TAMAGAWA NUMBERS OF DIAGONAL CUBIC SURFACES OF HIGHER RANK

EMMANUEL PEYRE AND YURI TSCHINKEL

ABSTRACT. We consider diagonal cubic surfaces defined by an equation of the form

$$ax^3 + by^3 + cz^3 + dt^3 = 0.$$

Numerically, one can find all rational points of height $\leqslant B$ for B in the range of up to 10^5, thanks to a program due to D. J. Bernstein. On the other hand, there are precise conjectures concerning the constants in the asymptotics of rational points of bounded height due to Manin, Batyrev and the authors. Changing the coefficients one can obtain cubic surfaces with rank of the Picard group varying between 1 and 4. We check that numerical data are compatible with the above conjectures. In a previous paper we considered cubic surfaces with Picard groups of rank one with or without Brauer-Manin obstruction to weak approximation. In this paper, we test the conjectures for diagonal cubic surfaces with Picard groups of higher rank.

CONTENTS

1.	Introduction	1
2.	Description of the conjectural constant	3
3.	The Galois module $\mathrm{Pic}(\overline{V})$	5
4.	Euler product for the good places	10
5.	Density at the bad places	11
6.	The constant $\alpha(V)$	13
7.	Some statistical formulae	18
8.	Presentation of the results	19
	References	25

1. INTRODUCTION

This paper is devoted to numerical tests of a refined version of a conjecture of Manin about the number of points of bounded height on Fano varieties (see [BM], [FMT], [Pe], or [BT] for a description of the conjectures). The choice of diagonal cubic surfaces to test these conjectures was motivated by the work of Heath-Brown [H-B] in which he treated the cases

$$X^3 + Y^3 + Z^3 + aT^3 = 0$$

for $a = 2$ or 3. The results he obtained were used as a benchmark for the subsequent attempts to interpret the asymptotic constants (see, in particular, [S-D], [Pe] and [PT]).

Date: January 18, 2001.

More precisely, we consider a diagonal cubic surface $V \subset \mathbf{P}^3_{\mathbf{Q}}$ given by an equation of the form

$$aX^3 + bY^3 + cZ^3 + dT^3 = 0.$$

Let H be the height function on $\mathbf{P}^3(\mathbf{Q})$ defined by the formula: for any $Q = (x : y : z : t)$ in $\mathbf{P}^3(\mathbf{Q})$, one has

$$H(Q) = \max\{|x|, |y|, |z|, |t|\} \text{ if } \begin{cases} (x, y, z, t) \in \mathbf{Z}^4, \\ \gcd(x, y, z, t) = 1. \end{cases}$$

Let U be the complement in V to the 27 lines. We are interested in the asymptotic behavior of the cardinal

$$N_{U,H}(B) = \#\{\, Q \in U(\mathbf{Q}) \mid H(Q) \leqslant B \,\}$$

as B goes to infinity.

Assume that $V(\mathbf{Q})$ is Zariski dense, which by a result of Segre (see [Man2, §29,§30]) is equivalent to $V(\mathbf{Q}) \neq \emptyset$. It is expected that

$$N_{U,H}(B) = BP(\log(B)) + o(B)$$

as B goes to $+\infty$, where P is a polynomial of degree $\operatorname{rk} \operatorname{Pic}(V) - 1$, with leading coefficient $\boldsymbol{\theta}_H(V)$. This constant has a conjectural description. The goal is to compute $\boldsymbol{\theta}_H(V)$ explicitly in the examples at hand and to compare it with the constant obtained from numerical data. Our previous paper [PT] was devoted to surfaces with Picard groups of rank one with or without Brauer-Manin obstruction to weak approximation. In this paper, we consider examples with Picard groups of higher rank.

Note that in these examples the relative error term

$$(N_{U,H}(B) - \boldsymbol{\theta}_H(V) B (\log B)^{\operatorname{rk}\operatorname{Pic}(V)-1}) / B (\log B)^{\operatorname{rk}\operatorname{Pic}(V)-1}$$

is expected to decrease more slowly. Indeed, if $\operatorname{rk} \operatorname{Pic}(V) = 1$ this error term is expected to decrease as $1/B^{\epsilon}$ for some $\epsilon > 0$, whereas for higher ranks it should be comparable to $1/\log B$. Thus we decided not only to compare the conjectural constant $\boldsymbol{\theta}_H(V)$ with the quotient

$$N_{U,H}(B) / B (\log B)^{\operatorname{rk}\operatorname{Pic}(V)-1}$$

with $B = 10^5$, but also to take into account that a polynomial P of degree $\operatorname{rk} \operatorname{Pic}(V) - 1$ should appear in the asymptotics and use a naive statistical formula to estimate its leading coefficient $\boldsymbol{\theta}_H^{\mathrm{stat}}(V)$ from the data. We observe a quite good accordance: the difference between $\boldsymbol{\theta}_H(V)$ and $\boldsymbol{\theta}_H^{\mathrm{stat}}(V)$ is less than 6% in the examples. Moreover the fact that $\boldsymbol{\theta}_H^{\mathrm{stat}}(V)$ is nearer to $\boldsymbol{\theta}_H(V)$ than the above quotient is in itself a point in favor of the conjecture: indeed there is no obvious purely statistical reason for which this should be true in general.

The paper is organized as follows: in section 2 we define $\boldsymbol{\theta}_H(V)$. Section 3 contains the description of the Galois action on the geometric Picard group $\operatorname{Pic}(\overline{V})$. In section 4 we compute the Euler product corresponding to good reduction places. In section 5 we explain how to compute the local densities at the places of bad reduction. In section 6 we determine in each case the value of the geometric constant $\alpha(V)$ defined in §2. Section 7 is devoted to the description of statistical tools we used to analyze the numerical data. In section 8 we present the results.

We would like to thank the referee for the improvements he suggested.

2. DESCRIPTION OF THE CONJECTURAL CONSTANT

In this section we give a short description of the conjectural asymptotic constant for heights defined by an adelic metrization of the anticanonical line bundle (see [Pe] for more details and [BT] for a discussion in a more general setting).

Notations 2.1. For any field E, we denote by $\overline{E}$ an algebraic closure of E. If X is a variety over E, then $X(E)$ denotes the set of rational points of X and $\overline{X}$ the product $X \times_{\operatorname{Spec}(E)} \operatorname{Spec} \overline{E}$. The cohomological Brauer group $\operatorname{Br}(X)$ is defined as the étale cohomology group $H^2_{\text{ét}}(X, \mathbf{G}_m)$. For any A in $\operatorname{Br}(X)$, any extension E' of E and any P in $V(E')$, we denote by $A(P)$ the evaluation of A at P.

For a number field F we denote by $\operatorname{Val}(F)$ the set of places of F and by $\operatorname{Val}_f(F)$ the set of finite places. The absolute discriminant of F is denoted by d_F. For any place v of F, let F_v be the v-adic completion of F. If v is finite, then $\mathscr{O}_v$ is the ring of v-adic integers and $\mathbf{F}_v$ the residue field. By global class field theory we have an exact sequence

$$0 \to \operatorname{Br}(F) \to \bigoplus_{v \in \operatorname{Val}(F)} \operatorname{Br}(F_v) \xrightarrow{\sum \operatorname{inv}_v} \mathbf{Q}/\mathbf{Z} \to 0. \tag{2.1}$$

In the following, V is a smooth projective geometrically integral variety over a number field F satisfying the conditions:

(i) $H^i(V, \mathscr{O}_V) = 0$ for $i = 1$ or 2,
(ii) $\operatorname{Pic}(\overline{V})$ has no torsion,
(iii) the anticanonical line bundle ω_V^{-1} belongs to the interior of the cone of classes of effective divisors $\Lambda_{\text{eff}}(V) \subset \operatorname{Pic}(V) \otimes_{\mathbf{Z}} \mathbf{R}$.

