Formation and development of singularities for the compressible Euler equations

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Abstract

In this paper we review the authors' recent work [4] which gives a complete description of the formation and development of singularities for the compressible Euler equations in two space dimensions, under azimuthal symmetry. This solves an open problem posed by Landau and Lifshitz, which was previously open even in one space dimension. Our proof applies mutatis mutandis in the drastically simpler situations of one-dimensional flows, or multi-dimensional flows with radial symmetry. We prove that for smooth and generic initial data with azimuthal symmetry, the 2D compressible Euler equations yield a local in time smooth solution, which in finite time forms a first gradient singularity, the so-called $C^{1/3}$ *pre-shock*. We then show that a discontinuous entropy producing *shock wave* instantaneously develops from the pre-shock. Simultaneous to the development of the shock, two other characteristic surfaces of higher-order cusp-type singularities emerge from the pre-shock. These surfaces have been termed *weak discontinuities* by Landau and Lifshitz [17, Chapter IX, §96], who conjectured their existence. We prove that along the characteristic surface moving with the fluid, a *weak contact discontinuity* is formed, while along the slowest surface in the problem, a *weak rarefaction wave* emerges. The constructed solution is the *unique* solution of the Euler equations in a certain class of entropy-producing weak solutions with azimuthal symmetry and with regularity determined by the fact that it arises from a generic pre-shock.

Contents

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1 Introduction

The compressible Euler equations are the fundamental mathematical model of fluid dynamics. Their mathematical analysis has a very rich history, see for instance, the classical books of Courant and Friedrichs [10], or Landau and Lifshitz [17]. The unknowns of the model are the velocity $u : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$, the mass density $\rho : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}_+$, the total energy $E : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}_+$, where $d \geqslant 1$ is the spatial dimension. The quasilinear system of conservation laws describing their evolution is given by

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u + pI) = 0, \tag{1.1a}$$

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{1.1b}$$

$$\partial_t E + \operatorname{div}((p + E)u) = 0, \tag{1.1c}$$

and represent the conservation of momentum, mass, and energy. Here $p : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}_+$ is the pressure which may be computed in terms of (u, ρ, E) as

$$p = (\gamma - 1)\left(E - \tfrac{1}{2}\rho|u|^2\right), \tag{1.1d}$$

where $\gamma > 1$ denotes the adiabatic exponent. The pressure may alternatively be computed in terms of the (specific) entropy $S : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ via

$$p(\rho, S) = \tfrac{1}{\gamma}\rho^\gamma e^S. \tag{1.2}$$

Note that in regions of spacetime where the fields (u, ρ, E) are smooth, one may replace (1.1c) by the transport of specific entropy

$$\partial_t S + u \cdot \nabla S = 0. \tag{1.3}$$

The system (1.1) is supplemented with *smooth* Cauchy data (u_0, ρ_0, E_0).

At least since the middle of the 19th century and the work of Riemann [21], it is known that the compressible Euler equations exhibit solutions which have smooth initial data and develop a *finite time singularity*. The nonlinear interactions in (1.1) cause a gradual steepening of the density and velocity profiles, eventually leading to a first spacetime point at which their slope becomes infinite (the *pre-shock*). A shock wave then forms and propagates through the fluid according to the so-called Rankine-Hugoniot jump conditions, which ensure that the evolution gives an entropy-producing weak solution of (1.1).

A rigorous mathematical understanding of the above described process of *shock formation* and *shock development*, from smooth initial data, is partially available only in one space dimension [10, 11, 17], or equivalently, in the presence of radial symmetry for $d \geqslant 2$. We emphasize however that even for $d = 1$ a complete understanding of these phenomena was not available as of 2019. Indeed, regarding the 1D shock formation process, a rigorous proof of the expectation (see Eggers and Fontelos [13]) that the first singularity is asymptotically self-similar, and a stability analysis of the associated self-similar profiles within the Euler evolution (1.1), was unavailable. This issue was settled in our work [1]. Regarding the shock development process, Landau and Lifshitz note in [17, Chapter IX, §96] that simultaneously to the development of the discontinuous shock wave, other surfaces of higher-order singularities are expected to form. Landau and Lifschitz termed these surfaces *weak discontinuities*, but stop short of describing their nature: "*The irregularity may be of various kinds. For example, the first spatial derivatives of ρ, p, u etc. may be discontinuous on a surface, or these derivatives may become infinite or higher derivatives may behave in the same manner.*" In spite of the huge literature on compressible flows, we are not aware of any analysis of these weak discontinuities for the Euler system (1.1). Providing a resolution to the problem raised by Landau and Lifschitz is the purpose of our work [4].

We emphasize that the arguments in our works [1] and [4] are able to treat not just the case $d = 1$, or $d \geqslant 2$ with radial symmetry, but a more general situation: $d = 2$ for flows with *azimuthal symmetry* and nonzero vorticity. We view the analysis of solutions with azimuthal symmetry as a key step in our program of understanding shock formation and development for the full Euler system (1.1) in multiple space dimensions ($d \geqslant 2$), from smooth initial data, in the absence of any symmetry assumptions, which is considered to be *the* outstanding open problem in the field.

2 Prior results for Euler shock formation and development

The mathematical literature on the compressible Euler equations is too vast to review here. The majority of results have been focused on either the one-dimensional problem, or on the theory of weak solutions, or on the Riemann problem. See for instance the book of Dafermos [11] for an extensive modern review. In spite of this, there are very few results devoted to the mathematical analysis of shock formation for smooth initial data, and even less so to the shock development problem.

For the one-dimensional p-system (which models 1D isentropic Euler), Lebaud [18] was the first to prove shock formation and development. Chen and Dong [5], and also Kong [16], revisited the proof of Lebaud and established the formation and development of shocks for the 1D p-system with slightly more general initial data. However, as explained in Remark 3.3 below, the use an isentropic system cannot produce weak solutions to the Euler equations, even for $d = 1$. The first work to address the formation and development problem for the non-isentropic Euler equations was Yin [22], who considered the 3×3 system under spherical symmetry (which makes the problem one-dimensional). Independently of Yin, shock development for the barotropic Euler equations under spherical symmetry was established by Christodoulou and Lisbach [8]. Since isentropic dynamics cannot yield weak solutions to the Euler equations (see Remark 3.3), the analysis in [8] has been termed the *restricted shock development*. Christodoulou [7] has established restricted shock development for *irrotational and isentropic* 3D Euler equations, outside of symmetry assumptions. We note however that besides the inability of the isentropic model to capture the correct shock jump conditions, outside of radial symmetry the usage of an irrotational model can also not be justified; regular shock solutions produce entropy and generically create vorticity (see Remark 4.1 below).

As noted above, Landau and Lifshitz conjectured in [17, Chapter IX, §96] that at the same time that the discontinuous shock wave develops, other surfaces of weak singularities are expected to simultaneously form. For the full Euler system (1.1), with or without symmetry, even in one space dimension, the analysis of these surfaces of weak singularities has been heretofore nonexistent. In [4] we have proven that for the Euler equations in azimuthal symmetry, two such surfaces emerge from the pre-shock and move with the slower sound-speed characteristic ($\mathfrak{s}_1$), and respectively with the fluid velocity ($\mathfrak{s}_2$). We shall refer to these the $\mathfrak{s}_2$ surface as a *weak contact* because it moves with the fluid velocity, and both the normal velocity and the pressure are one degree smoother than the density and entropy across this surface. The shall also refer to $\mathfrak{s}_1$ as a *weak rarefaction* because the normal velocity to this curve is decreasing in the direction of its motion.

The precise analysis of the shock development problem in [4] is made possible by a very detailed understanding of the pre-shock which arises from smooth generic initial data. In multiple space dimensions and in the absence of symmetries, such a comprehensive description of the first singularity is currently unavailable. The constructive proofs of shock formation by Christodoulou [6], Christodoulou and Miao [9], and by Luk and Speck [19, 20] yield the existence of at least one point in spacetime where a shock must form, and a bound is given for this blow up time; however, since the construction of the shock solution is a perturbation of a simple plane wave, there are numerous possibilities for the type of singularities that actually form; the blow up could potentially occur at one point, at multiple points, on a curve, or along a surface. The first step towards the precise characterization of the pre-shock in three space dimensions, without symmetries and for the full Euler equations, has been obtained recently by the first and last two authors [2, 3]. We prove in [2, 3] that the first singularity which arises from smooth and non-degenerate initial data develops at a single point in spacetime, it forms in an asymptotically self-similar way, and the corresponding similarity profiles are stable. This first singularity has been termed a *point-shock*, and it is given by the intersection of the pre-shock surface with the time slice $\{t = T_1\}$, where T_1 is the first time a gradient blowup occurs.

3 Classical vs Regular shock solutions

Given a sufficiently smooth initial datum (u_0, ρ_0, E_0) defined on $\mathbb{R}^d \times \{T_0\}$, the existence of a unique local in time smooth solution to the Euler system (1.1) defined $\mathbb{R}^d \times [T_0, T_0 + \delta)$ for some $\delta > 0$, is classical. For a proof, see for instance the H^s energy estimates of Kato [14]. This solution may be continued uniquely on a maximal time interval $[T_0, T_1)$, characterized by the fact that T_1 is the first time at which the solution has an infinite gradient. Thus, there is no ambiguity in the notion of solution to (1.1) on $\mathbb{R}^d \times [T_0, T_1)$ since all the fields are differentiable in space and time, and so the solution is *classical*. The evolution on the time interval $[T_0, T_1)$ is called *shock formation*, leading to a first singularity at time T_1, the so-called *pre-shock*,[1] which we shall prove is generically of cusp-type, with the solution retaining Hölder $1/3$ regularity.

The evolution (1.1) may be continued past the time of the first singularity, say on an interval $(T_1, T_2]$, in what is known as *shock development*. The pre-shock instantaneously evolves into a discontinuous entropy producing shock wave, and we shall prove that in addition two other families of weak characteristic singularities simultaneously emerge from the pre-shock. In order to discuss shock development, we first need a suitable notion of solution to (1.1) on $\mathbb{R}^d \times (T_1, T_2]$, which in turn requires the introduction of the *Rankine-Hugoniot jump conditions* and of the *entropy condition*.

