as is Lemma 2.1.

For the bounds on  $\nabla U$  and  $\nabla^2 U$ , we run the interior-regularity argument of Lemma 2.1, making sure that the constants only depend on  $C_{U,0}$ . Fix  $(y, s) \in \mathbb{R}^3 \times [0, S]$  and set  $x_* = R(\alpha s) e^{-\frac{s}{2}} y$ ,  $t_* = -e^{-s}$ ,  $r = \frac{1}{2} e^{-\frac{s}{2}} \max\{|y|, 1\}$ . Then, the function<sup>20</sup>  $\tilde{u}(\tilde{x}, \tilde{t}) := r u(x_* + r \tilde{x}, t_* + r^2 \tilde{t})$  solves (1.1) on the unit parabolic cylinder  $Q_1 = B_1 \times (-1, 0]$ . For  $(\tilde{x}, \tilde{t}) \in Q_1$  we have  $(x, t) = (x_* + r \tilde{x}, t_* + r^2 \tilde{t})$  satisfies<sup>21</sup>  $|x| + \sqrt{-t} \geq r$ , so that (1.10) gives  $\|\tilde{u}\|_{L^\infty(Q_1)} \leq C_{U,0}$ , a bound independent of  $y, s$  and  $\alpha$ . A quantitative version of the interior regularity for Navier-Stokes [55], gives for each  $k \geq 1$  a constant  $C_k = C_k(C_{U,0})$  with  $|\nabla_{\tilde{x}}^k \tilde{u}(0, 0)| \leq C_k$ . Undoing the rescaling we have  $\nabla_x^k u(x_*, t_*) = r^{-(k+1)} \nabla_{\tilde{x}}^k \tilde{u}(0, 0)$ , and recall that by the definition of  $(x_*, t_*)$  we have  $U(y, s) = e^{-\frac{s}{2}} R(-\alpha s) u(x_*, t_*)$ . Since each  $y$ -derivative contributes a factor  $e^{-\frac{s}{2}}$ , we deduce  $|\nabla_y^k U(y, s)| = e^{-\frac{(k+1)s}{2}} |\nabla_x^k u(x_*, t_*)| =$

<sup>20</sup>Here we are using the RDSS ansatz (1.13) to extend  $u$  backwards in time from  $\mathbb{R}^3 \times [-1, 0)$  to a solution of (1.1) on  $\mathbb{R}^3 \times (-\infty, 0)$ , while retaining the Type I bound  $|u(x, t)| \leq C_{U,0}(|x| + \sqrt{-t})^{-1}$  there, with the same constant  $C_{U,0}$ .

<sup>21</sup>To see this, note that  $-t = -t_* - r^2 \tilde{t} = e^{-s} - r^2 \tilde{t} \geq e^{-s}$ , hence  $\sqrt{-t} \geq e^{-\frac{s}{2}}$ . Similarly,  $|x| = |x_* + r \tilde{x}| \geq |x_*| - r = e^{-\frac{s}{2}} |y| - r$ . If  $|y| \geq 1$  then we obtain  $|x| \geq r$ . If  $|y| < 1$  then  $\sqrt{-t} \geq e^{-\frac{s}{2}} = 2r$ . In either case  $|x| + \sqrt{-t} \geq r$ .

$e^{-\frac{(k+1)s}{2}} r^{-(k+1)} |\nabla_{\tilde{x}}^k \tilde{u}(0, 0)| \leq 2^{k+1} C_k \max\{|y|, 1\}^{-(k+1)}$ . Since  $C_k$  only depends on  $C_{U,0}$ , this establishes the desired bound

$$|\nabla_y^k U(y, s)| \leq \frac{C_{U,k}}{(1 + |y|)^{k+1}}, \quad k \geq 0, \quad (7.2)$$

for all  $(y, s) \in \mathbb{R}^3 \times [0, S]$ , uniformly in  $\alpha$ . For  $k = 1, 2$  this is the claimed estimate (2.1).

Since  $P = R_i R_j (U^i U^j)$  plus a function of time, which without loss of generality we take to vanish identically, the bounds in (7.2) and the singular-integral representation of  $R_i R_j$  imply

$$|\nabla_y^k P(y, s)| \leq \frac{C_{P,k}}{(1 + |y|)^{2-\sigma+k}}, \quad k \geq 0, \quad (7.3)$$

where  $C_{P,k}$  depends on  $C_{U,0}$ ,  $k$ , and  $\sigma > 0$ , and is hence independent of  $s$  and  $\alpha$ .

In order to prove the first bound in (7.1), we use (1.14a) to write  $\partial_s U = \Delta U - \alpha(JU - (Jy \cdot \nabla)U) - \frac{1}{2}U - \frac{1}{2}(y \cdot \nabla)U - (U \cdot \nabla)U - \nabla P$ , and bound each term on the right side using (7.2) and (7.3). The  $\alpha$  independence of the upper bound is a consequence of the standing assumption  $|\alpha| \leq 1$ . For the second bound in (7.1) we differentiate (1.14a) with respect to  $y$ , and proceed similarly.  $\square$

**7.2. The time-averaged profile and fluctuations from the mean.** We use the following notation  $\langle f \rangle_s$  for the *time-average* of a function  $f$  over the period  $[0, S]$ , and denote by  $\tilde{f}$  the associated *fluctuation* from the mean:

$$\langle f \rangle_s := \frac{1}{S} \int_0^S f(\cdot, s) ds, \quad \tilde{f} := f - \langle f \rangle_s. \quad (7.4)$$

In particular,  $\langle \tilde{f} \rangle_s = 0$ . An immediate consequence of this definition and of the bounds in (7.1) is that the fluctuation  $\tilde{U} = U - \langle U \rangle_s$  is *linearly* small in the period.