The adelic space $V(\boldsymbol{A}_F)$ of V is the product $\prod_{v \in \operatorname{Val}(F)} V(F_v)$. By [CT, lemma 1], for any class A in $\operatorname{Br}(V)$, one has a map ρ_A defined as the composition

$$\begin{array}{ccccc} V(\boldsymbol{A}_F) & \to & \bigoplus_{v \in \operatorname{Val}(F)} \operatorname{Br}(F_v) & \xrightarrow{\sum \operatorname{inv}_v} & \mathbf{Q}/\mathbf{Z} \\ (P_v)_{v \in \operatorname{Val}(F)} & \mapsto & (A(P_v))_{v \in \operatorname{Val}(F)}. & & \end{array}$$

Then one defines

$$V(\boldsymbol{A}_F)^{\operatorname{Br}} = \bigcap_{A \in \operatorname{Br}(V)} \ker(\rho_A) \subset V(\boldsymbol{A}_F).$$

By the exact sequence (2.1), one has the inclusion

$$\overline{V(F)} \subset V(\boldsymbol{A}_F)^{\operatorname{Br}}$$

where $\overline{V(F)}$ denotes the topological closure of the set of rational points. Conjecturally both sets coincide for cubic surfaces. (See also the text of Swinnerton-Dyer in this volume). There is a *Brauer-Manin obstruction to weak approximation*, as described by Manin in [Man1] and by Colliot-Thélène and Sansuc in [CTS], if one has

$$V(\boldsymbol{A}_F)^{\operatorname{Br}} \neq V(\boldsymbol{A}_F).$$

Let us assume that the height H on V is defined by an adelic metric $(\|\cdot\|_v)_{v \in \operatorname{Val}(F)}$ on ω_V^{-1}. By definition, this means that we consider ω_V^{-1} as a line bundle, that the functions $\|\cdot\|_v$ are v-adically continuous metrics on $\omega_V^{-1}(F_v)$ which for almost all places v are given by a smooth model of V, and that the height of a rational point x of V is given by the formula

$$\forall y \in \omega_V^{-1}(x) - \{0\}, \quad H(x) = \prod_{v \in \operatorname{Val}(F)} \|y\|_v^{-1}$$

where $\omega_V^{-1}(x)$ is the fiber of ω_V^{-1} at x.

If $v \in \mathrm{Val}(F)$ the Haar measure $\mathrm{d}x_v$ on F_v is normalized as follows:

- $\int_{\mathscr{O}_v} \mathrm{d}x_v = 1$ if v is finite,
- $\mathrm{d}x_v$ is the usual Lebesgue measure if $F_v \xrightarrow{\sim} \mathbf{R}$,
- $\mathrm{d}x_v = \mathrm{d}z\,\mathrm{d}\overline{z} = 2\mathrm{d}x\,\mathrm{d}y$ if $F_v \xrightarrow{\sim} \mathbf{C}$.

The metric $\|\cdot\|_v$ defines a measure $\boldsymbol{\omega}_{H,v}$ on the locally compact space $V(F_v)$. In local v-adic analytic coordinates $x_{1,v} \dots x_{n,v}$ on $V(F_v)$ this measure is given by the formula

$$\boldsymbol{\omega}_{H,v} = \left\| \frac{\partial}{\partial x_{1,v}} \wedge \dots \wedge \frac{\partial}{\partial x_{n,v}} \right\|_v \mathrm{d}x_{1,v} \dots \mathrm{d}x_{n,v}\,.$$

If M is a discrete representation of $\mathrm{Gal}(\overline{F}/F)$ over $\mathbf{Q}$, then for any finite place $\mathfrak{p}$ of F, the local term of the corresponding Artin L-function is defined as follows: we choose an algebraic closure $\overline{F}_\mathfrak{p}$ of $F_\mathfrak{p}$ containing $\overline{F}$. We get an exact sequence

$$1 \to I_\mathfrak{p} \to D_\mathfrak{p} \to \mathrm{Gal}(\overline{\mathbf{F}}_\mathfrak{p}/\mathbf{F}_\mathfrak{p}) \to 1$$

where $D_\mathfrak{p}$ is the decomposition group and $I_\mathfrak{p}$ the inertia. We denote by $\widetilde{\mathrm{Fr}}_\mathfrak{p}$ a lifting of the Frobenius map to $D_\mathfrak{p} \subset \mathrm{Gal}(\overline{F}/F)$ (which up to conjugation depends only on $\mathfrak{p}$), and put

$$L_\mathfrak{p}(s, M) = \frac{1}{\mathrm{Det}(1 - (\#\mathbf{F}_\mathfrak{p})^{-s}\widetilde{\mathrm{Fr}}_\mathfrak{p} \mid M^{I_\mathfrak{p}})}.$$

We fix a finite set S of bad places containing the archimedean ones so that V admits a smooth projective model $\mathscr{V}$ over the ring of S-integers $\mathscr{O}_S$. For any $\mathfrak{p}$ in $\mathrm{Val}(F) - S$ we consider

$$L_\mathfrak{p}(s, \mathrm{Pic}(\overline{V})) = L_\mathfrak{p}(s, \mathrm{Pic}(\overline{V}) \otimes_\mathbf{Z} \mathbf{Q}).$$

The corresponding global L-function is given by the Euler product

$$L_S(s, \mathrm{Pic}(\overline{V})) = \prod_{\mathfrak{p} \in \mathrm{Val}(F) - S} L_\mathfrak{p}(s, \mathrm{Pic}(\overline{V}))$$

which converges for $\mathrm{Re}\, s > 1$ and has a meromorphic continuation to $\mathbf{C}$ with a pole of order $t = \mathrm{rk}\,\mathrm{Pic}(V)$ at 1. One introduces local convergence factors λ_v given by

$$\lambda_v = \begin{cases} L_v(1, \mathrm{Pic}(\overline{V})) \text{ if } v \in \mathrm{Val}(F) - S, \\ 1 \text{ otherwise.} \end{cases}$$

The Weil conjectures (proved by Deligne) imply that the Tamagawa measure

$$\prod_{v \in \mathrm{Val}(F)} \lambda_v^{-1} \boldsymbol{\omega}_{H,v}$$

converges on $V(\boldsymbol{A}_F)$ (see [Pe, proposition 2.2.2]).

Definition 2.2. The *Tamagawa measure* on $V(\boldsymbol{A}_F)$ corresponding to the adelic metric $(\|\cdot\|_v)_{v \in \mathrm{Val}(F)}$ is defined by

$$\boldsymbol{\omega}_H = \frac{1}{\sqrt{d_F}^{\dim V}} \lim_{s \to 1} (s-1)^t L_S(s, \mathrm{Pic}(\overline{V})) \prod_{v \in \mathrm{Val}(F)} \lambda_v^{-1} \boldsymbol{\omega}_{H,v}.$$

From the arithmetic standpoint, it seems more natural to integrate $\boldsymbol{\omega}_H$ over the closure $\overline{V(F)} \subset V(\boldsymbol{A}_F)$ (as in the original approach to the Tamagawa number). However, computationally, it is easier to work with $V(\boldsymbol{A}_F)^{\mathrm{Br}}$. Therefore, following a suggestion of Salberger, we define here

Definition 2.3.

$$\tau_H(V) = \boldsymbol{\omega}_H(V(\boldsymbol{A})^{\mathrm{Br}}). \tag{2.2}$$

Let $\mathrm{Pic}(V)^\vee$ be the dual lattice to $\mathrm{Pic}(V)$. We denote by $\mathrm{d}\boldsymbol{y}$ the corresponding Lebesgue measure on $\mathrm{Pic}(V)^\vee \otimes_{\mathbf{Z}} \mathbf{R}$ and by

$$\Lambda_{\mathrm{eff}}(V)^\vee = \{\, x \in \mathrm{Pic}(V)^\vee \otimes_{\mathbf{Z}} \mathbf{R} \mid \forall y \in \Lambda_{\mathrm{eff}}(V),\ \langle x, y\rangle \geqslant 0 \,\}$$

the dual cone of $\Lambda_{\mathrm{eff}}(V)$.

