Arithmetic of Fano varieties

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I. Rational points

Data:

- F number field;
- $X = X_F$ (smooth) Fano variety over F;
- $\mathrm{Pic}_{\mathbb{R}} \supset \Lambda_{\mathrm{eff}} \supseteq \Lambda_{\mathrm{ample}} \ni -K$;
- $f_{-K} : X \to \mathbb{P}^n$ - anticanonical embedding;
- $X(F)$ set of F-rational points.

1. Basic notions

Heights:

$$
\begin{array}{rrcl}
H: & \mathbb{P}^n(F) & \to & \mathbb{R}_{>0}, \\
& (x_0 : \cdots : x_n) & \mapsto & \prod_{v \in \mathrm{Val}(F)} \max_j (|x_j|_v), \\
H_{-K}: & x & \mapsto & H(f_{-K}(x)).
\end{array}
$$

Counting function:

$$N(U(F), B) := \#\{x \in U(F) \mid H_{-K}(x) \leq B\}.$$

$$U \subseteq X$$

Zariski open subset.

2. Problems

- $X(F) \neq \emptyset$?

- $X(F)$ Zariski dense?

Conjecture (Potential density):
$\exists\, F'/F$ (finite) s.t. $X(F')$ is Zariski dense.

- $N(U(F), B) \sim$?? for $B \to \infty$
  (and some Zariski open $U \subset X$)?

Working hypothesis (Linear growth):
$\exists\, U \subset X$ s.t. $N(U(F), B) = O(B^{1+\epsilon})$.

3. Invariants

Cones in geometry:

- (Mori) $\Lambda_{\text{ample}}(X)$ finitely generated;
- (Batyrev 92) $\Lambda_{\text{eff}}(X)$ finitely generated if $\dim(X) \leq 3$.

Example:

- $X = \bar{M}_{0,6} = \mathrm{Bl}_Z(\mathbb{P}^3)$,
  $Z = \cup_{j=1}^5 p_j \cup_{i \neq j} \ell_{ij}$, points and lines
  $\mathrm{Pic}(X) = \mathbb{Z}^{16} = \langle L, E_i, E_{ij} \rangle$;
  $\Lambda_{\text{eff}}(X)$ generated by
  E_i, E_{ij},
  $L - (E_i + E_j + E_k + E_{ij} + E_{ik} + E_{jk})$,
  $2L - (E_1 + ... + E_5) - (E_{ik} + E_{i\ell} + E_{jk} + E_{j\ell})$;
  $-K_X = 4L - 2(E_1 + ... + E_5) - (\sum_{i<j} E_{ij})$.
  $f_{-K_X} : X \to \mathbb{P}^4$ (Igusa quartic)
  $(x_0x_1 + x_0x_2 + x_1x_2 - x_3x_4)^2 = 4x_0x_1x_2(\sum_{j=0}^4 x_j)$.

Cones in analysis:

- (A, Λ) = (lattice, convex cone in $A_{\mathbb{R}}$);

- $(\check{A}, \check{\Lambda})$ = (dual lattice, dual cone);

- $\alpha(\Lambda, s) := \int_{\check{\Lambda}} e^{-\langle s, \check{a} \rangle} d\check{a}, \quad \Re(s) \in \Lambda^{\circ}$.

Example:

- $\alpha(\Lambda_{\mathrm{eff}}(S_3), -K) = 7/18$,
  $S_3 : x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$.

- $\alpha(\Lambda_{\mathrm{eff}}(\bar{M}_{0,6}), -K) = ??$,
  $\check{\Lambda}_{\mathrm{eff}}(\bar{M}_{0,6})$ has 3905 generators.

Brauer groups:

- $\mathrm{Br}(X)/\mathrm{Br}(F) = H^1(\Gamma, \mathrm{Pic}(\bar{X}))$.