The Rankine-Hugoniot jump conditions are a manifestation of the fact (u, ρ, E) is a weak solution of (1.1), and thus the shock speed is related to the jumps of various quantities across the shock surface. More precisely, suppose that the shock front $\mathcal{S} \subset \mathbb{R}^d \times (T_1, T_2]$ is an orientable spacetime hypersurface across which the velocity, density, and energy jump. For $t \in (T_1, T_2]$ the shock front at time t locally separates space into two sets $\Omega^\pm(t)$, and we denote the values of the fields in these sets by $(u^\pm, \rho^\pm, E^\pm)$. We consider the case where this surface is parametrized as $\mathcal{S} := \{\mathfrak{s}(x,t) = 0\}$, and denote the spacetime normal to this surface as $-(\nabla_x \mathfrak{s}, \partial_t \mathfrak{s})|_{\mathcal{S}} =: (\mathfrak{n}, -\dot{\mathfrak{s}})$. We let $\mathfrak{n}(\cdot, t)$ point from $\Omega^-(t)$ to $\Omega^+(t)$, which is the direction of propagation of the shock front. We denote by $\dot{\mathfrak{s}}$ the shock speed, while the *jump* of a quantity f across the shock is written as $[\![f]\!] = f^- - f^+$, where $f^\pm$ are the traces of f along $\mathcal{S}$ in the regions $\Omega^\pm$. Let $u_\mathfrak{n} = u \cdot \mathfrak{n}|\mathfrak{n}|^{-1}$ be the projection of the velocity field in the direction of the normal vector $\mathfrak{n}$. The tangential components of the velocity are continuous across the shock, i.e. $[\![u - u_\mathfrak{n}\mathfrak{n}|\mathfrak{n}|^{-1}]\!] = 0$. The Rankine-Hugoniot jump conditions state that

$$\dot{\mathfrak{s}}|\mathfrak{n}|^{-1}[\![\rho u_\mathfrak{n}]\!] = [\![\rho u_\mathfrak{n}^2 + pI]\!], \tag{3.1a}$$

$$\dot{\mathfrak{s}}|\mathfrak{n}|^{-1}[\![\rho]\!] = [\![\rho u_\mathfrak{n}]\!], \tag{3.1b}$$

$$\dot{\mathfrak{s}}|\mathfrak{n}|^{-1}[\![E]\!] = [\![(p + E) u_\mathfrak{n}]\!]. \tag{3.1c}$$

Note that only one of the equations in (3.1) are used to compute the shock speed, while the remaining equations yield two constraints for the variables $(u_\mathfrak{n}^+, \rho^+, E^+)|_\mathcal{S}$ and $(u_\mathfrak{n}^-, \rho^-, E^-)|_\mathcal{S}$.

The entropy condition is nothing but the second law of thermodynamics, and states that the entropy ρS, which in view of (1.3) satisfies the conservation law $\partial_t(\rho S) + \nabla \cdot (\rho u S) = 0$ as long as the solution is smooth, must *increase* in the presence of a shock singularity. With the above choice of orientation of the

[1] To be precise, this first singularity is called a *pre-shock* only for one-dimensional problems, or in the presence of azimuthal symmetry, discussed here. For $d \geqslant 2$ in the absence of any symmetry, this first singularity occurs at a single point in space-time, the *point-shock*. The point-shock is the intersection of the pre-shock with the time slice $\{t = T_1\}$.

normal vector $\mathfrak{n}$ the mass flux $j = \rho(u_{\mathfrak{n}} - \dot{\mathfrak{s}}|\mathfrak{n}|^{-1})$ is negative, mass is passing across the shock from $\Omega^+(t)$ into $\Omega^-(t)$, and so the physical entropy condition becomes

$$[\![S]\!] > 0\,. \tag{3.2}$$

Remark 3.1 (The physical entropy condition and the geometric Lax entropy conditions). The negativity of the mass flux $j = \rho(u_{\mathfrak{n}} - \dot{\mathfrak{s}}|\mathfrak{n}|^{-1})$ immediately gives

$$u^- \cdot \mathfrak{n} < \dot{\mathfrak{s}}, \qquad u^+ \cdot \mathfrak{n} < \dot{\mathfrak{s}}. \tag{3.3}$$

The *Lax geometric entropy conditions* are given by (3.3) along with

$$u^+ \cdot \mathfrak{n} + c^+ < \dot{\mathfrak{s}} < u^- \cdot \mathfrak{n} + c^-\,, \tag{3.4}$$

where c^- and c^+ are the sound speeds behind and in front of the shock. Condition (3.4) states that the shock discontinuity is supersonic relative to the state in front (the '$+$' phase) and subsonic relative to the state behind (the '$-$' phase) the shock. It turns out that for an ideal gas, and under the assumption that (u, ρ, E) has a *weak shock*, i.e.

$$\sup_{t \in [T_1, T_2]} |[\![u(t)]\!]| + |[\![\rho(t)]\!]| + |[\![E(t)]\!]| \ll 1\,,$$

the physical entropy condition (3.2) is *equivalent* to the Lax geometric entropy conditions. Moreover, in this setting one may show that the Rankine-Hugoniot jump conditions imply

$$[\![S]\!] = \mathcal{O}([\![p]\!]^3), \tag{3.5}$$

with a positive pre-factor; it follows that the entropy production postulated in (3.2) implies the positivity of the jumps $[\![p]\!] > 0$, $[\![\rho]\!] > 0$ and $[\![u_{\mathfrak{n}}]\!] > 0$. See Landau and Lifshitz [17, Chapter IX] or [4, Section 2] for details.

Having defined the Rankine-Hugoniot conditions (3.1) and the entropy condition (3.2) we are now ready to define the physically relevant notion of solution to the development problem for (1.1), evolving from the pre-shock data.

Definition 3.2 (Regular shock solution). *We say that* (u, ρ, E) *and a shock front* $\mathcal{S}$ *is a* regular shock solution *on* $\mathbb{R}^d \times [T_1, T_2]$ *if the following conditions hold:*

- (u, ρ, E) *is a weak solution of* (1.1) *and* $\rho \geqslant \rho_{\min} > 0$*;*
- *the shock front* $\mathcal{S} \subset \mathbb{R}^d \times [T_1, T_2]$ *is an orientable co-dimension* 1 *hypersurface;*
- (u, ρ, E) *are Lipschitz continuous in space and time on the complement of the shock surface* $(\mathbb{R}^d \times [T_1, T_2]) \backslash \mathcal{S}$*;*
- (u, ρ, E) *have discontinuities across the shock which satisfy the Rankine-Hugoniot jump conditions* (3.1)*;*
- *entropy is produced at the shock, so that* (3.2) *holds.*

Remark 3.3 (Regular shock solutions cannot be isentropic). Definition 3.2 shows that one *cannot* study the physical shock development problem within the isentropic Euler model ($S \equiv 0$). Indeed, while the isentropic Euler system is perfectly justifiable prior to the first singularity since $S|_{t=T_0} = 0$ implies by (1.3) that $S(\cdot, t) = 0$ for all $t \in [T_0, T_1]$, as soon as a shock front develops entropy must be generated according to (3.5). That is, the flow becomes non-isentropic in order to satisfy the Rankine-Hugoniot jump conditions, or equivalently, in order for (u, ρ, E) to be a weak solution of the Euler system (1.1). Consistency with the production of entropy (3.2) is a secondary condition, which is meant to rule out the physically incorrect weak solutions.

4 Azimuthal symmetry

It regions of spacetime where the fields (u, ρ, E) are differentiable, the divergence form of the Euler equations (1.1) is equivalent to a more symmetric version, in which the conservation of the energy is replaced by the transport of specific entropy S, and the conservation of mass is replaced by the evolution of the rescaled sound speed σ, defined as

$$\sigma = \tfrac{1}{\alpha}\sqrt{\partial p/\partial \rho} = \tfrac{1}{\alpha} e^{\frac{S}{2}} \rho^{\alpha}, \qquad \text{where} \qquad \alpha = \tfrac{\gamma-1}{2}. \tag{4.1}$$

With this notation, the ideal gas equation of state (1.2) becomes $p = \frac{\alpha^2}{\gamma}\rho\sigma^2$, while the Euler equations (1.1), as a system for (u, σ, S), are given by

$$\partial_t u + (u\cdot\nabla)u + \alpha\sigma\nabla\sigma = \tfrac{\alpha}{2\gamma}\sigma^2\nabla S, \tag{4.2a}$$

$$\partial_t \sigma + (u\cdot\nabla)\sigma + \alpha\sigma \operatorname{div} u = 0, \tag{4.2b}$$

$$\partial_t S + (u\cdot\nabla)S = 0. \tag{4.2c}$$

Note that the system (4.2) is valid away from the shock surface, and that the Rankine-Hugoniot conditions need to be determined from the conservation law form of the Euler equations (1.1). Additionally, we note that the Rankine-Hugoniot jump conditions defined in terms of the jumps of normal velocity, density, and energy (3.1), may be translated into jump conditions for the variables (u, σ, E), by appealing to (4.1) and $E = \frac{1}{2}\rho|u|^2 + \frac{\alpha}{2\gamma}\rho\sigma^2$.

A fundamental quantity to the analysis of (4.2) is the vorticity, defined as $\omega = \nabla^\perp \cdot u$ for $d = 2$ and $\omega = \nabla \times u$ for $d = 3$. Then, the *specific vorticity* $\zeta = \frac{\omega}{\rho}$ solves

$$\partial_t \zeta + (u\cdot\nabla)\zeta = \begin{cases} \frac{\alpha}{\gamma}\frac{\sigma}{\rho}\nabla^\perp\sigma\cdot\nabla S, & d=2, \\ (\zeta\cdot\nabla u) + \frac{\alpha}{\gamma}\frac{\sigma}{\rho}\nabla\sigma\times\nabla S, & d=3, \end{cases} \tag{4.3}$$

and the analysis of (4.3) is of fundamental importance to our works [1–4].

Remark 4.1 (Regular shock solutions generically create vorticity). The baroclinic torque term on the right side of (4.3) shows that a misalignment of density and entropy gradients creates vorticity. Combining this observation with Remark 3.3, it is thus expected that even when one starts the shock formation process with isentropic irrotational flow, as soon as the shock surface is formed, *generically* not just entropy is created, but vorticity is created as well. Thus, for generic smooth initial data the shock development problem cannot be studied in the class of irrotational flows. The only two exceptions we are aware of are $d = 1$ or the conceptually equivalent sittuation $d \geqslant 2$ under the reduction of *radial symmetry*, when there is no vorticity to speak of in the first place.