Definition 2.4. We define

$$\alpha(V) = \frac{1}{(t-1)!} \int_{\Lambda_{\mathrm{eff}}(V)^\vee} e^{-\langle \omega_V^{-1}, \boldsymbol{y}\rangle} \mathrm{d}\boldsymbol{y}$$

and

$$\beta(V) = \# H^1(k, \mathrm{Pic}(\overline{V})).$$

The theoretical constant attached to V and H is defined as

$$\boldsymbol{\theta}_H(V) = \alpha(V)\beta(V)\tau_H(V). \tag{2.3}$$

In the following sections we compute $\boldsymbol{\theta}_H(V)$ for various diagonal cubic surfaces.

3. The Galois module $\mathrm{Pic}(\overline{V})$

The description of this Galois module is based upon the study of the 27 lines of the cubic. We fix notations for these lines which are slightly different from those given by Colliot-Thélène, Kanevsky and Sansuc in [CTKS, p. 9].

Notations 3.1. From now on V is a diagonal cubic surface V given by an equation of the form

$$aX^3 + bY^3 + cZ^3 + dT^3 = 0 \tag{3.1}$$

where a, b, c and d are strictly positive integers with $\gcd(a, b, c, d) = 1$. Let

$$S = \{\infty, 3\} \cup \{\, p \mid \ p | abcd\}$$

We fix a cubic root α (resp. α', α'') of b/a (resp. c/a, d/a) (which is assumed to be in $\mathbf{Q}$ if b/a (resp. c/a, d/a) is a cube in $\mathbf{Q}$) and we put

$$\beta = \frac{\alpha''}{\alpha'} = \sqrt[3]{\frac{d}{c}}, \qquad \beta' = \frac{\alpha}{\alpha''} = \sqrt[3]{\frac{b}{d}} \qquad \text{and} \qquad \beta'' = \frac{\alpha'}{\alpha} = \sqrt[3]{\frac{c}{b}}.$$

We also consider

$$\gamma = \frac{\alpha}{\alpha'\alpha''} = \sqrt[3]{\frac{ab}{cd}}, \qquad \gamma' = \frac{\alpha'}{\alpha''\alpha} = \sqrt[3]{\frac{ac}{bd}} \qquad \text{and} \qquad \gamma'' = \frac{\alpha''}{\alpha\alpha'} = \sqrt[3]{\frac{ad}{bc}}.$$

We denote by θ a primitive third root of one. The 27 lines of the cubic surface (3.1) are given by the following equations, where i belongs to $\mathbf{Z}/3\mathbf{Z}$:

$$L(i){:}\begin{cases}X+\theta^i\alpha Y=0,\\ Z+\theta^i\beta T=0.\end{cases}\quad L'(i){:}\begin{cases}X+\theta^i\alpha Y=0,\\ Z+\theta^{i+1}\beta T=0.\end{cases}\quad L''(i){:}\begin{cases}X+\theta^i\alpha Y=0,\\ Z+\theta^{i+2}\beta T=0.\end{cases}$$

$$M(i){:}\begin{cases}X+\theta^i\alpha' Z=0,\\ T+\theta^i\beta' Y=0.\end{cases}\quad M'(i){:}\begin{cases}X+\theta^i\alpha' Z=0,\\ T+\theta^{i+1}\beta' Y=0.\end{cases}\quad M''(i){:}\begin{cases}X+\theta^i\alpha' Z=0,\\ T+\theta^{i+2}\beta' Y=0.\end{cases}$$

$$N(i){:}\begin{cases}X+\theta^i\alpha'' T=0,\\ Y+\theta^i\beta'' Z=0.\end{cases}\quad N'(i){:}\begin{cases}X+\theta^i\alpha'' T=0,\\ Y+\theta^{i+1}\beta'' Z=0.\end{cases}\quad N''(i){:}\begin{cases}X+\theta^i\alpha'' T=0,\\ Y+\theta^{i+2}\beta'' Z=0.\end{cases}$$

Let K be the field $\mathbf{Q}(\theta,\alpha,\alpha',\alpha'')$. It is a Galois extension of $\mathbf{Q}$. In the generic case, K is an extension of degree 54 with a Galois group isomorphic to

$$(\mathbf{Z}/3\mathbf{Z})^3 \rtimes \mathbf{Z}/2\mathbf{Z}.$$

It is generated by the elements c, τ, τ' and τ'' characterized by their action on θ, α, α' and α''.

	θ	α	α'	α''
c	θ^2	α	α'	α''
τ	θ	$\theta\alpha$	α'	α''
τ'	θ	α	$\theta\alpha'$	α''
τ''	θ	α	α'	$\theta\alpha''$

Their action on the 27 lines is given as follows: for τ we have

$$\begin{tikzcd}[column sep=small] L(i) \arrow[rr] & & L''(i+1) \arrow[dl] \\ & L'(i+2) \arrow[ul] & \end{tikzcd}\quad \begin{tikzcd}[column sep=small] M(i) \arrow[rr] & & M'(i) \arrow[dl] \\ & M''(i) \arrow[ul] & \end{tikzcd}\quad\text{and}\quad \begin{tikzcd}[column sep=small] N(i) \arrow[rr] & & N''(i) \arrow[dl] \\ & N'(i) \arrow[ul] & \end{tikzcd} \tag{3.2}$$

for τ':

$$\begin{tikzcd}[column sep=small] L(i) \arrow[rr] & & L''(i) \arrow[dl] \\ & L'(i) \arrow[ul] & \end{tikzcd}\quad \begin{tikzcd}[column sep=small] M(i) \arrow[rr] & & M''(i+1) \arrow[dl] \\ & M'(i+2) \arrow[ul] & \end{tikzcd}\quad\text{and}\quad \begin{tikzcd}[column sep=small] N(i) \arrow[rr] & & N'(i) \arrow[dl] \\ & N''(i) \arrow[ul] & \end{tikzcd} \tag{3.3}$$

for τ'':

$$\begin{tikzcd}[column sep=small] L(i) \arrow[rr] & & L'(i) \arrow[dl] \\ & L''(i) \arrow[ul] & \end{tikzcd}\quad \begin{tikzcd}[column sep=small] M(i) \arrow[rr] & & M''(i) \arrow[dl] \\ & M'(i) \arrow[ul] & \end{tikzcd}\quad\text{and}\quad \begin{tikzcd}[column sep=small] N(i) \arrow[rr] & & N''(i+1) \arrow[dl] \\ & N'(i+2) \arrow[ul] & \end{tikzcd} \tag{3.4}$$

for c:

$$\begin{tikzcd}[column sep=small] L(0) & L'(0) \arrow[r, leftrightarrow] & L''(0) & M(0) & M'(0) \arrow[r, leftrightarrow] & M''(0) & N(0) & N'(0) \arrow[r, leftrightarrow] & N''(0) \\ L(1) \arrow[d, leftrightarrow] & L'(1) \arrow[dr, leftrightarrow] & L''(1) \arrow[dl, leftrightarrow] & M(1) \arrow[d, leftrightarrow] & M'(1) \arrow[dr, leftrightarrow] & M''(1) \arrow[dl, leftrightarrow] & N(1) \arrow[d, leftrightarrow] & N'(1) \arrow[dr, leftrightarrow] & N''(1) \arrow[dl, leftrightarrow] \\ L(2) & L'(2) & L''(2) & M(2) & M'(2) & M''(2) & N(2) & N'(2) & N''(2). \end{tikzcd} \tag{3.5}$$