Example:

- $\mathrm{Br}(S_3)/\mathrm{Br}(\mathbb{Q}) = \mathbb{Z}/3$,
  $S_3 : x_0^3 + x_1^3 + x_2^3 + 2x_3^3 = 0$.

- (Kresch-T. 02) $\mathrm{Br}(S_2)/\mathrm{Br}(\mathbb{Q}) = \mathbb{Z}/2$,
  $S_2 : w^2 = ax_0^4 + bx_1^4 + cx_2^4$ (generic (a, b, c)).

- (Urabe, 96) $G \subset W(E_7)$ or $W(E_8)$ cyclic subgroup.

Tamagawa numbers (Peyre)

$-\mathcal{K} := \{-K, (\|\cdot\|_v)_{v\in \mathsf{Val}(F)}\}$ - metrization:

- (local) height:

$$H_{-K,g,v}(x) := \|g(x)\|_v^{-1}$$

  $g(x) \neq 0$, g - F-rational section of $-K$;

- global height: $H_{-K}(x) := \prod_v H_{-K,g,v}(x)$;

- measure ω_v on $X(F_v)$:

- (regularized) measure on $X(\mathbb{A}_F)$:

  $$\omega_{\mathcal{K}} := L_S^*(1, \mathrm{Pic}(X))|\mathrm{disc}(F)|^{-d/2} \prod_v \lambda_v^{-1}\omega_v,$$

  $\lambda_v := L_v(1, \mathrm{Pic}(X))$, $v \notin S$, $\lambda_v = 1, v \in S$,
  $L_S^*(1, \mathrm{Pic}(X)) = \lim_{s\to 1}(s-1)^r L_S(s, \mathrm{Pic}(X))$
  (partial Artin L-function).

4. Working hypothesis

(Manin, Batyrev, Peyre)

$\exists\ F_0/F$ (finite) and $U_0 \subset X$ (Zariski open) s.t.
$\forall\ F'/F_0$ (finite) and $\forall\ U \subset U_0$ (Zariski open)

$$N(U(F'), B) \sim \alpha(X)\beta(X)\tau(\mathcal{K})B \log(B)^{r-1},$$

for $B \to \infty$, where

- $r = \mathrm{rk}\,\mathrm{Pic}(X)$;
- $\alpha(X) := \alpha(\Lambda_{\mathrm{eff}}(X), -K)$;
- $\beta(X) := \#\mathrm{Br}(X)/\mathrm{Br}(F)$;
- $\tau(\mathcal{K}) := \int_{\overline{X(F)} \subset X(\mathbb{A}_F)} \omega_{\mathcal{K}}$.

(over F').

5. Numerical data

- $S_1 : w^2 = 5x_0^4 + 7x_1^4 - 3x_2^4$, $r = 1$;
- $S_2 : w^2 = x_0^4 + x_1^4 - 2x_2^4$, $r = 2$;

S_1	S_2
108	505
124	636
118	652
127	672
113	740
116	780
125	866
136	798
115	822
115	808
117	838
121	766
116	942
129	880
142	826
134	796
129	926
116	896
128	816
129	940

(in steps of 250).

6. “Soft” analysis

Lower bounds:

- X_F (split) del Pezzo surface $\Rightarrow$

$$N(U(F), B) \gg B$$

  $\forall\ U \subset X$ (Zariski open);

- (Swinnerton-Dyer 98) $X_{\mathbb{Q}}$ del Pezzo with two $\mathbb{Q}$-rational skew lines $\Rightarrow$

$$N(U(\mathbb{Q}), B) \gg B \log(B)^{r-1};$$

- (Manin 93) X_F Fano 3-fold $\Rightarrow$
  $\forall\, U \subset X\ \exists\ F'/F$ (finite) s.t.

$$N(U(F'), B) \gg B.$$

Upper bounds - del Pezzo surfaces over $\mathbb{Q}$:

$U =$ complement to exceptional curves.