The above remark motivates our introduction of the class of solutions to the Euler equations with *azimuthal symmetry*. This class of solutions may be defined for $d = 2$ by the requirement that the velocity and sound speed are linear functions of r with nonlinear dependence of (θ, t), while the entropy is only a function of (θ, t). Here (r, θ) are the polar coordinates on $\mathbb{R}^2$. This class of solutions is formally maintained under the Euler evolution (1.1). These solutions have nonzero vorticity, both velocity components are nontrivial and strongly affect the shock formation and development, and the system has three distinct wave-speed families. As such, we view azimuthal symmetry as a multi-dimensional intermediary case between one-dimensional problems, and multi-dimensional problems without any symmetry. More precisely, by introducing the unknowns (a, b, c, k) via

$$(u_r, u_\theta, \sigma, S)(r, \theta, t) =: (ra(\theta, t), rb(\theta, t), rc(\theta, t), k(\theta, t)), \tag{4.4}$$

and cancelling all powers of r, the Euler system (4.2) becomes

$$(\partial_t + b\partial_\theta)\, a + a^2 - b^2 + \alpha c^2 = 0, \tag{4.5a}$$
$$(\partial_t + b\partial_\theta)\, b + \alpha c \partial_\theta c + 2ab = \tfrac{\alpha}{2\gamma} c^2 \partial_\theta k, \tag{4.5b}$$
$$(\partial_t + b\partial_\theta)\, c + \alpha c \partial_\theta b + \gamma a c = 0, \tag{4.5c}$$
$$(\partial_t + b\partial_\theta)\, k = 0. \tag{4.5d}$$

For smooth initial data (u_0, ρ_0, E_0) or (u_0, σ_0, S_0) at $t = T_0$ which has azimuthal symmetry, one may define via (4.4) suitable initial data (a_0, b_0, c_0, k_0) for the system (4.5). Then, solving (4.5) gives a unique solution (a, b, c, k) on a maximal time interval $[T_0, T_1)$ on which the solution remains smooth. On this time interval, the unique solution (u, σ, S) to (4.2) is then given by the identification (4.4). That is, as long as solutions remain smooth, the azimuthal symmetry of the data is preserved, and systems (1.1), (4.4), and (4.5) are all equivalent. As we shall see below, we may in fact continue the solution (a, b, c, k) of (4.5) past $t = T_1$ in a ***unique*** way as a physical shock solution by translating the Rankine-Hugoniot jump conditions (3.1) and the entropy condition (3.2) into corresponding azimuthal jump/entropy conditions. The resulting solution (u, σ, S) (or equivalently (u, ρ, E)) obtained via the identification (4.4) can be shown to be a regular weak solution of the full Euler system (4.4) (equivalently (1.1)) in the sense of Definition 3.2. The uniqueness of this regular weak solution to (1.1) is only known to hold if we assume that the solution has azimuthal symmetry.

4.1 Riemann-like variables in azimuthal symmetry

For simplicity of presentation, for the remainder of this review, as was done in [4], we shall work with the adiabatic exponent

$$\gamma = 2, \qquad \text{or equivalently} \qquad \alpha = \tfrac{1}{2}. \tag{4.6}$$

We also note that it is convenient to rescale time, letting

$$t = \tfrac{3}{4}\widetilde{t}, \qquad \text{so that} \qquad \partial_t \mapsto \tfrac{4}{3}\partial_{\widetilde{t}}, \tag{4.7}$$

and for notational simplicity, we continue to write t for $\widetilde{t}$. More importantly, it is convenient for the subsequent analysis to work with Riemann-like variables w and z which symmetrize (in a certain sense) the b and c evolutions (4.5). These Riemann variables are defined by

$$w = b + c, \qquad z = b - c, \tag{4.8}$$

so that $b = \frac{1}{2}(w + z)$ and $c = \frac{1}{2}(w - z)$. We shall refer to w as the *dominant Riemann variable*, and to z as the *subdominant Riemann variable*.

With the adiabatic exponent from (4.6), the temporal rescaling (4.7), and using the Riemann variables from (4.8), the system (4.5) can be equivalently written as

$$\partial_t w + \lambda_3 \partial_\theta w = -\tfrac{8}{3} a w + \tfrac{1}{24}(w - z)^2 \partial_\theta k, \tag{4.9a}$$
$$\partial_t z + \lambda_1 \partial_\theta z = -\tfrac{8}{3} a z + \tfrac{1}{24}(w - z)^2 \partial_\theta k, \tag{4.9b}$$
$$\partial_t k + \lambda_2 \partial_\theta k = 0, \tag{4.9c}$$
$$\partial_t a + \lambda_2 \partial_\theta a = -\tfrac{4}{3} a^2 + \tfrac{1}{3}(w + z)^2 - \tfrac{1}{6}(w - z)^2. \tag{4.9d}$$

where the three distinct wave speeds are given by

$$\lambda_1 = \tfrac{1}{3} w + z\,, \qquad \lambda_2 = \tfrac{2}{3} w + \tfrac{2}{3} z\,, \qquad \lambda_3 = w + \tfrac{1}{3} z\,. \tag{4.10}$$

The Cauchy problem for (4.9) is considered with initial conditions given by

$$(w_0, z_0, a_0, k_0)(\theta) = (w, z, a, k)(\theta, T_0).$$

We shall henceforth refer to (4.9)–(4.10) as the *azimuthal Euler system*.

Remark 4.2 (Specific vorticity in azimuthal symmetry). Using the azimuthal symmetry ansatz (4.4), the specific vorticity ζ may be written as

$$\zeta(r, \theta, t) = \varpi(\theta, t) = \left(4(w + z - \partial_\theta a)c^{-2}e^k\right)(\theta, t)\,, \tag{4.11}$$

and we may show that it solves the evolution equation

$$\partial_t \varpi + \lambda_2 \partial_\theta \varpi = \tfrac{8}{3} a \varpi + \tfrac{4}{3} e^k \partial_\theta k\,. \tag{4.12}$$

Remark 4.3 (Motivation for the choice of γ in (4.6)). The choice of adiabatic exponent $\gamma = 2$ was made in order to emphasize that the shock wave produces not just entropy, but it also generates the subdominant Riemann variable z. In order to clearly emphasize this, for the shock formation process we choose initial data at time $t = T_0$ which satisfies

$$k(\theta, T_0) = 0, \qquad \text{and} \qquad z(\theta, T_0) = 0. \tag{4.13}$$

The entropy transport (4.9c) ensures that for any $t \in [T_0, T_1]$, where T_1 is the time of the first singularity, we have $k(\cdot, t) = 0$. The Rankine-Hugoniot conditions (cf. (4.16) below) guarantee that entropy *must be produced* at the shock, resulting in $k(\theta, t) > 0$ in a certain region of points $(\theta, t) \in \mathbb{T} \times (T_1, T_2]$. The choice of $k_0 = 0$ in (4.13) emphasizes the production of entropy in the clearest possible way. The choice $\gamma = 2$ ($\alpha = \frac{1}{2}$) is related to the evolution of the subdominant Riemann variable z. Since we have that $k \equiv 0$, the right side of (4.9b) simplifies to $-\frac{8}{3}az$, but we note that for general values of γ, this term would simplify to $-\frac{3+2\alpha}{1+\alpha}az - \frac{1-2\alpha}{1+\alpha}aw$. As such, even if $z_0 = 0$, the term $-\frac{1-2\alpha}{1+\alpha}aw$ would ensure that $z \not\equiv 0$ for $t > T_0$. For $\alpha = \frac{1}{2}$ this term however does not exist, and so the choice of $k_0 = 0$ in (4.13) ensures that $z(\cdot, t) = 0$ for all $t \in [T_0, T_1]$. The remarkable fact is that the Rankine-Hugoniot conditions (cf. (4.16) below) imply that we must have $z < 0$ for a certain region of points $(\theta, t) \in \mathbb{T} \times (T_1, T_2]$. Thus, the choice $z_0 = 0$ is made in order to most clearly emphasize the breaking of the symmetry $b = c$ at the shock.

As noted in Remark 4.3, the choice of initial datum in (4.13) implies that during the shock formation process, we have that $k \equiv 0$ and $z \equiv 0$, so that the system (4.9) becomes

$$\partial_t w + w \partial_\theta w = -\tfrac{8}{3} a w, \tag{4.14a}$$

$$\partial_t a + \tfrac{2}{3} w \partial_\theta a = -\tfrac{4}{3} a^2 + \tfrac{1}{6} w^2. \tag{4.14b}$$

The pre-shock, which will be shown to be smooth away from a unique blowup point $\theta_* \in \mathbb{T}$, inherits the property that $k(\theta, T_1)$ and $z(\theta, T_1)$ vanish on $\mathbb{T}$, but these symmetries are broken instantaneously during the shock development process. The presence of a shock necessitates that we supplement the system (4.9) with Rankine-Hugoniot jump and entropy conditions.