**LEMMA 7.2.** *Under the assumptions of Theorem 1.7 with  $|\alpha| \leq 1$ , let  $C_{U,s} = C_{U,s}(C_{U,0}) > 0$  be the constant from (7.1). Then, the time-fluctuation of the profile satisfies*

$$|\tilde{U}(y, s)| \leq \frac{C_{U,s} S}{1 + |y|}, \quad |\nabla \tilde{U}(y, s)| \leq \frac{C_{U,s} S}{1 + |y|^2}, \quad (7.5)$$

for all  $(y, s) \in \mathbb{R}^3 \times [0, S]$ . In particular, we have

$$\|\tilde{\Omega}(\cdot, s)\|_{L^2(\mathbb{R}^3)} \leq 10C_{U,s} S \quad \text{for every } s \in [0, S]. \quad (7.6)$$

Additionally, the time-averaged profile  $\langle U \rangle_s = \langle U \rangle_s(y)$  is smooth, divergence-free, and satisfies the bounds (1.9) and (2.1), with the same  $C_{U,0}$ -dependent constants.

**PROOF OF LEMMA 7.2.** Since  $\langle \tilde{U} \rangle_s = 0$ , we may write  $\tilde{U}(\cdot, s) = \frac{1}{S} \int_0^S (U(\cdot, s) - U(\cdot, s')) ds' = \frac{1}{S} \int_0^S \int_{s'}^s \partial_\tau U(\cdot, \tau) d\tau ds'$ , whence  $|\tilde{U}(\cdot, s)| \leq S \sup_\tau |\partial_\tau U|$ . The bounds (7.5) now follow from (7.1).

Since  $\tilde{\Omega} = \nabla \times \tilde{U}$ , we have  $\|\tilde{\Omega}(\cdot, s)\|_{L^2} \leq \sqrt{2} \|\nabla \tilde{U}(\cdot, s)\|_{L^2} \leq \sqrt{2} C_{U,s} S \|(1 + |y|^2)^{-1}\|_{L^2(\mathbb{R}^3)} \leq 10C_{U,s} S$ , which is (7.6).

Finally, the claims about  $\langle U \rangle_s$  follow directly from the definition (7.4) and from Lemma 7.1.  $\square$

**7.3. The time-dependent equation satisfied by the Bernoulli function.** We use the same definition for the head-pressure (Bernoulli function) as in (4.1), except that all fields are now time dependent:

$$\Pi(y, s) := P(y, s) + \frac{1}{2}|U(y, s)|^2 + \frac{1}{2}y \cdot U(y, s).$$

Analogously to (4.2), we denote by  $L$  the elliptic operator with time-dependent coefficients

$$L := L(s) = -\Delta + (U(y, s) + \frac{1}{2}y) \cdot \nabla,$$

and define a time-independent version associated to the time-averaged profile

$$\bar{L} := -\Delta + (\langle U \rangle_s(y) + \frac{1}{2}y) \cdot \nabla.$$

The difference between the above two elliptic operators is a pure transport derivative:

$$L - \bar{L} = \tilde{U}(y, s) \cdot \nabla.$$

With this notation, we derive a time-dependent version of (4.3), by computing  $L\Pi$ ; the resulting identity acquires exactly one new term when compared to the time-independent setting. Assume that  $(U, P)$  are smooth solutions of (1.14) which are  $[0, S]$ -periodic in time. Then, by repeating the computation in (4.3) we obtain<sup>22</sup> that

$$L\Pi + |\Omega|^2 = \alpha E - (U + \tfrac{1}{2}y) \cdot \partial_s U, \quad (7.7)$$

where  $E$  is the error term defined in (4.4). Recalling (4.6), we may write (7.7) in the more informative form

$$L\Pi + |\Omega|^2 = \tfrac{1}{2}(\alpha\partial_\theta - \partial_s)(|U|^2 + U \cdot y).$$

The most convenient form of the above identity involves replacing the operator  $L$  with its time-independent version  $\bar{L}$ , leading to

$$\bar{L}\Pi + |\Omega|^2 = -\tilde{U} \cdot \nabla\Pi + \tfrac{1}{2}(\alpha\partial_\theta - \partial_s)(|U|^2 + U \cdot y). \quad (7.8)$$

At this point the proof strategy becomes clear: make the terms on the right side of (7.8) small by either taking  $|\alpha|$  small or because by taking the period to be small, and use the left side of (7.8) to deduce a weighted  $L^2$  bound for the vorticity, where the weight is taken to lie in the kernel of  $\bar{L}^*$ .