- (Hooley 80) $x_0^3+x_1^3+x_2^3+x_3^3 = 0$: $O(B^{5/3+\epsilon})$;
- (Manin-T. 93) split degree 5: $O(B^{1+\epsilon})$;
- (Heath-Brown 02) cubic: $O(B^{52/27+\epsilon})$;
- (Heath-Brown 02) cubic with 3 coplaner lines: $O(B^{4/3+\epsilon})$;
- (Salberger 02) degree 4 with a conic: $O(B^{1+\epsilon})$.

Universal torsors:

- $\mathcal{T}_X \xrightarrow{T} X^0 \subset X$;
  with $\mathfrak{X}^*(T) = \mathrm{Pic}(X)$;

- $N(U(F), B) =$

  $\#\{$lattice points in a bounded domain$\}$

  (subject to coprimality conditions).

Idea: $N(X^0, B) \sim$ volume of the domain.

Example:

- $\mathbb{A}^{n+1} \setminus 0 \xrightarrow{\mathbb{G}_m} \mathbb{P}^n$;

- $N(\mathbb{P}^n, B) =$

$$\#\{(x_0, ..., x_n) \in \mathbb{Z}^{n+1}_{\mathrm{prim}}/\pm,\ |x_j| \leq B \quad \forall j\}.$$

Implementation:

- (Salberger 98, de la Breteche 01) toric Fano over $\mathbb{Q}$;

- (de la Breteche 02) $X = \bar{M}_{0,5}$, $\mathcal{T}_X = \mathrm{Gr}(2,5)$: count $(x_{ij} \in \mathbb{Z} \setminus 0, i, j \in [1, ..., 5])$ s.t.
  - $(x_{ij}, x_{ik}) = 1$;
  - $\max(|x_{ij}x_{ik}x_{lk}x_{lm}x_{jm}|) \leq B$;
  - $x_{ij}x_{kl} - x_{ik}x_{jl} + x_{jk}x_{il} = 0$ ($\mathrm{Gr}(2,5)$-equations).

Problem - Cayley cubic:

$$S : x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2x_3 = 0$$

$\mathcal{T}_S$ given by:

$$\begin{array}{rcrcl}
y_1y_{13}y_{14} & + & y_2y_{23}y_{24} & = & y_{34}y_{12,34} \\
y_1y_{12}y_{14} & + & y_3y_{23}y_{34} & = & y_{24}y_{13,24} \\
y_2y_{12}y_{24} & + & y_3y_{13}y_{34} & = & y_{14}y_{14,23} \\
-y_3y_{13}y_{23} & + & y_4y_{14}y_{24} & = & y_{12}y_{12,34} \\
-y_2y_{12}y_{23} & + & y_4y_{14}y_{34} & = & y_{13}y_{13,24} \\
-y_1y_{12}y_1 & + & y_4y_{24}y_{34} & = & y_{23}y_{14,23} \\
y_2y_4y_{24}^2 & + & y_1y_3y_{13}^2 & = & y_{12,34}y_{14,23} \\
-y_1y_2y_{12}^2 & + & y_3y_4y_{34}^2 & = & y_{13,24}y_{14,23} \\
y_1y_4y_{14}^2 & - & y_2y_3y_{23}^2 & = & y_{12,34}y_{13,24}
\end{array}$$

Height: $\max(|y_iy_jy_ky_{ij}y_{ik}y_{kj}|) \leq B$.

Coprimality: y_{ij} all pairwise coprime,

$(y_{ij}, y_k) = (y_{ij,kl}, y_k) = (y_{ij,kl}, y_{jl}) = 1$.

7. "Hard" analysis

Results:

- $X_d \subset \mathbb{P}^n$ (circle method);

- (Franke 89) G/P, (Strauch 01) twisted products of G/P;

- (Batyrev-T. 95) $X \supset T$;

- (Strauch-T. 99) $X \supset G/U$;

- (Chambert-Loir-T. 02) $X \supset \mathbb{G}_a^n$;

- (Shalika-T. 02) $X \supset U$ (bi-equivariant);

- (Shalika-Takloo-Bighash-T. 02) $X \supset G$ De Concini-Procesi varieties.

