Homework 2

1. Consider the homogeneous ideal $J := \langle x_0^2 x_1, x_1^3, x_1 x_2 \rangle$. For $i = 0, 1, 2$ let I_i be the dehomogenization of J with respect to x_i and J_i the homogenization of I_i. Compute each I_i and J_i. Show that $J \subsetneq J_0 \cap J_1 \cap J_2$. Show that $(J : \langle x_0, x_1, x_2 \rangle) \neq J$.

2. Compute the Hilbert polynomial of
$$J := \langle x_0 x_1, x_2^2 \rangle \subset \mathbb{C}[x_0, x_1, x_2].$$

3. Let $C \subset \mathbb{P}^2$ be a curve given by $f(x_0, x_1, x_2) = 0$. Let $C \to \mathbb{P}^2$ be a map given by
$$(x_0, x_1, x_2) \mapsto (\frac{\partial f}{\partial x_0}(x_0, x_1, x_2), \frac{\partial f}{\partial x_1}(x_0, x_1, x_2), \frac{\partial f}{\partial x_2}(x_0, x_1, x_2)).$$
Show that the image is a point iff C is a line. The image is called the *dual curve*.

4. Let $K = \mathbb{Q}(\sqrt[3]{a}, \sqrt[4]{b})$, with $a \neq m^3$, i.e., a is an integer which is not a cube, and $b \neq \pm n^2$. Determine the Galois group $\mathrm{Gal}(\tilde{K}/\mathbb{Q})$ of the Galois closure $\tilde{K}$ of K.

5. Determine generators of $\mathcal{O}_K$, where $K = \mathbb{Q}(\sqrt[3]{d})$, with $d \in \mathbb{Z}$.