To describe the relations between the classes of these divisors in $\mathrm{Pic}(\overline{V})$, which shall be useful for the computation of $\alpha(V)$, we consider $\overline{V}$ as the blow-up of a plane $\mathbf{P}^2_{\overline{\mathbf{Q}}}$ in six points P_1, P_2, P_3, P_4, P_5 and P_6. The 27 lines may then be described as the 6 exceptional divisors $E_1, \dots, E_6$, the 15 strict transforms $L_{i,j}$ of the projective lines (P_iP_j) for $1 \leqslant i < j \leqslant 6$ and the 6 strict transforms of the conics Q_i going through all points except P_i. Let Λ be the preimage of a line of $\mathbf{P}^2_{\mathbf{Q}}$ which does not contain any of the points $P_1, \dots, P_6$. Then

$$([\Lambda], [E_1], [E_2], [E_3], [E_4], [E_5], [E_6])$$

is a basis of $\mathrm{Pic}(\overline{V})$ and we have the following relations in $\mathrm{Pic}(\overline{V})$:

$$\begin{aligned} [L_{i,j}] &= [\Lambda] - [E_i] - [E_j] \qquad \text{for } 1 \leqslant i < j \leqslant 6, \\ [Q_i] &= 2[\Lambda] - \sum_{j \neq i} [E_j]. \end{aligned} \tag{3.6}$$

In the following, we choose the projection of $\overline{V}$ to $\mathbf{P}^2_{\mathbf{Q}}$ so that we have the equalities:

$$\begin{aligned}
E_1 &= L(0), & E_2 &= L(1), & E_3 &= L(2), \\
E_4 &= M(1), & E_5 &= M'(2), & E_6 &= M''(0), \\
Q_1 &= L'(1), & Q_2 &= L'(2), & Q_3 &= L'(0), \\
Q_4 &= M(0), & Q_5 &= M'(1), & Q_6 &= M''(2), \\
L_{1,2} &= L''(1), & L_{2,3} &= L''(2), & L_{3,1} &= L''(0), \\
L_{4,5} &= M''(1), & L_{5,6} &= M(2), & L_{6,4} &= M'(0), \\
L_{1,4} &= N(0), & L_{1,5} &= N(1), & L_{1,6} &= N(2), \\
L_{2,4} &= N'(1), & L_{2,5} &= N'(2), & L_{2,6} &= N'(0), \\
L_{3,4} &= N''(2), & L_{3,5} &= N''(0), & L_{3,6} &= N''(1).
\end{aligned} \tag{3.7}$$

Notations 3.2. We consider the étale algebra E_1 over $\mathbf{Q}$ defined as $\mathbf{Q}(\gamma)$ if ab/cd is not a cube in $\mathbf{Q}$ and as $\mathbf{Q}(\theta) \times \mathbf{Q}$ otherwise. Similarly, we define the algebra E_2 (resp. E_3) corresponding to γ' (resp. γ'') and we put

$$E = E_1 \times E_2 \times E_3.$$

We also consider the following elements of $\mathrm{Pic}(\overline{V})$

$$\begin{aligned}
e_0^1 &= [M(0)] + [M(1)] + [M(2)], & e_1^1 &= [M'(0)] + [M'(1)] + [M'(2)], \\
e_2^1 &= [M''(0)] + [M''(1)] + [M''(2)], & e_0^2 &= [N(0)] + [N(1)] + [N(2)], \\
e_1^2 &= [N'(0)] + [N'(1)] + [N'(2)], & e_2^2 &= [N''(0)] + [N''(1)] + [N''(2)], \\
e_0^3 &= [L(0)] + [L(1)] + [L(2)], & e_1^3 &= [L'(0)] + [L'(1)] + [L'(2)], \\
e_2^3 &= [L''(0)] + [L''(1)] + [L''(2)]
\end{aligned}$$

and the sets

$$\mathscr{E}_1 = \{e_0^1, e_1^1, e_2^1\}, \quad \mathscr{E}_2 = \{e_0^2, e_1^2, e_2^2\}, \quad \mathscr{E}_3 = \{e_0^3, e_1^3, e_2^3\},$$

and $\mathscr{E} = \mathscr{E}_1 \cup \mathscr{E}_2 \cup \mathscr{E}_3$.

Lemma 3.3. *The sets $\mathscr{E}_1$, $\mathscr{E}_2$ and $\mathscr{E}_3$ are globally invariant under the action of* $\mathrm{Gal}(K/\mathbf{Q})$ *and the étale algebra corresponding to the set $\mathscr{E}_i$ is isomorphic to E_i.*

Proof. The fact that the sets $\mathscr{E}_1$, $\mathscr{E}_2$ and $\mathscr{E}_3$ are globally invariant follows immediately from the descriptions (3.2)–(3.5). The étale algebra F corresponding to a finite $\mathrm{Gal}(K/\mathbf{Q})$-set $\mathscr{F}$ may be defined as the algebra

$$(K[\mathscr{F}])^{\mathrm{Gal}(K/\mathbf{Q})}$$

where $K[\mathscr{F}]$ is the algebra $K^{\mathscr{F}}$ and where $\mathrm{Gal}(K/\mathbf{Q})$ acts simultaneously on K and $\mathscr{F}$. In the generic case, let us consider

$$\sigma = \tau'\tau'', \quad \sigma' = \tau''\tau \quad \text{and} \quad \sigma'' = \tau\tau'.$$

Then σ sends γ on $\theta\gamma$ and acts trivially on γ', γ'' and θ. We may describe similarly the actions of σ' and σ''. The action of $\mathrm{Gal}(K/\mathbf{Q})$ on $\mathscr{E}_1$ in the generic case is given by the table

	e^1_0	e^1_1	e^1_2
c	e^1_0	e^1_2	e^1_1
σ	e^1_1	e^1_2	e^1_0
σ'	e^1_0	e^1_1	e^1_2
σ''	e^1_0	e^1_1	e^1_2

This implies that if ab/cd is not a cube in $\mathbf{Q}$, then $\mathscr{E}_1$ is isomorphic to

$$\mathrm{Gal}(K/\mathbf{Q})/\,\mathrm{Gal}(K/\mathbf{Q}(\gamma))$$

as a $\mathrm{Gal}(K/\mathbf{Q})$-set. Then the corresponding étale algebra is

$$(K[\mathrm{Gal}(K/\mathbf{Q})/\,\mathrm{Gal}(K/\mathbf{Q}(\gamma))])^{\mathrm{Gal}(K/\mathbf{Q})} \xrightarrow{\sim} K^{\mathrm{Gal}(K/\mathbf{Q}(\gamma))} = \mathbf{Q}(\gamma) = E_1.$$

Similarly if ab/cd is a cube in $\mathbf{Q}$, then we may decompose $\mathscr{E}_1$ into two orbits and we see that the corresponding étale algebra is $\mathbf{Q}(\theta) \times \mathbf{Q} = E_1$. The proofs for $\mathscr{E}_2$ and $\mathscr{E}_3$ are similar. □

Lemma 3.4. *There exists an exact sequence of* $\mathrm{Gal}(K/\mathbf{Q})$ *modules*

$$0 \to \mathbf{Q}^2 \to \mathbf{Q}[\mathscr{E}] \to \mathrm{Pic}(\overline{V}) \otimes_{\mathbf{Z}} \mathbf{Q} \to 0.$$