4.2 Rankine-Hugoniot jump and entropy conditions

In azimuthal symmetry, with the adiabatic exponent from (4.6) and the temporal rescaling (4.7), the shock hypersurface is given as

$$\mathcal{S} = \{(r, \theta, t) \colon \mathfrak{s}(t) - \theta = 0\}.$$

The spatial normal to this hypersurface is $\mathfrak{n} = \frac{1}{r}\vec{e}_\theta$. We have that $\dot{\mathfrak{s}} > 0$ and so the shock is moving from left to right when the angular variable θ is viewed as being defined on $[-\pi, \pi)$. To see this, note that since $z = 0$ by (4.8) we have that $w = 2c$, and since we wish to stay away from vacuum we must have $c \geqslant c_{\min} > 0$ on $\mathbb{T}$; therefore, w is strictly positive on $\mathbb{T}$, which implies that the three wave speeds defined in (4.10) are all strictly positive, and ordered as $\lambda_1 < \lambda_2 < \lambda_3$ on $\mathbb{T} \times [T_0, T_1]$ (by continuity this also holds on $\mathbb{T} \times (T_1, T_2]$ if $T_2 - T_1 \ll 1$). The negativity of the mass flux in (3.3) then yields $\dot{\mathfrak{s}} > 0$. According to the orientation of $\mathfrak{n}$, we denote by $(w_+, z_+, a_+, k_+)(t)$ the limiting values on the shock curve $\mathfrak{s}(t)$ from the right (or front) of the shock, and by $(w_-, z_-, a_-, k_-)(t)$ the limiting values from the left (or back) of the shock. As discussed in [4, Remark 2.5], the Lax geometric entropy inequalities (3.3)–(3.4) imply that the characteristics of the three wave speeds $\{\lambda_i\}_{i=1}^3$ in front of the shock (the '+' phase) impinge on the shock front, carrying with them the data from the $\{t = T_1\}$ Cauchy hypersurface. In particular, since $k(\cdot, T_1) = z(\cdot, T_1) = 0$, this implies that during the development process we have

$$k_+(t) = z_+(t) = 0, \qquad \text{for all} \qquad t \in (T_1, T_2], \tag{4.15}$$

so that $[\![k]\!] = k_-$ and $[\![z]\!] = z_-$. Using (4.15) and the observation that $u_{\mathfrak{n}}^{\pm} = rb_{\pm}(\mathfrak{s}(t), t)$ the Rankine-Hugoniot jump conditions (3.1) may be shown to be equivalent to a system of two equations which are used to determine the values of z_- and k_- in terms of w_+ and w_-

$$\begin{aligned} &(e^{k_-} - 1)(w_- - z_-)^4 \Big(3w_+^2 e^{k_-} - (w_- - z_-)^2\Big) \\ &\qquad = \Big((w_- - z_-)^2 - e^{k_-} w_+^2\Big)^3, \end{aligned} \tag{4.16a}$$

$$\begin{aligned} &\Big((w_- - z_-)^2(w_- + z_-)^2 + \tfrac{1}{8}(w_- - z_-)^4 - \tfrac{9}{8}e^{k_-} w_+^4\Big)\Big((w_- - z_-)^2 - e^{k_-} w_+^2\Big) \\ &\qquad = \Big((w_- - z_-)^2(w_- + z_-) - e^{k_-} w_+^3\Big)^2, \end{aligned} \tag{4.16b}$$

and an evolution equation for $\dot{\mathfrak{s}}$ given by

$$\dot{\mathfrak{s}}(t) = \frac{2}{3}\frac{e^{-k_-}(w_- - z_-)^2(w_- + z_-) - w_+^3}{e^{-k_-}(w_- - z_-)^2 - w_+^2}. \tag{4.16c}$$

To summarize, the values of the dominant Riemann variable, w_+ in the front and w_- in the back of the shock, determine the values of z_- and k_- via (4.16a)–(4.16b), which in turn allows one to compute the location of the evolving shock front. We note that the dominant Riemann variable w travels according to the fastest wave-speed in the system (4.9), namely λ_3. Thus, the values of w_+ and w_- are carried from the $\{t = T_1\}$ Cauchy hypersurface via the characteristics of λ_3, which impinge on the shock front from the left and right.

Remark 4.4 (The entropy condition in azimuthal symmetry). The system of three equations (4.16) is in one to one correspondence with the Rankine-Hugoniot jump conditions (3.1). So the natural question is: what is the correspondent of the physical entropy condition (3.2) in azimuthal symmetry? To answer this question we first note that (4.16a)–(4.16b) are a coupled system of sixth order polynomials in the variables w_+, w_-, z_-, e^{k_-}. The second observation is that at the pre-shock we have $w_+(T_1) = w_-(T_1)$ and $z_-(T_1) = k_-(T_1) = 0$, which solves (4.16a)–(4.16b). The natural question then is whether in the weak shock regime $0 < [\![w]\!] = w_- - w_+ \ll 1$, with $\langle\!\langle w \rangle\!\rangle = \frac{1}{2}(w_- + w_+) > 0$, the system (4.16a)–(4.16b) has an *unique* solution or not. Being sixth order equations with real coefficients, the presence of one real solution implies the presence of at least one more solution. Indeed, one may verify that in the weak shock regime the system (4.16a)–(4.16b) has exactly two real solutions with $|z_-| + |k_-| \ll 1$, the other roots being complex. The remarkable fact is that only one of these two solutions is entropy producing: $k_- > 0$. Thus, the role of the physical entropy condition (3.2), which is equivalent in view of (4.15) to $k_- > 0$, is to *select the unique physically relevant root* of the system of equations (4.16a)–(4.16b).

We conclude this section by revisiting the notion of a regular shock solution, as defined in Definition 3.2, in the context of the azimuthal Euler equations. During the *formation part* of our result, i.e. for $t \in [T_0, T_1)$, we have that the solution (w, z, k, a) of (4.9)–(4.10) is smooth, so that the notion of solution is the classical one: the system (4.9) is satisfied in the sense of C^1 functions of space and time. On the time interval $[T_1, T_2]$, which covers the *development part* of our result, the notion of *regular shock solution* becomes:

Definition 4.5 (Regular azimuthal shock solution). *We say that (w, z, k, a) and a shock front parametrized as $\mathcal{S} = \{\mathfrak{s}(t) = \theta\}$ is a* regular azimuthal shock solution *on $\mathbb{T} \times [T_1, T_2]$ if*

- *(w, z, k, a) are $C^1_{\theta,t}$ smooth, and ϖ is $C^0_{\theta,t}$ smooth, on the complement of $\mathcal{S}$;*
- *on the complement of the shock curve (w, z, k, a) solve the equations (4.9)–(4.10) pointwise, and ϖ solves (4.12) pointwise;*
- *(w, z, k) have jump discontinuities across the shock curve which satisfy the algebraic equations (4.16a)–(4.16b);*
- *the shock location $\mathfrak{s} : [T_1, T_2] \to \mathbb{T}$ is C^1_t smooth and solves (4.16c);*
- *entropy is produced at the shock so that $[\![k]\!](t) > 0$ for $t \in (T_1, T_2]$.*

5 Main results

The the main result of [4] is stated first in terms of the azimuthal variables (w, z, k, a). The result may be best visualized by inspecting Figures 1, 2, 3, 4. A condensed statement is as follows; for details, see [4, Theorems 3.2, 5.5, 6.1].

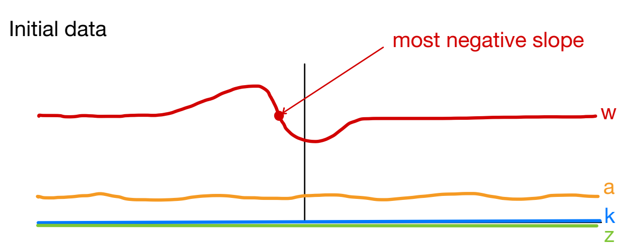

Figure 1: The initial conditions $(w, z, k, a)|_{t=T_0}$ satisfying (4.13) are represented in (red, green, blue, orange) as functions of the angular variable $\theta \in [-\pi, \pi)$. The function $w(\cdot, T_0)$ is strictly positive and has has a non-degenerate most negative slope of size $\approx -\frac{1}{\varepsilon}$ at a unique point in $\mathbb{T}$. The function $a(\cdot, T_0)$ is $\mathcal{O}(1)$ in $C^4(\mathbb{T})$.

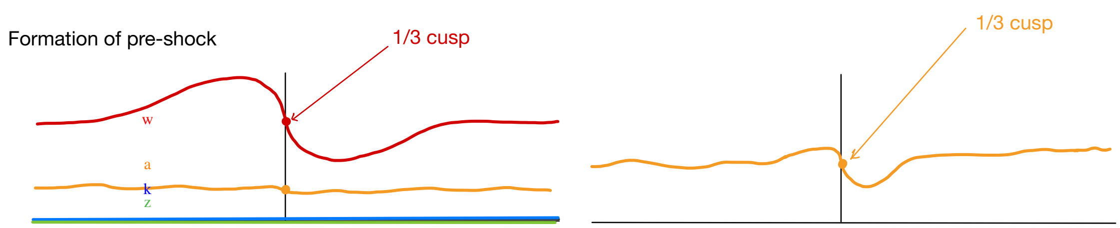

Figure 2: At the time of the first singularity, the functions $(w, z, k, a)|_{t=T_1}$ are sketched in the figure on the left, using the same color scheme as in Figure 1. In the image on the right we have plotted the function $\partial_\theta a$, which also develops a singularity at $t = T_1$. More precisely, the shock formation process for the system (4.14) results in the formation of the pre-shock at time T_1, manifested as a $C^{\frac{1}{3}}$ cusp at a unique distinguished angle $\theta_* \in \mathbb{T}$ for the functions w and $\partial_\theta a$. At T_1 we have that z and k remain equal to 0.

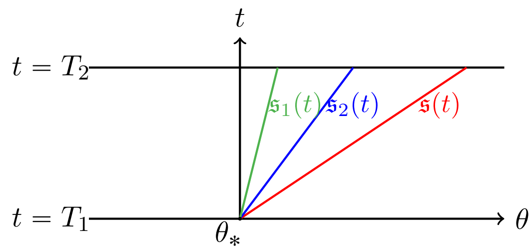

Figure 3: Three distinct families of singularities instantaneously emerge from the the pre-shock located at (θ_*, T_1). Across the classical shock curve $\mathfrak{s}$ the fields $(w, z, k, \partial_\theta a)$ jump, and the Rankine-Hugoniot conditions are satisfied. A weak rarefaction singularity develops across the curve $\mathfrak{s}_2$ which travels along characteristics of λ_2. Here the quantities (w, z, k) have regularity $C^{1,1/2}$ and no better. A weak contact singularity forms across the curve $\mathfrak{s}_1$ which travels with the characteristics of λ_1. Here the function z has regularity $C^{1,1/2}$ and no better. The functions z and k are equal to 0 on the left side of $\mathfrak{s}_1$ and on the right side of $\mathfrak{s}$.

