Homework 7

1. Let p be a prime $= 1 \mod 4$. Show that

$$\sum_{j=0}^{p-1} \sin\left(\frac{2\pi j^2}{p}\right) = 0 \quad \text{and} \quad \left|\sum_{j=0}^{p-1} \cos\left(\frac{2\pi j^2}{p}\right)\right| = \sqrt{p}.$$

2. Compute the residue of $\Gamma(s)$ at each pole.

3. Show that

$$\sum_{0 \le j \le n/3} \binom{n}{3j} = \frac{1}{3}(2^n + 2\cos(\frac{\pi n}{3})).$$

4. Show that

$$\prod_p \frac{p^2+1}{p^2-1} = \frac{5}{2}.$$

5. Show that

$$\int_0^\infty \frac{\sin(x)}{x^s} dx = \cos(\pi s/2)\Gamma(1-s).$$