Proof. By (3.6) and (3.7), we have in $\mathrm{Pic}(\overline{V})$ the relations

$$\begin{aligned}
e^1_0 &= 3[\Lambda] - [E_1] - [E_2] - [E_3] + [E_4] - 2[E_5] - 2[E_6],\\
e^1_1 &= 3[\Lambda] - [E_1] - [E_2] - [E_3] - 2[E_4] + [E_5] - 2[E_6],\\
e^1_2 &= 3[\Lambda] - [E_1] - [E_2] - [E_3] - 2[E_4] - 2[E_5] + [E_6],\\
e^2_0 &= 3[\Lambda] - 3[E_1] - [E_4] - [E_5] - [E_6],\\
e^2_1 &= 3[\Lambda] - 3[E_2] - [E_4] - [E_5] - [E_6],\\
e^2_2 &= 3[\Lambda] - 3[E_3] - [E_4] - [E_5] - [E_6],\\
e^3_0 &= [E_1] + [E_2] + [E_3],\\
e^3_1 &= 6[\Lambda] - 2[E_1] - 2[E_2] - 2[E_3] - 3[E_4] - 3[E_5] - 3[E_6],\\
e^3_2 &= 3[\Lambda] - 2[E_1] - 2[E_2] - 2[E_3]
\end{aligned}$$

which proves that the natural projection from $\mathbf{Q}[\mathscr{E}]$ to $\mathrm{Pic}(\overline{V})\otimes_{\mathbf{Z}}\mathbf{Q}$ is surjective. Moreover one has the relations

$$3\omega_V^{-1} = \sum_{x\in\mathscr{E}_1} x = \sum_{x\in\mathscr{E}_2} x = \sum_{x\in\mathscr{E}_3} x,$$

which gives a homomorphism of $\mathrm{Gal}(K/\mathbf{Q})$-modules

$$\mathbf{Q}^2 \to \mathbf{Q}[\mathscr{E}]$$

and the exact sequence of the lemma. □

Notations 3.5. For any prime p and any finite field extension F of $\mathbf{Q}$, we consider the local factor $\zeta_{F,p}$ of the function ζ_F at p which is defined by

$$\zeta_{F,p}(s) = \prod_{\{\, v\in\mathrm{Val}(F)\,|\,v|p\,\}} (1-\#\mathbf{F}_v^{-s})^{-1}.$$

Let F be an étale algebra over $\mathbf{Q}$ and $F=\prod_{i\in I} F_i$ its decomposition in fields. Put

$$\zeta_F(s) = \prod_{i\in I}\zeta_{F_i}(s) \quad \text{and} \quad \zeta_{F,p}(s) = \prod_{i\in I}\zeta_{F_i,p}(s).$$

For any prime p, we denote by $\nu_F(p)$ the number of components of $F\otimes_{\mathbf{Q}}\mathbf{Q}_p$ of degree one over $\mathbf{Q}_p$.

Proposition 3.6. *With notation as above, for any prime p not in S, one has*

(i) $L_p(s,\mathrm{Pic}(\overline{V})) = \dfrac{\zeta_{E,p}(s)}{\zeta_{\mathbf{Q},p}(s)^2}$,

(ii) $\mathrm{Tr}(\widetilde{\mathrm{Fr}}_p \mid \mathrm{Pic}(\overline{V})) = \nu_E(p)-2$.

Proof. By lemma 3.4, we have

$$L_p(s,\mathrm{Pic}(\overline{V})) = \frac{L_p(s,\mathbf{Q}[\mathscr{E}])}{L_p(s,\mathbf{Q})^2}.$$

Thus it is enough to prove that if E is an arbitrary étale algebra over $\mathbf{Q}$ corresponding to a $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-set $\mathscr{E}$ and if p is a prime such that $E/\mathbf{Q}$ is not ramified at p, then

$$\zeta_{E,p}(s) = L_p(s,\mathbf{Q}[\mathscr{E}]).$$

This well-known assertion follows from the fact that the components of $E\otimes\mathbf{Q}_p$ are in bijection with the orbits of $\widetilde{\mathrm{Fr}}_p$ in $\mathscr{E}$, and the degree of each component is the length of the corresponding orbit. This proves (i).

But this also shows that

$$\mathrm{Tr}(\widetilde{\mathrm{Fr}}_p \mid \mathbf{Q}[\mathscr{E}]) = \nu_E(p)$$

which implies (ii). □

Remark 3.7. Thus the factor λ'_p which was defined in proposition 5.1 in [PT] coincides with $L_p(1,\mathrm{Pic}(\overline{V}))$ at the good places (as suggested by the referee of that paper).

4. Euler product for the good places

We need to compute the number of solutions of (3.1) modulo p for all primes not in S.

Proposition 4.1. *For any prime p not in S, one has*

$$\frac{\#V(\mathbf{F}_p)}{p^2} = 1 + \frac{\nu_E(p) - 2}{p} + \frac{1}{p^2}$$

where E is the étale algebra defined in §3.

Proof. By a result of Weil (see [Man2, theorem 23.1]),

$$\#V(\mathbf{F}_p) = 1 + \mathrm{Tr}(\mathrm{Fr}_p \mid \mathrm{Pic}(\overline{V}))p + p^2.$$

Proposition 3.6 implies that

$$\mathrm{Tr}(\mathrm{Fr}_p \mid \mathrm{Pic}(\overline{V})) = \nu_E(p) - 2. \quad \square$$

Remark 4.2. We could have proved this result directly as in [PT]. Let $N(p)$ be the number of solutions of (3.1) in $\mathbf{F}_p^4$. We have

$$\#V(\mathbf{F}_p) = \frac{N(p) - 1}{p - 1}.$$

By [IR, §8.7 theorem 5], one has

$$N(p) = p^3 + \sum \overline{\chi}_1(a)\overline{\chi}_2(b)\overline{\chi}_3(c)\overline{\chi}_4(d)J_0(\chi_1, \chi_2, \chi_3, \chi_4),$$

where the sum is taken over all quadruples $(\chi_1, \ldots, \chi_4)$ of nontrivial cubic characters from $\mathbf{F}_p^*$ to $\mathbf{C}^*$ such that $\chi_1\chi_2\chi_3\chi_4 = 1$ and where

$$J_0(\chi_1, \chi_2, \chi_3, \chi_4) = \sum_{t_1+\cdots+t_4=0} \prod_{i=1}^{4} \chi_i(t_i),$$

the characters being extended by $\chi_i(0) = 0$. For $p \equiv 2$ mod 3 there are no nontrivial characters and the formula is obvious. Otherwise there are exactly two nontrivial conjugated characters χ and $\overline{\chi}$. By [PT, proof of prop. 4.1], we have

$$J_0(\chi_1, \chi_2, \chi_3, \chi_4) = p(p - 1)$$

and

$$\#V(\mathbf{F}_p) = 1 + p(1 + \sum \chi_1(a)\chi_2(b)\chi_3(c)\chi_4(d)) + p^2$$

where the sum is taken over the same quadruples as above. The formula

$$\sum \chi_1(a)\chi_2(b)\chi_3(c)\chi_4(d) = \chi\Big(\frac{ab}{cd}\Big) + \overline{\chi}\Big(\frac{ab}{cd}\Big) + \chi\Big(\frac{ac}{bd}\Big) + \overline{\chi}\Big(\frac{ac}{bd}\Big) + \chi\Big(\frac{ad}{bc}\Big) + \overline{\chi}\Big(\frac{ad}{bc}\Big)$$

implies the result.