**7.4. A pointwise-in-time local enstrophy bound.** By Lemma 7.2, the time-averaged profile  $\langle U \rangle_s$  is smooth, divergence-free, and satisfies (1.9)–(2.1). Proposition 5.1 thus applies to the operator  $\bar{L} := -\Delta + (\langle U \rangle_s + \tfrac{1}{2}y) \cdot \nabla$  and its formal adjoint operator  $\bar{L}^*$  (with respect to  $L^2(\mathbb{R}^3)$ ). With  $\varepsilon = \frac{1}{4}$ , Proposition 5.1 yields constants  $m, M > 0$  depending only on  $C_{U,0}$ , and a strictly positive  $C^2$  weight  $\bar{w}: \mathbb{R}^3 \rightarrow \mathbb{R}_+$  with

$$\bar{L}^* \bar{w} = 0, \quad m e^{-\frac{5}{16}|y|^2} \leq \bar{w}(y) \leq M e^{-\frac{3}{16}|y|^2}, \quad (7.9)$$

where  $\bar{L}^* := -\Delta - (\langle U \rangle_s + \tfrac{1}{2}y) \cdot \nabla - \frac{3}{2}$ . We stress that  $\bar{w} = \bar{w}(y)$  is independent of  $s$  and  $\alpha$ , and so are the constants  $m$  and  $M$ .

We multiply (7.8) by the weight  $\bar{w}$  (fixed as in (7.9)), and then integrate in  $y$  over  $\mathbb{R}^3$ , to obtain

$$\int_{\mathbb{R}^3} |\Omega(\cdot, s)|^2 \bar{w} dy = - \int_{\mathbb{R}^3} \tilde{U} \cdot \nabla\Pi \bar{w} dy + \frac{\alpha}{2} \int_{\mathbb{R}^3} \partial_\theta (|U|^2 + U \cdot y) \bar{w} dy - \frac{1}{2} \frac{d}{ds} \int_{\mathbb{R}^3} (|U|^2 + U \cdot y) \bar{w} dy,$$

pointwise for  $s \in [0, S]$ . Here we have crucially used that  $\bar{w}$  is independent of  $s$ .

The periodicity in  $s$  of the profile  $U$  is now used to make the last term in the above identity disappear, upon integration in  $s$ :

$$\int_0^S \int_{\mathbb{R}^3} |\Omega(\cdot, s)|^2 \bar{w} dy ds = - \int_0^S \int_{\mathbb{R}^3} \tilde{U} \cdot \nabla\Pi \bar{w} dy ds + \alpha \int_0^S \int_{\mathbb{R}^3} E \bar{w} dy ds.$$

Here we have also recalled that the error term  $E$  is given in (4.6). Using (4.5), (7.2), (7.3), (7.5), (7.9), and recalling that  $\nabla\Pi = \nabla P + \nabla U \cdot U + \frac{1}{2}U + \frac{1}{2}\nabla U \cdot y$ , we deduce from the above identity that

$$\frac{1}{S} \int_0^S \int_{\mathbb{R}^3} |\Omega(\cdot, s)|^2 \bar{w} dy ds \leq \frac{1}{S} \int_0^S \int_{\mathbb{R}^3} |\tilde{U}| |\nabla\Pi| \bar{w} dy ds + \frac{|\alpha|}{S} \int_0^S \int_{\mathbb{R}^3} |E| \bar{w} dy ds \leq C'(S + |\alpha|), \quad (7.10)$$

where

$$C' := M(C_{P,1} + C_{U,0} + C_{U,1} + C_{U,0}C_{U,1})C_{U,s} \int_{\mathbb{R}^3} \frac{1}{(1+|y|)^2} e^{-\frac{3}{16}|y|^2} dy + C_E M \int_{\mathbb{R}^3} e^{-\frac{3}{16}|y|^2} dy$$

<sup>22</sup>The only difference is that the  $\nabla P$  term carries the extra factor of  $-\partial_s U$ , which in turn causes the new term  $-(U + \frac{1}{2}y) \cdot \partial_s U$  to appear in (7.7).

is a constant that only depends on  $C_{U,0}$ ; it is independent of  $\alpha$  and  $S$ .

Restricting the space integral on the left side of (7.10) to  $B_{\bar{R}}$  (where  $\bar{R} = \sqrt{8C_{U,1}}$  is as in Proposition 3.1), and using the Gaussian lower bound on  $\bar{w}$  from (7.9), we obtain the *time-averaged* ball estimate

$$\frac{1}{S} \int_0^S \|\Omega(\cdot, s)\|_{L^2(B_{\bar{R}})}^2 ds \leq e^{\frac{5}{16}\bar{R}^2} m^{-1} \cdot \frac{1}{S} \int_0^S \int_{\mathbb{R}^3} |\Omega|^2 \bar{w} dy ds \leq C' e^{\frac{5}{16}\bar{R}^2} m^{-1} (S + |\alpha|).$$

Jensen's inequality then gives

$$\|\langle \Omega \rangle_s\|_{L^2(B_{\bar{R}})}^2 \leq \frac{1}{S} \int_0^S \|\Omega\|_{L^2(B_{\bar{R}})}^2 ds \leq C' e^{\frac{5}{16}\bar{R}^2} m^{-1} (S + |\alpha|).$$