Idea: Harmonic analysis and zeta functions.

Example:

- $X \supset G$ (unipotent), $d = \dim(X)$,
  $X \setminus G = \cup_{\alpha \in \mathcal{A}} D_\alpha$, $-K = \sum_\alpha \kappa_\alpha D_\alpha$;

- Height: H : $\mathrm{Pic}(X)_{\mathbb{C}} \times G(\mathbb{A}_F) \to \mathbb{C}$;

- Zeta function: $\mathcal{Z}(\mathbf{s}, g) := \sum_{\gamma \in G(F)} H(\mathbf{s}; \gamma g)^{-1}$;

- Fourier expansion (in L^2): $\mathcal{Z}(\mathbf{s}, g) = \sum_\pi \mathcal{Z}_\pi(\mathbf{s}; g)$.

- Trivial representation (main pole):
  $$\prod_{v \nmid \infty} \left(\sum_{A \subset \mathcal{A}} \frac{\# D_A^0(\mathbb{F}_q)}{q^d} \prod_{\alpha \in A} \frac{q - 1}{q^{s_\alpha - \kappa_\alpha + 1} - 1} \right),$$
  where $D_A = \cap_{\alpha \in A} D_\alpha$, $D_A^0 = D_A \setminus \cup_{A' \supsetneq A} D_{A'}$.

II. Integral points

Data:

- F, $S \subset \mathrm{Val}(F)$ finite, non-archimedean; $\mathfrak{o}_S$;
- (X, D), $U_0 := X \setminus D$;
- $(\mathcal{X}, \mathcal{D})$ models over $\mathfrak{o}_S$;
- $U_0(\mathfrak{o}_S)$ - set of $(\mathcal{D}, S)$-integral points;
- $-(\mathcal{K} + \mathcal{D})$ metrized (very) ample;
- $H_{-(K+D)} : U_0(\mathfrak{o}_S) \subset X(F) \to \mathbb{R}_{>0}$ height.

Counting function:

$$N(U(\mathfrak{o}_S), B) := \#\{x \in U(\mathfrak{o}_S) \,|\, H_{-(K+D)}(x) \le B\}$$

$U \subset U_0$ Zariski open .

1. Problems

- $U_0(\mathfrak{o}_S) \neq \emptyset$?

- $U_0(\mathfrak{o}_S)$ Zariski dense?

Conjecture (Potential density):
$\exists\ F'/F$ (finite) $S' \supset S$, models over $\mathfrak{o}_{S'}$, s.t. $U_0(\mathfrak{o}_{S'})$ is Zariski dense.

- $N(U(\mathfrak{o}_S), B) \sim$?? for $B \to \infty$
  (and some Zariski open $U \subset U^0$)?

Working hypothesis (Linear growth):
$\exists\ U \subset U_0$ s.t. $N(U(\mathfrak{o}_S), B) = O(B^{1+\epsilon})$.

2. ???

$$N(U(\mathfrak{o}_S), B) \sim \alpha\beta\tau B \log(B)^{r_S - 1},$$

where

- $r_S = \text{rk Pic}(U_0) + \sum_{v \in S \cup S_\infty} r_v$;
- $r_\nu = \dim \text{Cl}(\mathcal{D}_\nu)$ (Clemens polytope);
- $\alpha \in \mathbb{Q}$;
- $\beta \in \mathbb{N}$;
- $\tau = \tau^S(U_0(\mathfrak{o}_S)) \cdot \prod_{v \in S \cup S_\infty} \left(\sum_{\sigma \in \text{Cl}_{\max}(\mathcal{D}_\nu)} \tau_v(\sigma) \right)$
- $\tau_v(\sigma)$ Tamagawa volume of σ, (adjunction!).