Notations 4.3. For any place v of $\mathbf{Q}$, we put

$$\lambda_v = \begin{cases} \frac{\zeta_{E,v}(s)}{\zeta_{\mathbf{Q},v}(s)^2} & \text{if } v \text{ is finite,} \\ 1 & \text{otherwise.} \end{cases}$$

Remark 4.4. By proposition 3.6, $\lambda_p = L_p(1, \mathrm{Pic}(\overline{V}))$ if $p \in \mathrm{Val}(\mathbf{Q}) - S$. Thus the Tamagawa measure $\boldsymbol{\omega}_H$ is given by the formula

$$\boldsymbol{\omega}_H = \lim_{s \to 1} (s-1)^{\mathrm{rk}\,\mathrm{Pic}(V)} \Big(\frac{\zeta_E(s)}{\zeta_{\mathbf{Q}}(s)^2} \Big) \times \prod_{v \in \mathrm{Val}(\mathbf{Q})} \lambda_v^{-1} \boldsymbol{\omega}_{H,v}.$$

By lemmata 3.2 and 3.4 in [PT] and lemma 5.4.6 in [Pe], for any p in $\mathrm{Val}(\mathbf{Q}) - S$ one has

$$\boldsymbol{\omega}_{H,p}(V(\mathbf{Q}_p)) = \frac{\#V(\mathbf{F}_p)}{p^2}$$

(see also [Pe, lemma 2.2.1] and [PT, remark 5.2]). Therefore, the local factor at a good place p is given by

$$\begin{array}{ll}
\Big(1-\frac{1}{p}\Big)^7 \Big(1+\frac{7}{p}+\frac{1}{p^2}\Big) & \text{if } p \equiv 1 \bmod 3 \text{ and } \nu_E(p) = 9 \\
\Big(1-\frac{1}{p}\Big)^4 \Big(1-\frac{1}{p^3}\Big) \Big(1+\frac{4}{p}+\frac{1}{p^2}\Big) & \text{if } p \equiv 1 \bmod 3 \text{ and } \nu_E(p) = 6 \\
\Big(1-\frac{1}{p}\Big) \Big(1-\frac{1}{p^3}\Big)^2 \Big(1+\frac{1}{p}+\frac{1}{p^2}\Big) & \text{if } p \equiv 1 \bmod 3 \text{ and } \nu_E(p) = 3 \\
\Big(1-\frac{1}{p}\Big)^{-2} \Big(1-\frac{1}{p^3}\Big)^3 \Big(1-\frac{2}{p}+\frac{1}{p^2}\Big) & \text{if } p \equiv 1 \bmod 3 \text{ and } \nu_E(p) = 0 \\
\Big(1-\frac{1}{p}\Big) \Big(1-\frac{1}{p^2}\Big)^3 \Big(1+\frac{1}{p}+\frac{1}{p^2}\Big) & \text{if } p \equiv 2 \bmod 3.
\end{array}$$

We get (for the good places) the factors C_0, C_1, C_2 and C_3 where

$$C_0 = \prod_{\substack{p \nmid 3abcd, \\ p \equiv 2 \bmod 3.}} \Big(1 - \frac{1}{p^3}\Big)\Big(1 - \frac{1}{p^2}\Big)^3,$$

$$C_1 = \prod_{\substack{p \nmid 3abcd, \\ p \equiv 1 \bmod 3, \\ \nu_E(p) = 9.}} \Big(1 - \frac{1}{p}\Big)^7 \Big(1 + \frac{7}{p} + \frac{1}{p^2}\Big),$$

$$C_2 = \prod_{\substack{p \nmid 3abcd, \\ p \equiv 1 \bmod 3, \\ \nu_E(p) = 6.}} \Big(1 - \frac{1}{p^3}\Big)\Big(1 - \frac{1}{p}\Big)^4 \Big(1 + \frac{4}{p} + \frac{1}{p^2}\Big),$$

$$C_3 = \prod_{\substack{p \nmid 3abcd, \\ p \equiv 1 \bmod 3, \\ \nu_E(p) = 0 \text{ or } 3.}} \Big(1 - \frac{1}{p^3}\Big)^3.$$

These products converge rapidly and are easily approximated.

5. Density at the bad places

In this section we restrict to cubic surfaces with equations of the form

$$X^3 + Y^3 + qZ^3 + q^2T^3 = 0 \tag{5.1}$$

with q prime and

$$aX^3 + aY^3 + qZ^3 + qT^3 = 0 \tag{5.2}$$

with q prime and a an integer coprime to q.

Notations 5.1. If V is defined by the equation (3.1), and p is a prime, then we consider

$$N^*(p^r) = \#\{(x,y,z,t) \in (\mathbf{Z}/p^r\mathbf{Z})^4 - (p\mathbf{Z}/p^r\mathbf{Z})^4 \mid ax^3+by^3+cz^3+dt^3=0 \text{ in } \mathbf{Z}/p^r\mathbf{Z}\}$$

Remark 5.2. By [PT, lemmata 3.2 and 3.4], there is an explicit integer r_0 such that

$$\boldsymbol{\omega}_{H,p}(V(\mathbf{Q}_p)) = \frac{1}{1-p^{-1}} \times \frac{N^*(p^{r_0})}{p^{3r_0}}.$$

If $p = 3$ and $3 \not| \, abcd$, then a direct computation in $(\mathbf{Z}/9\mathbf{Z})^4$ gives the value of $N^*(9)$ and thus of $\boldsymbol{\omega}_{H,p}(V(\mathbf{Q}_p))$. Thus, in the following lemma we restrict to the case when V is given by (5.1) or (5.2) and $p = q$.

Lemma 5.3. *If V is given by the equation*

$$X^3 + Y^3 + pZ^3 + p^2T^3 = 0$$

then for $r \geqslant 2$,

$$\frac{N^*(p^r)}{p^{3r}} = \begin{cases} 1 - \frac{1}{p} & \text{if } p \equiv 2 \bmod 3, \\ 3\left(1 - \frac{1}{p}\right) & \text{if } p \equiv 1 \bmod 3, \\ \frac{2}{3} & \text{if } p = 3. \end{cases}$$

If V is given by the equation

$$aX^3 + aY^3 + pZ^3 + pT^3 = 0,$$

with $p \not| \, a$, then for $r \geqslant 3$,

$$\frac{N^*(p^r)}{p^{3r}} = \begin{cases} 1 - \frac{1}{p^2} & \text{if } p \equiv 2 \bmod 3, \\ 3\left(1 - \frac{1}{p^2}\right) & \text{if } p \equiv 1 \bmod 3, \\ \frac{4}{3} & \text{if } p = 3. \end{cases}$$

Remark 5.4. This lemma implies that if V is given by the first equation then the local factor at p is given by

$$\lambda_p \boldsymbol{\omega}_{H,p}(V(\mathbf{Q}_p)) = \begin{cases} \left(1 - \frac{1}{p^2}\right)\left(1 - \frac{1}{p}\right) & \text{if } p \equiv 2 \bmod 3, \\ 3\left(1 - \frac{1}{p}\right)^3 & \text{if } p \equiv 1 \bmod 3, \\ \frac{4}{9} & \text{if } p = 3, \end{cases}$$

and if V is given by the second equation then this factor is

$$\lambda_p \boldsymbol{\omega}_{H,p}(V(\mathbf{Q}_p)) = \begin{cases} \left(1 - \frac{1}{p^2}\right)^3 & \text{if } p \equiv 2 \bmod 3, \\ 3\left(1 - \frac{1}{p}\right)^4\left(1 - \frac{1}{p^2}\right) & \text{if } p \equiv 1 \bmod 3, \\ \frac{16}{27} & \text{if } p = 3. \end{cases}$$

Proof. Let us consider the set of quadruples (x,y,z,t) in $(\mathbf{Z}/p^r\mathbf{Z})^4 - (p\mathbf{Z}/p^r\mathbf{Z})^4$ such that

$$x^3 + y^3 + pz^3 + p^2t^3 = 0 \qquad \text{in } \mathbf{Z}/p^r\mathbf{Z}. \tag{5.3}$$