Finally, writing  $\Omega(\cdot, s) = \langle \Omega \rangle_s + \tilde{\Omega}(\cdot, s)$  and invoking the fluctuation bound (7.6), we conclude that for *every*  $s$ ,

$$\|\Omega(\cdot, s)\|_{L^2(B_{\bar{R}})} \leq \|\langle \Omega \rangle_s\|_{L^2(B_{\bar{R}})} + \|\tilde{\Omega}(\cdot, s)\|_{L^2(\mathbb{R}^3)} \leq \sqrt{C'} e^{\frac{5}{32}\bar{R}^2} m^{-\frac{1}{2}} \sqrt{S + |\alpha|} + 10C_{U,s}S.$$

Since  $S \leq 1$  and  $|\alpha| \leq 1$ , the above estimate then gives the pointwise-in-time local enstrophy bound

$$\|\Omega(\cdot, s)\|_{L^2(B_{\bar{R}})} \leq C'' \sqrt{S + |\alpha|}, \quad (7.11)$$

for all  $s \in [0, S]$ , where

$$C'' := \sqrt{C'} e^{\frac{5}{32}\bar{R}^2} m^{-\frac{1}{2}} + 10C_{U,s}.$$

Note that  $C'$ ,  $\bar{R}$ ,  $m$ , and  $C_{U,s}$  only depend on  $C_{U,0}$ , and thus so does  $C''$ .

**7.5. Proof of the main result for RDSS solutions when  $|\alpha| + S$  is small.** Analogously to (3.3), taking the curl of (1.14a) gives the vorticity evolution, which is just (3.3) with an extra  $\partial_s \Omega$  term on the left side:

$$\partial_s \Omega + \alpha(J\Omega - Jy \cdot \nabla \Omega) + \Omega + \frac{1}{2}y \cdot \nabla \Omega - \Delta \Omega + U \cdot \nabla \Omega = \Omega \cdot \nabla U.$$

Integrating this evolution against  $\Omega$  (justified in light of (2.1)) leads to

$$\frac{1}{2} \frac{d}{ds} \|\Omega\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{4} \|\Omega\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \Omega\|_{L^2(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} \Omega^i \partial_i U^j \Omega^j dy, \quad (7.12)$$

for all  $s \in [0, S]$ . As in the proof of Proposition 3.1, by (2.1) (which holds due to Lemma 7.1) we have that  $|\nabla U(y, s)| \leq \frac{1}{8}$  for  $|y| \geq \bar{R} = \sqrt{8C_{U,1}}$  and all  $s \in [0, S]$ . Absorbing the corresponding part of the vortex-stretching integral appearing on the right side of (7.12), and estimating the remainder over  $B_{\bar{R}}$  exactly as in (3.1), pointwise for  $s \in [0, S]$  we obtain

$$\frac{1}{2} \frac{d}{ds} \|\Omega\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{8} \left( \|\Omega\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \Omega\|_{L^2(\mathbb{R}^3)}^2 \right) \leq C_\Omega \|\Omega(\cdot, s)\|_{L^2(B_{\bar{R}})} \left( \|\Omega\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \Omega\|_{L^2(\mathbb{R}^3)}^2 \right),$$

where  $C_\Omega > 0$  is a universal constant. Integrating the above inequality in  $s \in [0, S]$ , using the periodicity in time of  $\Omega$ , and appealing to the pointwise-in- $s$  bound (7.11), we arrive at

$$\int_0^S \left( \|\Omega\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \Omega\|_{L^2(\mathbb{R}^3)}^2 \right) ds \leq 8C_\Omega C'' \sqrt{S + |\alpha|} \int_0^S \left( \|\Omega\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \Omega\|_{L^2(\mathbb{R}^3)}^2 \right) ds.$$

Therefore, if  $|\alpha|, S \leq 1$  satisfy

$$2 \log(\lambda) + |\alpha| = S + |\alpha| < \frac{1}{(8C_\Omega C'')^2},$$

with  $C_\Omega$  a universal constant and  $C''$  a constant that only depends on  $C_{U,0}$ , by the previous inequality we obtain

$$\int_0^S \left( \|\Omega(\cdot, s)\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \Omega(\cdot, s)\|_{L^2(\mathbb{R}^3)}^2 \right) ds = 0.$$

Hence  $\Omega \equiv 0$  on  $\mathbb{R}^3 \times [0, S]$ . Since  $\nabla \cdot U(\cdot, s) = 0$ ,  $\nabla \times U(\cdot, s) = \Omega(\cdot, s) = 0$ , and  $U(\cdot, s) \in L^4(\mathbb{R}^3)$  for each  $s$ , we obtain that  $U \equiv 0$  on  $\mathbb{R}^3 \times [0, S]$ . This concludes the proof of Theorem 1.7 when  $|\alpha|$  is sufficiently small, and  $\lambda > 1$  is sufficiently close to 1.

## 8. DSS and RDSS: proof of the main result for $\alpha$ large and $\lambda$ close to 1

The proof of the large  $|\alpha|$  case in Theorem 1.7 follows primarily by making minor adjustments to the proof of the small  $|\alpha|$  case in Theorem 1.7, and combining these arguments with some of the components of the large  $|\alpha|$  portion of Theorem 1.4.