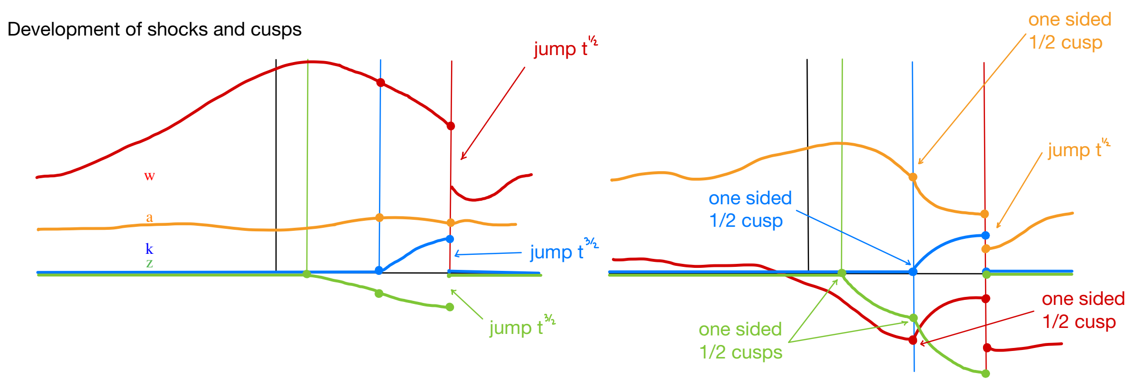

Figure 4: On the left side, we have a schematic representation of the functions $(w, z, k, a)|_{t=T_2}$ using the color scheme from Figure 1. On the right side, a schematic represetation of the functions $(\partial_\theta w, \partial_\theta z, \partial_\theta k, \partial_\theta a)|_{t=T_2}$ is given. In both images, the vertical lines represent the location of $\mathfrak{s}_1(T_2) < \mathfrak{s}_2(T_2) < \mathfrak{s}(T_2)$ using the color scheme from Figure 3. The image on the left emphasizes that all quantities except for a jump across the shock, and that z and k remain equal to 0 on $\mathbb{T}\backslash[\mathfrak{s}_1(T_2), \mathfrak{s}(T_2)]$. The image on the right emphasizes the one sided cusps form at the weak contact and weak rarefaction, and that $\partial_\theta a$ jumps across the shock.

Theorem 5.1 (Main result in azimuthal symmetry). *From smooth isentropic initial data at time T_0 with vanishing subdominant Riemann variable, as described in the first paragraph of Section 6, there exist smooth solutions to the azimuthal Euler system (4.9) that form a pre-shock singularity, at a time $T_1 > T_0$. The first singularity occurs at a single point in space, θ_*, and this first singularity is shown to have an asymptotically self-similar shock profile exhibiting a $C^{1/3}$ cusp in the dominant Riemann variable and a $C^{1,1/3}$ cusp in the radial velocity. A series expansion for $w(\cdot, T_1)$ in terms of $(\theta - \theta_*)^{1/3}$ may be computed explicitly.*

After the pre-shock is formed, the solution to (4.9)–(4.10) is continued uniquely *for a short time $(T_1, T_2]$ as a regular azimuthal shock solution (cf. Definition 4.5) with the following properties:*

- *Across the shock curve $\mathfrak{s}$, for all $t \in (T_1, T_2]$ the state variables jump*
$$[\![w]\!] \sim (t - T_1)^{\frac{1}{2}}, \quad [\![\partial_\theta a]\!] \sim (t - T_1)^{\frac{1}{2}}, \quad [\![z]\!] \sim (t - T_1)^{\frac{3}{2}}, \quad [\![k]\!] \sim (t - T_1)^{\frac{3}{2}}.$$
- *Across the characteristic $\mathfrak{s}_2$ emanating from the pre-shock and moving with the fluid velocity, the Riemann variables and the entropy make $C^{1,1/2}$ cusps approaching from the right side. Limiting from the left side, these variables are C^2 smooth.*
- *Across the characteristic $\mathfrak{s}_1$ emanating from the pre-shock and moving with the sound speed minus the fluid velocity, the entropy is zero while the subdominant Riemann variable makes a $C^{1,1/2}$ cusp from the right. Limiting from left, all fields are C^2 smooth.*

We note that the proof of Theorem 5.1, which is the bulk of our paper [4] applies with minor modifications to the case of the Euler equations for $d = 1$, or in the case of radial symmetry $d \geqslant 2$. In fact, as mentioned already in Remark 4.1, these two cases are simpler than the azimuthal symmetry considered here, since the vorticity vanishes identically.

Via the identification (4.4), Theorem 5.1 implies the following result for the Euler system in terms of hydrodynamic variables. We only state a condensed result here, and refer the interested reader to [4, Theorems 1.2, 7.1, 7.2] for details. The pictorial representation of this result is given in Figure 5 below.

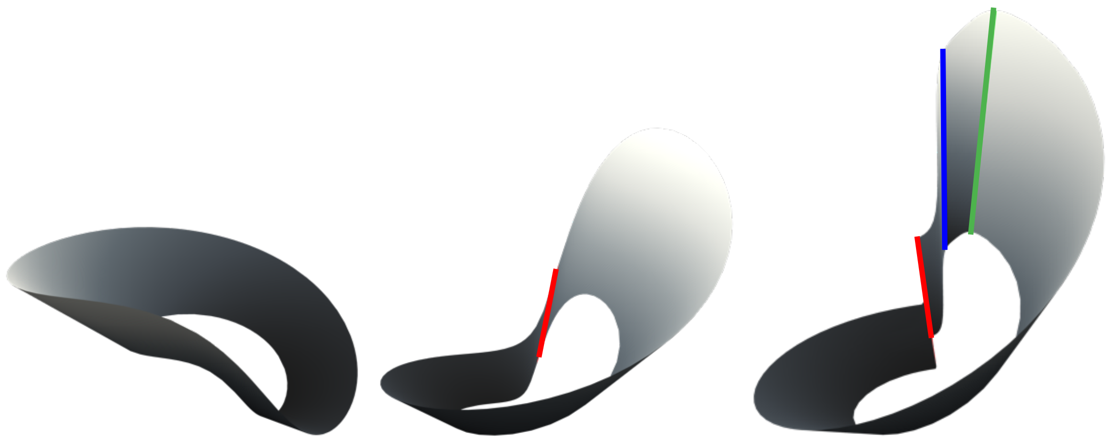

Figure 5: Values of the density written in polar coordinates $\rho(r, \theta, t)$, and plotted for $r \in [1, 2]$. The image on the left represents the smooth data at time T_0. The center image shows the pre-shock formed at time T_1, at one specific value of the angular coordinate; we marked the corresponding line in red. The image on the right represents the density at time T_2, where we have represented in red the line along which the shock discontinuity occurs, in blue the line containing the weak contact, and in green the line corresponding to the weak rarefaction.

Theorem 5.2 (Main result for 2D Euler). *From smooth isentropic initial data at time T_0 with azimuthal symmetry, there exist smooth solutions to the 2D Euler equations (1.1) that form a pre-shock singularity at a time $T_1 > T_0$. The first singularity occurs along a half-infinite ray and the blowup is asymptotically self-similar, exhibiting a $C^{1/3}$ cusp in the angular velocity and mass density, and a $C^{1,1/3}$ cusp in the radial velocity. Moreover, the blowup is given by a series expansion whose coefficients are computed as a function of the initial data.*

Past the pre-shock, the solution is continued on $(T_1, T_2]$, as an entropy–producing regular shock solution (cf. Definition 3.2) of the full 2D Euler equations (1.1). The solution is unique in the class of entropy producing weak solutions with azimuthal symmetry, with a certain weak shock structure and suitable regularity off the shock (see the space $\mathcal{X}_{\bar{\varepsilon}}$ defined in (7.8) below). The following properties are established for $t \in (T_1, T_2]$:

- *Across the classical shock hypersurface, all the state variables jump:*

$$[\![u_\theta]\!] \sim (t - T_1)^{\frac{1}{2}}, \;\; [\![\rho]\!] \sim (t - T_1)^{\frac{1}{2}}, \;\; [\![\partial_\theta u_r]\!] \sim (t - T_1)^{\frac{1}{2}}, \;\; [\![S]\!] \sim (t - T_1)^{\frac{3}{2}}.$$

- *Across the characteristic emanating from the pre-shock and moving with the fluid velocity, the entropy, density and radial velocity all have a $C^{1,1/2}$ one-sided cusp from the right, while from the left, they are all C^2 smooth. The second derivative of the angular velocity and of the pressure is bounded across this curve, justifying the name weak rarefaction.*

- *Across the characteristic emanating from the pre-shock and moving with sound speed minus the fluid velocity, the entropy is zero while the angular velocity and density have $C^{1,1/2}$ one-sided cusps from the* *right, while from the left, they are C^2 smooth. The second derivative of the radial velocity is bounded across this curve, justifying the name weak contact singularity.*

Theorem 5.2 yields a full propagation of singularities result for regular shock solutions of the Euler equations, capturing both the jump discontinuity and the weak singularities emanating from the initial cusp in the pre-shock. This gives an answer to the problem raised by Landau and Lifschitz in [17, Chapter IX, §96], at least in the context of flows with azimuthal symmetry (or one-dimensional flows).

Remark 5.3 (Anomalous entropy production). Theorem 5.2 provides an example of an entropy producing weak solution $(u, \rho, E) \in L^\infty_t(BV \cap L^\infty)_{\mathrm{loc}} \subset L^\infty_t(B^{1/p}_{p,\infty})_{\mathrm{loc}}$, for all $p \geqslant 1$. This regularity class encodes the emergence of a regular shock, obtained by continuing the past the first singularity. This proves that the Onsager-criterion proven by the second author and Eyink in [12, Theorem 3], which states that if $(u, \rho, E) \in L^\infty_t(B^{1/3+}_{3,\infty} \cap L^\infty)_{\mathrm{loc}}$ then there is no entropy production, is in fact sharp.

Remark 5.4 (Uniqueness and entropy). Theorem 5.2 establishes the uniqueness of solutions in a class of weak solutions with azimuthal symmetry, with *weak shock structure*, and which have regularity consistent with the fact that they emanate from a $C^{1/3}$ pre-shock (cf. (7.8) below), which in turn is the generic regularity that should be expected to arise at the first singularity from a smooth initial datum. The role of the entropy condition in establishing this uniqueness was explained in Remark 4.4. We contrast our uniqueness statement to the ill-posedness of the Euler system within the class of bounded, entropy-producing weak solutions emanating from 1D Riemann data, cf. Klingenberg et al. [15] and references therein.