If $p \,|\, x$ then $p \,|\, y$, $p \,|\, z$ and $p \,|\, t$. Therefore, for any (x,y,z,t) as above, $p \not| \, x$ and $p \not| \, y$. But for any triple (y,z,t) in $(\mathbf{Z}/p^r\mathbf{Z} - p\mathbf{Z}/p^r\mathbf{Z}) \times (\mathbf{Z}/p^r\mathbf{Z})^2$, there exists exactly one x verifying (5.3) if $p \equiv 2 \bmod 3$ and exactly three of them if $p \equiv 1 \bmod 3$. If $p = 3$ and y belongs to $\mathbf{Z}/3^r\mathbf{Z} - 3\mathbf{Z}/3^r\mathbf{Z}$ then (5.3) implies that $3 \,|\, z$. For any triple (y,z,t) with y

in $(\mathbf{Z}/3^r\mathbf{Z}) - (3\mathbf{Z}/3^r\mathbf{Z})$, z in $(3\mathbf{Z}/3^r\mathbf{Z})$ and t in $(\mathbf{Z}/3^r\mathbf{Z})$ there exist exactly three x in $\mathbf{Z}/3^r\mathbf{Z}$ which satisfy (5.3). We get that

$$\frac{N^*(p^r)}{p^{3r}} = \begin{cases} \frac{(p-1)p^{r-1}\times p^r\times p^r}{p^{3r}} = 1-\frac{1}{p} & \text{if } p \equiv 2 \bmod 3, \\ 3\frac{(p-1)p^{r-1}\times p^r\times p^r}{p^{3r}} = 3\Big(1-\frac{1}{p}\Big) & \text{if } p \equiv 1 \bmod 3, \\ 3\frac{2\times 3^{r-1}\times 3^{r-1}\times 3^r}{3^{3r}} = \frac{2}{3} & \text{if } p = 3. \end{cases}$$

Let us now turn to the set of (x,y,z,t) in $(\mathbf{Z}/p^r\mathbf{Z})^4 - (p\mathbf{Z}/p^r\mathbf{Z})^4$ such that

$$ax^3 + ay^3 + pz^3 + pt^3 = 0.$$

We decompose this set as follows

$$N_1^*(p^r) = \#\left\{ (x,y,z,t) \in (\mathbf{Z}/p^r\mathbf{Z})^4 - (p\mathbf{Z}/p^r\mathbf{Z})^4 \,\middle|\, \begin{cases} p \nmid x, \\ ax^3+ay^3+pz^3+pt^3 = 0. \end{cases} \right\}$$

$$N_2^*(p^r) = \#\left\{ (x,y,z,t) \in (\mathbf{Z}/p^r\mathbf{Z})^4 - (p\mathbf{Z}/p^r\mathbf{Z})^4 \,\middle|\, \begin{cases} p | x, \quad p \nmid z, \\ ax^3+ay^3+pz^3+pt^3 = 0. \end{cases} \right\}$$

As above we have the formula

$$\frac{N_1^*(p^r)}{p^{3r}} = \begin{cases} \frac{(p-1)p^{r-1}\times p^r\times p^r}{p^{3r}} = 1-\frac{1}{p} & \text{if } p \equiv 2 \bmod 3, \\ 3\times\frac{(p-1)p^{r-1}\times p^r\times p^r}{p^{3r}} = 3\Big(1-\frac{1}{p}\Big) & \text{if } p \equiv 1 \bmod 3, \\ 3\frac{2\times 3^{r-1}\times 3^{r-1}\times 3^r}{3^{3r}} = \frac{2}{3} & \text{if } p = 3, \end{cases}$$

where for $p = 3$ we use the equality

$$3^{r-1}\times 3^r = \#\{\, (z,t) \in (\mathbf{Z}/3^r\mathbf{Z})^2 \mid z^3 \equiv t^3 \bmod 3 \,\}.$$

On the other hand,

$$N_2^*(p^r) = p^2 \left\{ (x,y,z,t) \in (\mathbf{Z}/p^{r-1}\mathbf{Z})^4 \,\middle|\, \begin{cases} p \nmid z \\ ap^2x^3 + ap^2y^3 + z^3 + t^3 = 0. \end{cases} \right\}$$

and

$$\frac{N_2^*(p^r)}{p^{3r}} = \frac{p^2}{p^3}\times\begin{cases} \frac{(p-1)p^{r-2}\times p^{r-1}\times p^{r-1}}{p^{3(r-1)}} = 1-\frac{1}{p} & \text{if } p \equiv 2 \bmod 3, \\ 3\frac{(p-1)p^{r-2}\times p^{r-1}\times p^{r-1}}{p^{3(r-1)}} = 3\Big(1-\frac{1}{p}\Big) & \text{if } p \equiv 2 \bmod 3, \\ 3\frac{2\times 3^{r-2}\times 3^{r-1}\times 3^{r-1}}{3^{3(r-1)}} = 2 & \text{if } p = 3. \end{cases}$$

We conclude:

$$\frac{N^*(p^r)}{p^{3r}} = \begin{cases} 1-\frac{1}{p}+\frac{1}{p}-\frac{1}{p^2} = 1-\frac{1}{p^2} & \text{if } p \equiv 2 \bmod 3, \\ 3\Big(1-\frac{1}{p^2}\Big) & \text{if } p \equiv 1 \bmod 3, \\ \frac{2}{3}+\frac{2}{3} = \frac{4}{3} & \text{if } p = 3. \quad \Box \end{cases}$$

6. The constant $\alpha(V)$

Since the cubic surfaces we consider in this paper are **Q**-rational (which implies that $\beta(V) = 1$), it remains to compute the rank t of the Picard group and the value of $\alpha(V)$.

Proposition 6.1. *If V is given by the equation*

$$X^3 + Y^3 + aZ^3 + a^2T^3 = 0, \tag{6.1}$$

where a is not a cube in **Q**, *then* $\operatorname{rk}\operatorname{Pic}(V) = 2$ *and* $\alpha(V) = 2$.

If V is given by the equation

$$aX^3 + aY^3 + bZ^3 + bT^3 = 0, \tag{6.2}$$

where a and b are strictly positive integers and b/a is not a cube in $\mathbf{Q}$, then $\operatorname{rk}\operatorname{Pic}(V) = 3$ *and* $\alpha(V) = 1$.

If V is given by the equation

$$X^3 + Y^3 + Z^3 + T^3 = 0 \tag{6.3}$$

then $\operatorname{rk}\operatorname{Pic}(V) = 4$ *and* $\alpha(V) = 7/18$.

Proof. To compute $\alpha(V)$ we shall use its original definition [Pe, §2]:

$$\alpha(V) = \operatorname{Vol}\{\, x \in \Lambda_{\text{eff}}(V)^\vee \mid \langle \omega_V^{-1}, x\rangle = 1 \,\}$$

where the Lebesgue measure on the affine hyperplane

$$\mathscr{H}(\lambda) = \{\, x \in \operatorname{Pic}(V)^\vee \otimes_{\mathbf{Z}} \mathbf{R} \mid \langle \omega_V^{-1}, x\rangle = \lambda \,\}$$

is defined by the $(t-1)$-form $\mathrm{d}\boldsymbol{x}$ which is characterized by the relation

$$\mathrm{d}\boldsymbol{x} \wedge \mathrm{d}\omega_V^{-1} = \mathrm{d}\boldsymbol{y}$$

(where $\mathrm{d}\omega_V^{-1}$ is the linear form defined by ω_V^{-1} on $\operatorname{Pic}(V)^\vee$ and $\mathrm{d}\boldsymbol{y}$ is the form corresponding to the natural Lebesgue measure on $\operatorname{Pic}(V)^\vee \otimes_{\mathbf{Z}} \mathbf{R}$). More explicitely, let $(e_1, \dots, e_t)$ be a basis of $\operatorname{Pic}(V)$ and $(e_1^\vee, \dots, e_t^\vee)$ be the dual basis. Write

$$\omega_V^{-1} = \sum_{i=1}^{t} \lambda_i e_i$$

with $\lambda_t \neq 0$. Let $f_1, \dots, f_{t-1}$ be the projection of $e_1^\vee, \dots, e_{t-1}^\vee$ on $\mathscr{H}(0)$ along $e_t^\vee$. Then

$$\mathrm{d}\boldsymbol{x} = \frac{1}{\lambda_t} \mathrm{d}f_1^\vee \wedge \dots \wedge \mathrm{d}f_{t-1}^\vee .$$

By [SK, pages 14 and 55], if $O_1, \dots, O_m$ are the orbits of the action of $\operatorname{Gal}(K/\mathbf{Q})$ on the 27 lines, then $\Lambda_{\text{eff}}(V)$ is generated by the classes $[O_i] = \sum_{x \in O_i} [x]$.