We remark that if  $\lambda$  is close to 1, the assumption  $1 < \lambda < \lambda^{\frac{1}{1+\alpha^2}}$  appearing in Theorem 1.7 is equivalent to the statement that the period  $S = 2 \log(\lambda)$  satisfies

$$(1 + \alpha^2)S < 2 \log \lambda \ll 1. \quad (8.1)$$

In other words, the condition on  $\lambda$  ensures that the temporal period  $S$  is sufficiently small relative to  $\alpha$ . This assumption is required for proving the following variation of Lemma 7.1, which is the main additional technical input required for the proof of the large  $|\alpha|$  case in Theorem 1.7.

LEMMA 8.1. *Under the assumptions of Theorem 1.7, the profile  $U(\cdot, s)$  satisfies the pointwise bound (1.9) for all  $s \in [0, S]$ , with the same constant  $C_{U,0} > 0$  from the Type I upper bound (1.10). The space-gradients of  $U$  satisfy (2.1) and the pressure  $P(\cdot, s)$  obeys estimate (2.2) for every  $s \in [0, S]$ , with the same  $(\alpha, S)$ -independent constants  $C_{U,1}, C_{U,2}, C_{P,0}$  as in Lemma 2.1. Moreover, there exists an  $(\alpha, S)$ -independent constant  $C_{U,s} = C_{U,s}(C_{U,0}) > 0$  such that*

$$|\partial_s U(y, s)| \leq \frac{C_{U,s} S (1 + |\alpha|)^2}{1 + |y|}, \quad |\nabla \partial_s U(y, s)| \leq \frac{C_{U,s} S (1 + |\alpha|)^2}{1 + |y|^2}, \quad (8.2)$$

for all  $(y, s) \in \mathbb{R}^3 \times [0, S]$ .

PROOF OF LEMMA 8.1. The starting point of the proof is the same as Lemma 7.1. By the same argument, one can deduce the bounds

$$|\nabla_y^k U(y, s)| \leq \frac{C_{U,k}}{(1 + |y|)^{k+1}}, \quad |\nabla_y^k P(y, s)| \leq \frac{C_{P,k}}{(1 + |y|)^{2-\sigma+k}}, \quad k \geq 0, \quad (8.3)$$

for all  $(y, s) \in \mathbb{R}^3 \times [0, S]$ , uniformly in  $\alpha$ . Using these bounds, we deduce from the equation (1.14a) the bound

$$|\partial_s \nabla^k U(y, s)| \leq \frac{C_{U,s} (1 + |\alpha|)}{1 + |y|^{k+1}}, \quad k \in \{0, 1, 2, 3\}.$$

Arguing as in Lemma 2.1, these bounds then imply

$$|\nabla^k \partial_s P(y, s)| \leq \frac{C_{U,s} (1 + |\alpha|)}{(1 + |y|)^{2-\sigma+k}}, \quad k \in \{0, 1, 2\}.$$

A-priori, these bounds are worse than (8.2) because we do not have the smallness factor  $S$  to compensate for the large  $\alpha$  parameter. However, by applying  $\partial_s$  to (1.14a) and using the above bounds, along with (8.3), we obtain

$$|\partial_s^2 U(y, s)| \leq \frac{C_{U,s} (1 + |\alpha|)^2}{1 + |y|}, \quad |\nabla \partial_s^2 U(y, s)| \leq \frac{C_{U,s} (1 + |\alpha|)^2}{1 + |y|^2}.$$

Recalling that  $U$  is  $[0, S]$ -periodic in time, Poincaré's inequality in  $s$  then gives the bound (8.2).  $\square$

The main purpose of the above lemma is to provide us with the following analogue of Lemma 6.3.

LEMMA 8.2. *Let  $u$  satisfy the conditions of Theorem 1.7 with parameters  $\alpha, \bar{\alpha}, \lambda$ , and  $\underline{\lambda}$ . Assume that the smooth incompressible vector field  $U: \mathbb{R}^3 \times [0, S] \rightarrow \mathbb{R}^3$  satisfies the bounds (8.3). Define*

$$\mathcal{N} := -(U \cdot \nabla)U - \nabla P - \partial_s U, \quad (8.4)$$

*where  $P = R_i R_j (U^i U^j)$  (as in the proof of Lemma 2.1). For any  $\varepsilon \in (0, 1]$ , there exist constants  $C_\varepsilon = C_\varepsilon(\varepsilon, C_{U,0}) > 0$  and  $\delta_\varepsilon = \delta(\varepsilon, C_{U,0}) > 0$  such that*

$$\|(\mathcal{N})_a\|_{L_\mu^2} \leq \varepsilon + C_\varepsilon \|(U)_a\|_{L_\mu^2} + C_\varepsilon \|\nabla(U)_a\|_{L_\mu^2}, \quad (8.5)$$

*holds uniformly for all  $s \in [0, S]$ , provided that  $S(1 + |\alpha|)^2 \leq \delta_\varepsilon$ .*