6 Outline: the formation of the pre-shock

Fix a constant $\kappa_0 > 1$ sufficiently large and let $\varepsilon > 0$ be sufficiently small. Consider the azimuthal Euler system (4.9)–(4.10) with initial data given at time $T_0 = -\varepsilon$, satisfying (4.13), and with $w(\cdot, T_0)$ and $a(\cdot, T_0)$ which lie in a certain open subset of $C^4(\mathbb{T})$ described roughly as follows. The initial data for the radial velocity is taken to satisfy $\|a(\cdot, -\varepsilon)\|_{L^\infty} \leqslant \varepsilon$, $\|\partial_\theta a(\cdot, -\varepsilon)\|_{L^\infty} \lesssim \frac{1}{20}\kappa_0$, and $\|\partial^n_\theta a(\cdot, -\varepsilon)\|_{L^\infty} \lesssim 1$ for $2 \leqslant n \leqslant 4$. The initial data for the dominant Riemann variable is described in detail in [4, Equations (4.17)–(4.25)]. The most important property is that $w(\cdot, -\varepsilon) \in C^4(\mathbb{T})$ has a non-degenerate global minimum at a single point of $\mathbb{T}$, labeled for convenience by 0, where it holds that

$$w(0, -\varepsilon) = \kappa_0, \quad \partial_\theta w(0, -\varepsilon) = -\varepsilon^{-1}, \quad \partial^2_\theta w(0, -\varepsilon) = 0, \quad \partial^3_\theta w(0, -\varepsilon) = 6\varepsilon^{-4}. \tag{6.1}$$

Other conditions are that $\frac{7}{8}\kappa_0 \leqslant w(\cdot, -\varepsilon) \leqslant \frac{9}{8}\kappa_0$ which ensures that the density is bounded away from vacuum, that $w(\cdot, -\varepsilon) - \kappa_0$ is compactly supported $B_{\varepsilon^{1/2}}(0)$, and that the function $W(y) := \varepsilon^{-1/2}(w(y\varepsilon^{3/2}, -\varepsilon) - \kappa_0)$ lies in a certain ε-dependent open ball in the C^4 topology centered at the stable global self-similar solution of the 1D Burgers equation, $\overline{W}$, which is defined implicitly as the analytic solution of $\overline{W}(y) + \overline{W}(y)^3 + y = 0$.

For such datum, the formation of the first gradient singularity for (4.9)–(4.10) was previously established in [1]. This singularity is characterized as a *stable asymptotically self-similar* $C^{1/3}_\theta$ cusp for the dominant Riemann variable w, the so-called *pre-shock*, which occurs at a precisely computable spacetime location (θ_*, T_1), with $\theta_* \approx \kappa_0\varepsilon$ and $T_1 = \mathcal{O}(\varepsilon^3)$. The subdominant Riemann variable z and entropy κ remain identically equal to 0 on $\mathbb{T} \times [-\varepsilon, T_1]$, while radial velocity and specific vorticity satisfy $a \in L^\infty(-\varepsilon, T_1; C^{1,1/3}(\mathbb{T}))$ and $\varpi \in L^\infty(-\varepsilon, T_1; C^{0,1}(\mathbb{T}))$. From here, one may show that asymptotically as $\theta \to \theta_*$:

$$w(\theta, T_1) = \kappa - \mathsf{b}(\theta - \theta_*)^{\frac{1}{3}} + \mathrm{o}((\theta - \theta_*)^{\frac{1}{3}}), \tag{6.2a}$$

$$a(\theta, T_1) = \mathsf{a}_0 + \mathsf{a}_1(\theta - \theta_*) + \mathsf{a}_2(\theta - \theta_*)^{\frac{4}{3}} + \mathrm{o}((\theta - \theta_*)^{\frac{4}{3}}), \tag{6.2b}$$

for suitable constants computable constants $\mathsf{b} \approx 1$, a_i, and κ such that $|\kappa - \kappa_0| \lesssim \varepsilon^2$.

While the description of the pre-shock given by (6.2) would be likely sufficient to describe the classical shock singularity $\mathfrak{s}$ emerging from the pre-shock, in order to rigorously capture the formation of higher order characteristic singularities emerging along the curves $\mathfrak{s}_1$ and $\mathfrak{s}_2$ in Figure 3, a much finer understanding of the dominant Riemann variable w at the pre-shock is required. This information is not available in [1], and it is the subject of the analysis in [4, Section 4]. In particular, [4, Theorem 4.1] proves that

$$w(\theta, T_1) = \kappa - \mathsf{b}(\theta - \theta_*)^{\frac{1}{3}} + \mathsf{c}_1(\theta - \theta_*)^{\frac{2}{3}} + \mathsf{c}_2(\theta - \theta_*) + \mathcal{O}((\theta - \theta_*)^{\frac{4}{3}}), \tag{6.3}$$

holds for all θ in an ε-dependent ball around θ_*, for explicitly computable constants c_i. More importantly, we prove that *the fractional series expansion* (6.3) *holds in a C^3 sense*, meaning that the first three derivatives of the left side in (6.3) equal the first three derivatives of the expansion on the right side, with error bounds stable under differentiation.

The proof of (6.3) is based on a fully-Lagrangian characterization of the pre-shock, and a subtle interplay between the characteristics of the speeds $\lambda_3 = w$ and $\lambda_2 = \frac{2}{3}w$ present in (4.14), and which are defined by

$$\begin{aligned} \partial_t \eta &= \lambda_3(\eta(x,t),t) = w(\eta(x,t),t), \qquad \eta(x,-\varepsilon) = x, \\ \partial_t \phi &= \lambda_2(\phi(x,t),t) = \tfrac{2}{3} w(\phi(x,t),t), \qquad \phi(x,-\varepsilon) = x. \end{aligned}$$

By (4.14a) it is clear that η is the natural flow of the w evolution, while (4.14b) and (4.12), which simplifies here to $\partial_t \varpi + \frac{2}{3} w \partial_\theta \varpi = \frac{8}{3} a \varpi$, show that ϕ is the natural flow for a and ϖ.

The first and most important observation is that the spacetime location of the first singularity (θ_*, T_1) is characterized by $\theta_* = \eta(x_*, T_1)$, where (x_*, T_1) are the unique Lagrangian label, respectively the first time, which simultaneously solve the system

$$\partial_x \eta(x_*, T_1) = \partial_{xx} \eta(x_*, T_1) = 0. \tag{6.4}$$

In fact, as part of the proof it is crucial that we establish that

$$\begin{aligned} \partial_x \eta(x,t) &= \left(1 + \mathcal{O}(\varepsilon^{\frac{1}{2}})\right) \varepsilon^{-1}(T_* - t) + \left(3 + \mathcal{O}(\varepsilon^{\frac{1}{8}})\right) \varepsilon^{-3}(x - x_*)^2, \\ \partial_{xx} \eta(x,t) &= (T_* - t)\mathcal{O}(\varepsilon^{-2}) + \left(6 + \mathcal{O}(\varepsilon^{\frac{1}{8}})\right) \varepsilon^{-3}(x - x_*), \\ \partial_{xxx} \eta(x,t) &= \left(6 + \mathcal{O}(\varepsilon^{\frac{1}{8}})\right) \varepsilon^{-3}, \end{aligned}$$

for all labels $|x - x_*| \leqslant \varepsilon^2$ and all $t \in [-\varepsilon, T_1]$. This asymptotic description of the Lagrangian flow may be traced back to the initial datum assumption (6.1).

The second ingredient in the proof is that the fields $\eta, w \circ \eta, a \circ \eta, \varpi \circ \eta$ remain C^4 smooth as functions of the Lagrangian label x, *uniformly* in time on the interval $[-\varepsilon, T_1]$. Roughly speaking, this is achieved by appealing to the identities

$$\eta(x,t) = x + \int_{-\varepsilon}^{t} w \circ \eta(x,s) ds, \tag{6.5a}$$

$$w \circ \eta(x,t) = w(x,-\varepsilon) e^{-\frac{8}{3}\int_{-\varepsilon}^{t} a \circ \eta(x,s) ds}, \tag{6.5b}$$

which show that the regularity of $a \circ \eta$ implies the regularity of η and $w \circ \eta$, and to the one-derivative gains provided by the relations $\partial_\theta a = w - \frac{1}{16} w^2 \varpi$ and

$$\partial_x \phi(x,t) = \left(\tfrac{w(x,-\varepsilon)}{w \circ \phi(x,t)}\right)^2 e^{-\frac{16}{3}\int_{-\varepsilon}^{t} a \circ \phi(x,s) ds},$$

$$\varpi \circ \phi(x,t) = \varpi_0(x,-\varepsilon) e^{\frac{8}{3}\int_{-\varepsilon}^{t} a\circ\phi(x,s)ds},$$

which in turn allows us to establish the desired higher order regularity of a and ϖ.

The third ingredient in the proof concerns the invertibility of the map $x \mapsto \eta(x, T_1)$. Using (6.4) and a Taylor series expansion justified by the regularity of η, we have that

$$\theta = \eta(x,T_1) = \theta_* + \tfrac{1}{6}\partial_{xxx}\eta(x_*,T_1)(x-x_*)^3 + \tfrac{1}{24}\partial_{xxxx}\eta(\overline{x},T_1)(x-x_*)^4\,,$$

where $\theta_* = \eta(x_*, T_1)$, and $\overline{x}$ is a point between x_* and x. As such, with $\Theta = \theta - \theta_*$ and $X = x - x_*$, we are left to invert the quartic polynomial $\Theta = \mathrm{g}_1 X^3 + \mathrm{g}_2 X^4$, where $\mathrm{g}_1 \approx \varepsilon^{-3} > 0$ and $|\mathrm{g}_2| = \mathcal{O}(\varepsilon^{-4})$. This inversion, via a Newton iteration results in a fractional power series $X = \mathrm{f}_1\Theta^{1/3} + \mathrm{f}_2\Theta^{2/3} + \mathrm{f}_3\Theta + \mathcal{O}(\Theta^{4/3})$, with explicitly computable real coefficients f_i. This fractional power series is then directly translated into a power series expansion for the inverse map $\eta^{-1}(\theta, T_1)$ in powers of $(\theta - \theta_*)^{1/3}$, valid for θ sufficiently close to θ_*. At last, we insert this expansion into (6.5b), to obtain

$$w(\theta,T_1) = w(\eta^{-1}(\theta,T_1),-\varepsilon) e^{-\frac{8}{3}\int_{-\varepsilon}^{T_1} a\circ\eta(\eta^{-1}(\theta,T_1),s)ds}\,.$$

Using the known expansion for $\eta^{-1}(\cdot, T_1)$ and the regularity of $a \circ \eta$, we deduce (6.3).

