Homework 6

1. Let p be a prime, $p = 1 \mod 4$. Show that
$$[\sqrt{p}] + [\sqrt{2p}] + \cdots + [\sqrt{\frac{(p-1)}{4}p}] = \frac{p^2-1}{12}$$

2. Let A_n (resp. G_n) be the arithmetic (resp. geometric) mean of
$$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}.$$
Show that
$$\lim_{n\to\infty} \sqrt[n]{A_n} = 2 \quad \text{and} \quad \lim_{n\to\infty} \sqrt[n]{G_n} = \sqrt{e}.$$

3. Set $b_j := \int_{\mathbb{Z}_p} B_j(t_p)\, dt_p$, where dt_p is the standard Haar measure on $\mathbb{Q}_p$ and B_j are the Bernoulli polynomials. Let
$$F(T) := \sum_{j\geq 0} b_j T^j / j!.$$
Show that $F(T) = (T/(\exp(T)-1))^2$ and that $b_j = -(jB_{j-1} + (j-1)B_j)$.

4. Show that
$$\sum_{(a,n)=1} e^{\frac{2\pi i a}{n}} = \mu(n).$$

5. Let
$$f(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$
be absolutely convergent for $\Re(s) > \sigma > 0$. Put
$$g(x) := \sum_{n=1}^{\infty} a_n e^{-nx}.$$
Show that, for $\Re(s) > \sigma$, one has
$$f(s)\Gamma(s) = \int_0^{\infty} g(x) x^{s-1} dx.$$