PROOF OF LEMMA 8.2. The estimate for  $-(U \cdot \nabla)U - \nabla P$  is identical to that in Lemma 6.3, except that we take smallness factor  $\frac{\varepsilon}{2}$  instead of  $\varepsilon$ . To deal with the remaining term  $\partial_s U$ , we simply apply Lemma 8.1 to obtain (say),  $\|(\partial_s U)_a\|_{L_\mu^2} \leq \|\partial_s U\|_{L_\mu^2} \leq C_{U,s} S(1 + |\alpha|)^2 \|(1 + |y|)^{-1}\|_{L_\mu^2} \leq \frac{\varepsilon}{2}$ .  $\square$

By arguing exactly as in the proof of Proposition 6.5, we obtain smallness of  $\alpha \mathcal{R}U$  in  $L_\mu^2$ ,

PROPOSITION 8.3. *Let  $u$  satisfy the conditions of Theorem 1.7 with parameters  $\alpha, \bar{\alpha}, \lambda$  and  $\underline{\lambda}$ . Suppose that the profiles  $(U, P)$  satisfy the bounds (8.3). For any  $\varepsilon \in (0, 1]$ , there exist constants  $A_\varepsilon = A(\varepsilon, C_{U,0}) \geq 1$  and  $\delta_\varepsilon = \delta(\varepsilon, C_{U,0})$ , such that for all  $|\alpha| \geq A_\varepsilon$ , if  $S(1 + |\alpha|)^2 \leq \delta_\varepsilon$  we have*

$$|\alpha| \|\mathcal{R}U\|_{L_\mu^2} \leq \varepsilon, \quad (8.6)$$

*uniformly for all  $s \in [0, S]$ .*

To conclude the proof of the large  $|\alpha|$  case in Theorem 1.7, one simply follows the same reasoning as in Section 7; we only sketch the minor adjustments that are needed. By applying Lemma 8.1, one obtains the bounds (7.5) and (7.6), but with the constant  $S$  replaced by  $S(1 + |\alpha|)^2$  in each instance. For a parameter  $0 < \varepsilon = \varepsilon(C_{U,0}) \ll 1$  to be fixed later (similarly to the last displayed formula in Subsection 6.5), by first choosing  $\bar{\alpha}$  to be sufficiently large to ensure  $|\alpha| \geq \bar{\alpha} \geq A_\varepsilon$  (as in Proposition 8.3), and then choosing  $\underline{\lambda}$  to be sufficiently close to 1 (in terms of  $\varepsilon$  and  $C_{U,0}$  only), allows us through condition  $1 < \lambda < \underline{\lambda}^{\frac{1}{1+\alpha^2}}$  to treat  $S(1 + |\alpha|)^2 \leq \delta_\varepsilon$  (see (8.1)) as a *smallness parameter* in the proof, instead of  $S$  itself. One then proceeds with the same estimates as in Subsection 7.4 to conclude  $\|\Omega(\cdot, s)\|_{L^2(B_{\bar{R}})}$  is small, uniformly for  $s \in [0, S]$ ; here we use both smallness parameters  $\varepsilon$  and  $S(1 + |\alpha|)^2$ . The only substantial difference is that the term involving  $\alpha E$  is treated similarly to the proof of the large  $\alpha$  part of Theorem 1.4, by applying Proposition 8.3. One then concludes that  $U$  vanishes identically by repeating the argument in Subsection 7.5.

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## References

- [1] D. Albritton, E. Brué, M. Colombo, Non-uniqueness of Leray solutions of the forced Navier-Stokes equations, *Ann. of Math. (2)* **196** (1):415–455, 2022.
- [2] J. Bedrossian, V. Vicol. *The mathematical analysis of the incompressible Euler and Navier-Stokes equations: an introduction*. Graduate Studies in Mathematics, Vol. 225. American Mathematical Society, Providence, RI, 2022.
- [3] H. Berestycki, L. Nirenberg, S.R.S. Varadhan. The principal eigenvalue and maximum principle for second-order elliptic operators in general domains. *Comm. Pure Appl. Math.* **47**: 47–92, 1994.
- [4] V.I. Bogachev, N.V. Krylov, M. Röckner, S.V. Shaposhnikov. *Fokker–Planck–Kolmogorov equations*. Mathematical Surveys and Monographs, vol. 207. American Mathematical Society, Providence, RI, 2015.
- [5] Z. Bradshaw, T.-P. Tsai. Rotationally corrected scaling invariant solutions to the Navier-Stokes equations. *Comm. Partial Differential Equations* **42** (7):1065–1087, 2017.