7 Outline: the development of shocks and weak singularities

We next turn to the development problem, within the class of regular azimuthal shock solutions, cf. Definition 4.5. The initial datum for this development problem are the functions (w, z, k, a) at which we have arrived in the formation process at time T_1. For simplicity of the presentation let us shift the pre-shock location (θ_*, T_1) to $(0, 0)$, and let us denote the values of the azimuthal fields at the pre-shock by (w_0, z_0, k_0, a_0). By the analysis in Section 6, we have that $z_0 \equiv k_0 \equiv 0$ on $\mathbb{T}$, $a_0 \in C^{1,1/3}(\mathbb{T})$ with $\|a_0\|_{W^{1,\infty}} \lesssim \kappa_0$, $\varpi_0 \in \mathrm{Lip}(\mathbb{T})$ with $1 < \kappa_0\varpi_0(\theta) \lesssim 1$, and the dominant Riemann variable is given by

$$w_0(\theta) = \kappa - \mathsf{b}\theta^{\frac{1}{3}} + \mathsf{c}_1\theta^{\frac{2}{3}} + \mathsf{c}_2\theta + \mathcal{O}(\theta^{\frac{4}{3}}), \tag{7.1}$$

equality which holds in a C^3 sense, with $\kappa \approx \kappa_0 > 1$, $\mathsf{b} \approx 1$, and $\mathsf{c} = \mathcal{O}(\varepsilon^{1/2})$. The shock development problem from this initial data is solved on the interval $[0, \overline{\varepsilon}]$, i.e. $T_2 = T_1 + \overline{\varepsilon}$ in the language of Theorem 5.1, for a $\overline{\varepsilon}$ which is sufficiently small in terms of the data. The detailed analysis is carried out in [4, Sections 5 and 6], and here we only give the main ideas.

Given a smooth shock curve $\mathfrak{s}\colon [0, \overline{\varepsilon}] \to \mathbb{T}$, we shall denote the spacetime complement of the shock as $\mathcal{D}_{\overline{\varepsilon}} = (\mathbb{T} \times [0, \overline{\varepsilon}]) \backslash (\mathfrak{s}(t), t)_{t\in[0,\overline{\varepsilon}]}$, and for any function $f\colon \mathcal{D}_{\overline{\varepsilon}} \to \mathbb{R}$ we denote the left and right traces at the shock by $f_\pm(t) = \lim_{\theta\to\mathfrak{s}(t)^\pm} f(\theta, t)$, and the jump and mean across the shock as $[\![f]\!](t) = f_-(t) - f_+(t)$, respectively $\langle\!\langle f\rangle\!\rangle(t) = \frac{1}{2}(f_-(t) + f_+(t))$. Note that since $\overline{\varepsilon}$ is chosen to be sufficiently small, we have that $t \ll 1$ is a small parameter.

To leading order in $0 < t \ll 1$ and for $|\theta| \ll 1$, the *intuition* behind the shock developement problem is as follows. First, from the Rankine-Hugoniot jump conditions one has that to leading order the speed of propagation of weak shock waves (relative to the fluid) is equal to the sound speed, which in the context of azimuthal symmetry means that

$$\begin{aligned}\dot{\mathfrak{s}} &\approx b + c = w \approx w_0 + (\text{small error respect to } t)\\ &\qquad\approx \kappa + (\text{small error respect to } |\theta| \ll 1) + (\text{small error respect to } t \ll 1)\,.\end{aligned}$$

Thus, to leading order we may expect that $\mathfrak{s}(t) \approx \kappa t$.

Second, we note that although entropy k and the subdominant Riemann variable z are strictly positive for $t > 0$, for short time they are expected to be small. As such, to leading order one may expect that the evolution of the dominant Riemann variable w (cf. (4.9a)) may be approximated as

$$\partial_t w + (w + \text{small error})\,\partial_\theta w = (\text{small errors involving entropy gradients})\,,$$
$$w_0 = \kappa - \mathsf{b}\theta^{\frac{1}{3}} + (\text{small error near for } |\theta| \ll 1)\,.$$

Thus, we may hope to view the dominant Riemann variable w as being a perturbation of an inviscid Burgers solution w_{B} with associated Lagrangian η_{B}, namely

$$w_{\mathsf{B}}(\theta,t) = w_0({\eta_{\mathsf{B}}}^{-1}(\theta,t)), \qquad \eta_{\mathsf{B}}(x,t) = x + tw_0(x). \tag{7.2}$$

Here we denote Eulerian space variable by θ and the Lagrangian label by x. There is an important caveat in the standard-looking definition (7.2). Since the initial data w_0 is a pre-shock (recall (7.1)), the map ${\eta_{\mathsf{B}}}^{-1}(\theta,t)$ is not well-defined for θ which is very close to $\mathfrak{s}(t)$; indeed, in this region the map is two-valued. This is natural since these characteristics are expected to impinge upon the shock from either the left or right sides, which ensures that the Lax entropy conditions (3.4) are satisfied. To overcome this, given any $t \in (0,\bar{\varepsilon}]$, and *given a shock curve* $\mathfrak{s}(t)$, we compute two Lagrangian labels $x_\pm(t) = {\eta_{\mathsf{B}}}^{-1}(\mathfrak{s}(t)^\pm, t)$ such that the associated particle trajectories $\eta_{\mathsf{B}}(x_\pm(t),s)$ fall into the shock exactly at time $s = t$. This allows us to define ${\eta_{\mathsf{B}}}^{-1}(\cdot,t)\colon \mathbb{T}\backslash\{\mathfrak{s}(t)\} \to \mathbb{T}\backslash[x_-(t),x_+(t)]$ as a bijective map, giving a meaning to (7.2). Note that to leading order one may compute $\eta_{\mathsf{B}}(x,t) \approx x + \kappa t - (\mathsf{b}t)x^{1/3}$, and since to leading order $\mathfrak{s}(t) \approx \kappa t$, we deduce that $x_\pm(t) \approx (\mathsf{b}t)^{3/2}$. It follows that we may expect the jump of the dominant Riemann variable across the shock curve to be given, to leading order in t, by

$$[\![w]\!](t) \approx [\![w_{\mathsf{B}}]\!](t) = w_0(x_-(t)) - w_0(x_+(t)) \approx 2\mathsf{b}^{\frac{3}{2}}t^{\frac{1}{2}}. \tag{7.3}$$

Third, in analogy to how (3.5) was derived, we may show that in the weak shock regime $|[\![w]\!]| \ll 1$ (justified in view of (7.3)) the smallest root (in absolute value) of the system of equations (4.16a)–(4.16b) (which were derived from the azimuthal form of the Rankine-Hugoniot conditions) is given to leading order by

$$[\![z]\!](t) \approx -\tfrac{9[\![w]\!](t)^3}{16\langle\!\langle w\rangle\!\rangle(t)^2} \approx -\tfrac{9\mathsf{b}^{\frac{9}{2}}}{2\kappa^2}t^{\frac{3}{2}}, \qquad \text{and} \qquad [\![k]\!](t) \approx \tfrac{4[\![w]\!](t)^3}{\langle\!\langle w\rangle\!\rangle(t)^3} \approx \tfrac{32\mathsf{b}^{\frac{9}{2}}}{\kappa^3}t^{\frac{3}{2}}. \tag{7.4}$$

Just as (7.3), (7.4) may be shown to hold in a C^2_t sense. The jump relations show that positive entropy and negative subdominant Riemann variable must be *produced instantaneously* along the shock in order for mass, momentum, and energy not to be lost.

Fourth, we need to carefully analyze the three characteristic families present in the azimuthal Euler equations (4.9)–(4.10). These flows are defined naturally as

$$\partial_t\eta = \lambda_3 \circ \eta, \qquad \partial_t\phi = \lambda_2 \circ \phi, \qquad \partial_t\psi = \lambda_1 \circ \psi, \qquad (\eta,\phi,\psi)(x,0) = x.$$

Our heuristics indicate that to leading order in $t \ll 1$ and $|x| \ll 1$ we have that

$$\eta(x,t) \approx \eta_{\mathsf{B}}(x,t) \approx x + \kappa t - (\mathsf{b}t)x^{\frac{1}{3}}, \quad \phi(x,t) \approx x + \tfrac{2\kappa}{3}t, \quad \psi(x,t) \approx x + \tfrac{\kappa}{3}t, \tag{7.5}$$

which confirms our intuition that the λ_3 characteristic η impinges on the shock curve $\mathfrak{s}(t) \approx \kappa t$ only after we look at the next order term in t and x, and also that the λ_2 and λ_1 characteristics ϕ and ψ are transversal to the shock. Note that the two characteristic surfaces of weak singularities are nothing but the images under these slow flows of the point-shock

$$\mathfrak{s}_2(t) = \phi(0,t) \approx \tfrac{2\kappa}{3}t, \qquad \mathfrak{s}_1(t) = \psi(0,t) \approx \tfrac{\kappa}{3}t.$$

The transversality of characteristic families mentioned above plays a crucial role in our analysis: it may be combined with the fact that we stay away from the vacuum state in order to interchange a space derivative with a time derivative in terms which are composed with ϕ or ψ. For example, it allows us to heuristically replace the statements $[\![z]\!] \sim -t^{3/2}$ and $[\![k]\!] \sim t^{3/2}$ from (7.4), with asymptotic descriptions $z(\theta,t) \sim -(\theta - \mathfrak{s}_1(t))^{3/2}$ and $k(\theta,t) \sim (\theta - \mathfrak{s}_2(t))^{3/2}$ asymptotically as $\theta \to \mathfrak{s}_1(t)^+$, respectively $\theta \to \mathfrak{s}_2(t)^+$. Thus, the jump relations (7.4) and transversality imply that the fields z and k form $C^{1,1/2}$ cusps at $\mathfrak{s}_1$ and $\mathfrak{s}_2$, when approaching from the right.

Besides determining the location of the weak singularities, the flows η, ϕ, ψ, also paint a detailed picture as to how information is carried from the $\{t = 0\}$ initial data surface, respectively how information about the jumps at the shock are propagated through the fluid in spacetime. A schematic description is provided by Figure 6 below.

