Exercises 7 (November 7, 2005)

1. Show that \( \zeta(0) = -1/2 \) and \( \zeta'(0) = -\frac{\ln(2\pi)}{2} \).

2. Show that for \( s \in \mathbb{R}, \zeta(s) = 0 \) if and only if \( s = -2k \), with \( k \in \mathbb{N}, k \geq 1 \).

3. Find all solutions of \( x^2 + y^2 = 2z^2 \) in \( \mathbb{Q} \).

4. Let \( f(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i^m \), with \( a_i \in \mathbb{Z}_p, a_i \neq 0 \). Put
   \[ r := \nu_p(m), \quad s := \max_i (\nu_p(a_i)) \quad \text{and} \quad N := 2(r + s) + 1. \]
   Show that \( f(x_1, \ldots, x_n) \) has a nontrivial zero in \( \mathbb{Q}_p \) if and only if the congruence \( f(x_1, \ldots, x_n) = 0 \mod p^N \) has a solution \((x_0^1, \ldots, x_0^n)\) such that \( x_0^j \neq 0 \mod p \) for at least one \( j \).

5. Check that \( B_{2k} \neq 0 \mod 17 \) for \( k = 1, \ldots, 7 \), and that there exists a \( k \in [1, \ldots, 17] \) such that \( B_{2k} = 0 \mod 37 \).