- [6] Z. Bradshaw, T.-P. Tsai. Forward discretely self-similar solutions of the Navier-Stokes equations II, *Ann. Henri Poincaré* **18** (3):1095–1119, 2017.
- [7] J. Bricmont, A. Kupiainen. Universality in blow-up for nonlinear heat equations. *Nonlinearity* **7** (2):539–575, 1994.
- [8] T. Buckmaster, G. Cao-Labora, J. Gómez-Serrano. Smooth imploding solutions for 3D compressible fluids. *Forum Math. Pi* **13**:Paper No. e6, 139 pp., 2025.
- [9] T. Buckmaster, S. Shkoller, V. Vicol. Formation of point shocks for 3D compressible Euler. *Comm. Pure Appl. Math.* **76** (9):2073–2191, 2023.
- [10] L. Caffarelli, R. Kohn, L. Nirenberg. Partial regularity of suitable weak solutions of the Navier-Stokes equations. *Comm. Pure Appl. Math.* **35** (6):771–831, 1982.
- [11] D. Chae, J. Wolf. Removing discretely self-similar singularities for the 3D Navier-Stokes equations. *Comm. Partial Differential Equations* **42** (9):1359–1374, 2017.
- [12] D. Chae, J. Wolf. On the Liouville type theorems for self-similar solutions to the Navier-Stokes equations. *Arch. Ration. Mech. Anal.* **225** (1):549–572, 2017.
- [13] J. Chen, T. Y. Hou. Singularity formation in 3D Euler equations with smooth initial data and boundary. *Proc. Natl. Acad. Sci. USA* **122** (27):Paper No. e2500940122, 2025.
- [14] J. Chen, S. Shkoller, V. Vicol. Smooth and stable Euler implosions. arXiv:2605.00808, 2026.
- [15] C.-C. Chen, R.M. Strain, H.-T. Yau, T.-P. Tsai. Lower bound on the blow-up rate of the axisymmetric Navier-Stokes equations. *Int. Math. Res. Not. IMRN*, Art. ID rnn016, 2008.
- [16] D. Christodoulou. Examples of naked singularity formation in the gravitational collapse of a scalar field. *Ann. of Math. (2)* **140** (3):607–653, 1994.
- [17] C. Collot, T.-E. Ghoul, N. Masmoudi. Singularities and unsteady separation for the inviscid two-dimensional Prandtl system. *Arch. Ration. Mech. Anal.* **240** (3):1349–1430, 2021.
- [18] P. Constantin, C. Foias. *Navier-Stokes equations*. University of Chicago Press, Chicago, IL, 1988.
- [19] P. Constantin, M. Ignatova, V. Vicol. On putative self-similarity for incompressible 3D Euler. arXiv:2602.17570, 2026.
- [20] A.-L. Dalibard, N. Masmoudi. Separation for the stationary Prandtl equation. *Publ. Math. Inst. Hautes Études Sci.* **130**:187–297, 2019.
- [21] T. M. Elgindi. Finite-time singularity formation for  $C^{1,\alpha}$  solutions to the incompressible Euler equations on  $\mathbb{R}^3$ . *Ann. of Math. (2)* **194** (3):647–727, 2021.
- [22] L. Escauriaza, G. Seregin, V. Šverák.  $L_{3,\infty}$ -solutions of Navier-Stokes equations and backward uniqueness. *Russian Math. Surveys* **58** (2):211–250, 2003.
- [23] L.C. Evans. *Partial differential equations*, Second edition. Graduate Studies in Mathematics, Vol. 19. American Mathematical Society, Providence, RI, 2010.
- [24] C.L. Fefferman. Existence and smoothness of the Navier-Stokes equation. In *The Millennium Prize Problems*, 57–67. Clay Math. Inst., Cambridge, MA, 2006.
- [25] Y. Giga, R. V. Kohn. Asymptotically self-similar blow-up of semilinear heat equations. *Comm. Pure Appl. Math.* **38** (3):297–319, 1985.
- [26] D. Gilbarg, N.S. Trudinger. *Elliptic partial differential equations of second order*. Grundlehren der mathematischen Wissenschaften, Vol. 224. Springer-Verlag, Berlin, Heidelberg, New York, 1998.
- [27] J. Guillod, V. Šverák. Numerical investigations of non-uniqueness for the Navier-Stokes initial value problem in borderline spaces. *J. Math. Fluid Mech.* **25** (3):Paper No. 46, 2023.
- [28] J. Guillod, P. Wittwer. Generalized scale-invariant solutions to the two-dimensional stationary Navier-Stokes equations. *SIAM J. Math. Anal.* **47** (1):955–968, 2015.
- [29] Y. Guo, M. Hadžić, and J. Jang. Larson–Penston self-similar gravitational collapse. *Comm. Math. Phys.* **386** (3):1551–1601, 2021.
- [30] M.A. Herrero, J.J.L. Velázquez. Blow-up behaviour of one-dimensional semilinear parabolic equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **10** (2):131–189, 1993.
- [31] M.A. Herrero, J.J.L. Velázquez. Singularity patterns in a chemotaxis model. *Math. Ann.* **306** (3):583–623, 1996.
- [32] M.A. Herrero, E. Medina, J.J.L. Velázquez. Self-similar blow-up for a reaction-diffusion system. *J. Comput. Appl. Math.* **97** (1-2):99–119, 1998.
- [33] H. Jia, V. Šverák. Local-in-space estimates near initial time for weak solutions of the Navier-Stokes equations and forward self-similar solutions. *Invent. Math.* **196** (1):233–265, 2014.
- [34] H. Jia, V. Šverák. Are the incompressible 3d Navier-Stokes equations locally ill-posed in the natural energy space? *J. Funct. Anal.* **268** (12):3734–3766, 2015.
- [35] H. Jia, V. Šverák, T.-P. Tsai. Self similar solutions to the nonstationary Navier-Stokes equations. *Handbook of Mathematical Analysis in Mechanics of Viscous Fluids*:461–507. Y. Giga, A. Novotný (eds.), Springer, 2018.
- [36] R. Khasminskii. *Stochastic stability of differential equations*. Stochastic Modelling and Applied Probability, vol. 66. Springer, Heidelberg, second edition, 2012.