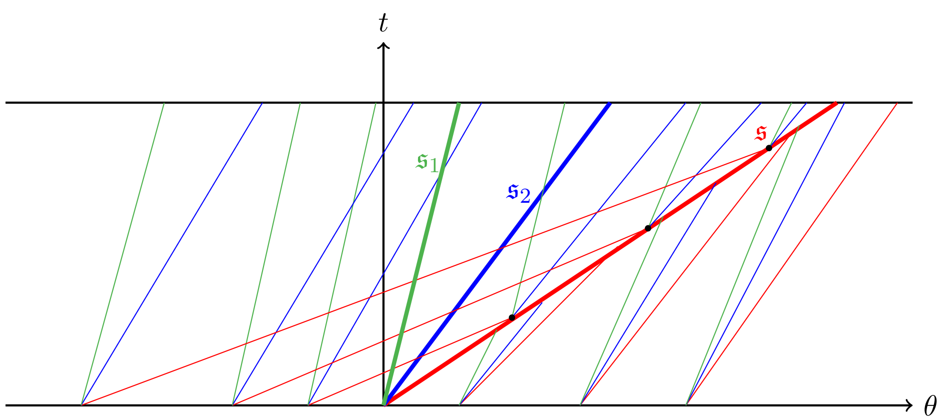

Figure 6: The three distinct wave families η, ϕ, and ψ are represented in red, blue, and respectively green, for various initial labels. The most interesting such labels are marked with black dots: these do not lie on the time-slice $\{t = 0\}$, but instead they lie on the shock curve $\mathfrak{s}$ at various values of time; at these points the values of k_- and z_- are computed according to (7.4). To leading order, the entropy k is propagated off the shock curve along the λ_2 characteristics ϕ, while the subdominant Riemann variable z is also propagated off the shock curve $\mathfrak{s}$, but along the λ_1 characteristics ψ. The λ_3 characteristics η initiated at $\{t = 0\}$, represented in red, impinge on the shock curve from the left side, determining w in terms of w_0 on both sides of the shock.

Fifth, we note that according to (4.9d) and (4.12), the fluid velocity λ_2 and its associated characteristic ϕ are the natural ones for carrying information about the radial velocity a and the specific entropy ϖ. In particular, since ϕ is transversal to $\mathfrak{s}$, we are able to use (4.12) in order to show that the specific vorticity is continuous across the shock curve. As such, the relation (4.11) implies that it is $\partial_\theta a$ and not a which has a jump discontinuity at $\mathfrak{s}$, and moreover to leading order we have

$$[\![\partial_\theta a]\!](t) \approx [\![w]\!](t) \approx 2\mathsf{b}^{\frac{3}{2}} t^{\frac{1}{2}} \,. \tag{7.6}$$

Sixth, concerning the characterization of the higher order singularities across the curves $\mathfrak{s}_1$ and $\mathfrak{s}_2$, the intuition regarding the precise regularity of the fields (w, z, k, a) stems from the jump relations (7.3), (7.4), (7.6), a detailed description of the Lagrangian flows ϕ and ψ similar to (7.5), and the structure of the forcing terms in (4.9) and (4.12). For instance, we have already mentioned in the paragraph below (7.5) that the transversality of ϕ and ψ to $\mathfrak{s}$, along with the jump relations (7.4) allow us to precisely compute the regularity of z and k approaching $\mathfrak{s}$ from the left. This matter is however more subtle near $\mathfrak{s}_1$ and $\mathfrak{s}_2$. To see this, we may inspect Figure 6 and note that an Eulerian point (θ, t) with $0 < \theta - \mathfrak{s}_2(t) \ll 1$ is traced backwards in time along the blue characteristics ϕ to a point which lies on the shock curve at some time $\mathcal{T}(\theta, t) \sim \theta - \mathfrak{s}_2(t) \ll 1$ (*shock-intersection times* are defined precisely in [4, Definitions 5.15 and 5.16]). Thus, singular information about the derivatives of the jumps of k at a time $\mathcal{T}(\theta, t) \ll t$ is carried via the ϕ characteristics to the point (θ, t), resulting in infinite terms as $\theta \to \mathfrak{s}_2(t)^+$.

An additional difficulty in analyzing the higher order singularities is that if one naively considers the evolution equations for $\partial_\theta w$ or $\partial_\theta z$ cf. (4.9a) and respectively (4.9b), we note the emergence of the forcing term $\frac{1}{24}(w-z)^2\partial_{\theta\theta}k$, resulting in what seems to be a derivative loss. In order to overcome this issue we introduce the *good unknowns*

$$q^w := \partial_\theta w - \tfrac{1}{4}c\partial_\theta k, \qquad q^z := \partial_\theta z + \tfrac{1}{4}c\partial_\theta k,$$

which satisfy the evolution equations

$$(\partial_t + \lambda_3\partial_\theta)q^w + (\partial_\theta\lambda_3 + \tfrac{8}{3}a)q^w = -\tfrac{8}{3}\partial_\theta a w + \Big(\tfrac{4}{3}ac + \tfrac{1}{6}c\partial_\theta\lambda_2\Big)\partial_\theta k, \tag{7.7a}$$

$$(\partial_t + \lambda_1\partial_\theta)q^z + (\partial_\theta\lambda_1 + \tfrac{8}{3}a)q^z = -\tfrac{8}{3}\partial_\theta a z - \Big(\tfrac{4}{3}ac + \tfrac{1}{6}c\partial_\theta\lambda_2\Big)\partial_\theta k. \tag{7.7b}$$

The remarkable feature of the system (7.7) is that the second derivatives of k do not appear in the equations, allowing us to close estimates. The unknowns q^w and q^z are useful because they involve only the first derivative of the entropy, $\partial_\theta k$, and this term makes a $C^{\frac{1}{2}}$ cusp along the curve $\mathfrak{s}_2$. On the other hand, the natural flows in the system (7.7) are η and respectively ψ, which are transversal to the flow ϕ along which the singularities of k are carried through the flow. This geometric structure of (7.7) and of the good unknowns q^w and q^z analytically result in a one-derivative regularization effect, which is not apparent if we were to inspect (4.9)–(4.12) directly. Another outcome of this derivative gain is that $q^w + q^z = \partial_\theta z + \partial_\theta w = \frac{2}{3}\partial_\theta u_\theta$ is smoother than the naive expectation $C^{\frac{1}{2}}$ because the $\partial_\theta k$ terms cancel. This translates into at least C^2 regularity for the angular velocity u_θ along the curve $\mathfrak{s}_2$; in contrast, the entropy S, the density ρ and the radial velocity u_r are precisely $C^{1,1/2}$ across $\mathfrak{s}_2$, which justifies the name *weak contact singularity*.

In closing, we note that in order to make the six-step heuristic outlined in this section rigorous, requires a good functional framework and a of number analytical tricks for the analysis of Lagrangian flows. In broad terms, we proceed as follows. We build an iteration scheme in which we start with a C^2 smooth shock curve $\mathfrak{s}$ with $|\mathfrak{s}(t) - \kappa t| \lesssim t^2$, use it to construct a Burgers solution w_{B} adapted to this particular shock curve (as described in Step 2), and then use a contraction mapping principle to build a solution (w,z,k,a) of the azimuthal Euler equations (4.9)–(4.12) which has jump discontinuities across $\mathfrak{s}$ that satisfy the algebraic system (4.16a)–(4.16b) resulting from the Rankine-Hugoniot jump conditions, and such that the regularity of the solution is consistent with the fact that the solution emanates from a $C^{1/3}$ pre-shock. More precisely, there exists $\overline{\varepsilon}$ sufficiently small such that the solution lies in the functional space

$$\mathcal{X}_{\overline{\varepsilon}} = \Big\{(w,z,k,a) \in C^1_{\theta,t}(\mathcal{D}_{\overline{\varepsilon}})\colon (w,z,k,a)|_{t=0} = (w_0,0,0,a_0), \|\!|(w-w_{\mathsf{B}},z,k,a)|\!\|_{\overline{\varepsilon}} \leqslant 1\Big\} \tag{7.8}$$

where the norm $\|\!|(v,z,k,a)|\!\|_{\overline{\varepsilon}}$ is defined by

$$\begin{aligned}\|\!|(v,z,k,a)|\!\|_{\overline{\varepsilon}} = \sup_{(\theta,t)\in\mathcal{D}_{\overline{\varepsilon}}} \max\Big\{ & \mathsf{m}_1 t^{-1}|v(\theta,t)|, \mathsf{m}_2\big(\mathsf{b}^3t^3 + (\theta-\mathfrak{s}(t))^2\big)^{\frac{1}{6}}|\partial_\theta v(\theta,t)|,\\ & \mathsf{m}_3 t^{-\frac{3}{2}}|z(\theta,t)|, \mathsf{m}_3 t^{-\frac{1}{2}}|\partial_\theta z(\theta,t)|, \mathsf{m}_4 t^{-\frac{3}{2}}|k(\theta,t)|,\\ & \mathsf{m}_4 t^{-\frac{1}{2}}|\partial_\theta k(\theta,t)|, \mathsf{m}_5|a(\theta,t)|, \mathsf{m}_5|\partial_\theta a(\theta,t)|\Big\}\end{aligned}$$

where m_i are sufficiently large constants. In particular, we note that the space $\mathcal{X}_{\overline{\varepsilon}}$ encodes precisely how close w is to the Burgers solution w_{B}.

So far, we have thus defined a map $\mathfrak{s} \mapsto (w,z,k,a)$, but we are missing one key ingredient: the shock curve was just a given curve with $|\mathfrak{s}(t) - \kappa t| \lesssim t^2$, it did not satisfy the evolution equation (4.16c) imposed by the Rankine-Hugoniot jump conditions. This however gives us a natural way of updating the shock curve: we solve for $\widetilde{\mathfrak{s}}$ the ODE (4.16c) with data $\widetilde{\mathfrak{s}}(0) = 0$ and fields (w_+, w_-, z_-, k_-) given by the restrictions of

(w, z, k, a) on the old curve $\mathfrak{s}$. Then, we prove that $\widetilde{\mathfrak{s}}$ is C^2 smooth and satisfies $|\widetilde{\mathfrak{s}}(t) - \kappa t| \lesssim t^2$. Lastly, we prove that above described iteration $\mathfrak{s} \mapsto \widetilde{\mathfrak{s}}$ is in fact a contraction in C^2, resulting in a unique fixed point which is the desired shock curve. Associated to this curve, we also prove that there is a unique regular azimuthal shock solution $(w, z, k, a) \in \mathcal{X}_{\bar{\varepsilon}}$, as soon as $\bar{\varepsilon} > 0$ is sufficiently small. This completes the proof of Theorem 5.1.

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