- [37] M. Korobkov, T.-P. Tsai. Forward self-similar solutions of the Navier-Stokes equations in the half space. *Anal. PDE* **9** (8):1811–1827, 2016.
- [38] I. Kukavica. Partial regularity results for solutions of the Navier-Stokes system. *Partial Differential Equations and Fluid Mechanics*:121–145. JC. Robinson, JL. Rodrigo (eds.), London Mathematical Society Lecture Note Series. Cambridge University Press, 2009.
- [39] P.G. Lemarié-Rieusset. *Recent developments in the Navier-Stokes problem*. Chapman & Hall/CRC Research Notes in Mathematics, 431. Boca Raton, FL, 2002. xiv+395 pp.
- [40] J. Leray. Sur le mouvement d’un liquide visqueux emplissant l’espace. *Acta Math.* **63** (1):193–248, 1934.
- [41] N. Masmoudi, H. Zaag. Blow-up profile for the complex Ginzburg–Landau equation. *J. Funct. Anal.* **255** (7):1613–1666, 2008.
- [42] F. Merle, P. Raphaël. On universality of blow-up profile for  $L^2$  critical nonlinear Schrödinger equation. *Invent. Math.* **156** (3):565–672, 2004.
- [43] F. Merle, P. Raphaël, I. Rodnianski, J. Szeftel. On the implosion of a compressible fluid I: Smooth self-similar inviscid profiles. *Ann. of Math. (2)* **196** (2):567–778, 2022.
- [44] F. Merle, P. Raphaël, I. Rodnianski, J. Szeftel. On the implosion of a compressible fluid II: Singularity formation. *Ann. of Math. (2)* **196** (2):779–889, 2022.
- [45] F. Merle, H. Zaag. Stability of the blow-up profile for equations of the type  $u_t = \Delta u + |u|^{p-1}u$ . *Duke Math. J.* **86** (1):143–195, 1997.
- [46] J. Nečas, M. Růžička, V. Šverák. On Leray’s self-similar solutions of the Navier-Stokes equations. *Acta Math.* **176**:283–294, 1996.
- [47] G. Perelman. The entropy formula for the Ricci flow and its geometric applications. arXiv:math/0211159, 2002.
- [48] P. Plecháč, V. Šverák. On self-similar singular solutions of the complex Ginzburg–Landau equation. *Comm. Pure Appl. Math.* **54** (10):1215–1242, 2001.
- [49] P. Raphaël, I. Rodnianski. Stable blow up dynamics for the critical co-rotational wave maps and equivariant Yang–Mills problems. *Publ. Math. Inst. Hautes Études Sci.* **115**:1–122, 2012.
- [50] P. Raphaël, R. Schweyer. Stable blowup dynamics for the 1-corotational energy critical harmonic heat flow. *Comm. Pure Appl. Math.* **66** (3):414–480, 2013.
- [51] G. Seregin. Remarks on Type II blowups of solutions to the Navier-Stokes equations. arXiv:2304.04045, 2023.
- [52] G. Seregin. A note on potential Type II blowups of axisymmetric solutions to the Navier-Stokes equations. arXiv:2402.13229, 2024.
- [53] G. Seregin. On potential Type II blowups for the Navier-Stokes equations. arXiv:2606.29468, 2026.
- [54] G. Seregin, V. Šverák. On type I singularities of the local axis-symmetric solutions of the Navier-Stokes equations. *Comm. Partial Differential Equations* **34** (2):171–201, 2009.
- [55] J. Serrin. On the interior regularity of weak solutions of the Navier-Stokes equations. *Arch. Rational Mech. Anal.* **9**:187–195, 1962.
- [56] T.-P. Tsai. On Leray’s self-similar solutions of the Navier-Stokes equations satisfying local energy estimates. *Arch. Rational Mech. Anal.* **143** (1):29–51, 1998.
- [57] T.-P. Tsai. Forward discretely self-similar solutions of the Navier-Stokes equations. *Comm. Math. Phys.* **328** (1):29–44, 2014.
- [58] T.-P. Tsai. *Lectures on Navier-Stokes equations*. Graduate Studies in Mathematics, 192. American Mathematical Society, Providence, RI, 2018. xii+224 pp.

COURANT INSTITUTE SCHOOL OF MATHEMATICS, COMPUTING, AND DATA SCIENCE, NEW YORK UNIVERSITY,  
NEW YORK, NY, 10012

Email address: [brp305@nyu.edu](mailto:brp305@nyu.edu)

COURANT INSTITUTE SCHOOL OF MATHEMATICS, COMPUTING, AND DATA SCIENCE, NEW YORK UNIVERSITY,  
NEW YORK, NY, 10012

Email address: [vicol@cims.nyu.edu](mailto:vicol@cims.nyu.edu)