When V is given by the equation (6.1) the Galois group $\operatorname{Gal}(K/\mathbf{Q})$ is

$$\mathbf{Z}/3\mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}$$

and the orbits of its action on the 27 lines are

$$\begin{aligned}
O_1 &= \{L(0), L'(0), L''(0)\},\\
O_2 &= \{L(1), L(2), L'(1), L'(2), L''(1), L''(2)\},\\
O_3 &= \{M(0), M(1), M(2)\},\\
O_4 &= \{M'(0), M'(1), M'(2), M''(0), M''(1), M''(2)\},\\
O_5 &= \{N(0), N'(1), N''(2)\},\\
O_6 &= \{N(1), N(2), N'(0), N'(2), N''(0), N''(1)\}.
\end{aligned}$$

In the basis $([\Lambda], [E_1], \dots, [E_6])$, a basis of $\operatorname{Pic}(V) = (\operatorname{Pic}\overline{V})^{\operatorname{Gal}(K:\mathbf{Q})}$ is given by

$$e_1 = \omega_V^{-1}, \qquad e_2 = -2[E_4] + [E_5] + [E_6].$$

In the basis (e_0, e_1), the effective cone $\Lambda_{\text{eff}}(V)$ is generated by the classes

$$\begin{aligned}
&[O_1] = e_1, && [O_2] = 2e_1, && [O_3] = e_1 - e_2,\\
&[O_4] = 2e_1 + e_2, && [O_5] = e_1 + e_2, && [O_6] = 2e_1 - e_2.
\end{aligned}$$

Therefore, this cone is generated by the elements $e_1 - e_2$ and $e_1 + e_2$ and $\alpha(V)$ is given as the volume of the domain

$$x = 1, \quad x + y > 0 \quad \text{and} \quad x - y > 0,$$

that is, as the volume of the segment $[-1, 1]$ and $\alpha(V) = 2$.

If V is given by the equation (6.2) then $\mathrm{Gal}(K/\mathbf{Q})$ is isomorphic to

$$\mathbf{Z}/3\mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}$$

and the orbits of the Galois action on the 27 lines are

$$\begin{aligned}
O_1 &= \{L(0)\},\\
O_2 &= \{L(1), L(2)\},\\
O_3 &= \{L'(0), L''(0)\},\\
O_4 &= \{L'(1), L''(2)\},\\
O_5 &= \{L'(2), L''(1)\},\\
O_6 &= \{M(0), M'(1), M''(2)\},\\
O_7 &= \{M(1), M(2), M'(0), M'(2), M''(0), M''(1)\},\\
O_8 &= \{N(0), N(1), N(2)\},\\
O_9 &= \{N'(0), N'(1), N'(2), N''(0), N''(1), N''(2)\}.
\end{aligned}$$

A basis of $\mathrm{Pic}(V)$ is given by

$$e_1 = \omega_V^{-1}, \quad e_2 = [E_1], \quad e_3 = [E_2] + [E_3],$$

and the cone $\Lambda_{\text{eff}}(V)$ is generated by

$$\begin{aligned}
&[O_1] = e_2, && [O_2] = e_3, && [O_3] = e_1 - e_2,\\
&[O_4] = e_1 + e_2 - e_3, && [O_5] = e_1 - e_2, && [O_6] = 2e_1 - e_2 - e_3,\\
&[O_7] = e_1 + e_2 + e_3, && [O_8] = e_1 - 2e_2 + e_3, && [O_9] = 2e_1 + 2e_2 - e_3,
\end{aligned}$$

that is, by

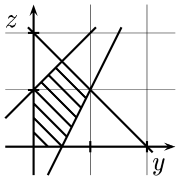
$$e_2, \quad e_3, \quad e_1 + e_2 - e_3, \quad 2e_1 - e_2 - e_3, \quad e_1 - 2e_2 + e_3$$

(since $3[O_3] = [O_6] + [O_8]$). Thus $\alpha(V)$ is the volume of the domain given by

$$\begin{cases}
x = 1,\ y > 0,\ z > 0,\\
x + y - z > 0,\\
2x - y - z > 0,\\
x - 2y + z > 0.
\end{cases}$$

Using the description above, $\alpha(V)$ is the volume of

$$\begin{cases}
0 < y,\ 0 < z,\\
z - y < 1,\\
y + z < 2,\\
2y - z < 1.
\end{cases}$$



Therefore $\alpha(V) = 1$.

If V is given by the equation (6.3), then $\mathrm{Gal}(K/\mathbf{Q}) = \mathbf{Z}/2\mathbf{Z}$ and the orbits of the Galois action on the 27 lines are given by

$$\begin{aligned}
&O_1 = \{L(0)\}, && O_2 = \{L(1), L(2)\}, && O_3 = \{L'(0), L''(0)\},\\
&O_4 = \{L'(1), L''(2)\}, && O_5 = \{L'(2), L''(1)\}, &&\\
&O_6 = \{M(0)\}, && O_7 = \{M(1), M(2)\}, && O_8 = \{M'(0), M''(0)\},\\
&O_9 = \{M'(1), M''(2)\}, && O_{10} = \{M'(2), M''(1)\}, &&\\
&O_{11} = \{N(0)\}, && O_{12} = \{N(1), N(2)\}, && O_{13} = \{N'(0), N''(0)\},\\
&O_{14} = \{N'(1), N''(2)\}, && O_{15} = \{N'(2), N''(1)\}. &&
\end{aligned}$$

A basis of the Picard group is given by

$$e_1 = [\Lambda] - [E_4], \quad e_2 = [E_1], \quad e_3 = [E_2] + [E_3], \quad e_4 = -2[E_4] + [E_5] + [E_6].$$

The effective cone $\Lambda_{\text{eff}}(V)$ is generated by

$$\begin{aligned}
&[O_1] = e_2, && [O_2] = e_3,\\
&[O_3] = 3e_1 - 2e_2 - e_3 - e_4, && [O_4] = 3e_1 - 2e_3 - e_4,\\
&[O_5] = 3e_1 - 2e_2 - e_3 - e_4, && [O_6] = 2e_1 - e_2 - e_3 - e_4,\\
&[O_7] = e_1 - e_4, && [O_8] = e_1,\\
&[O_9] = 4e_1 - 2e_2 - 2e_3 - e_4, && [O_{10}] = e_1,\\
&[O_{11}] = e_1 - e_2, && [O_{12}] = 2e_1 - 2e_2 - e_4,\\
&[O_{13}] = 2e_1 - e_3 - e_4, && [O_{14}] = 2e_1 - e_3,\\
&[O_{15}] = 2e_1 - e_3 - e_4. &&
\end{aligned}$$

Since $[O_3] = [O_5] = [O_6] + [O_{11}]$ and $[O_{13}] = [O_{15}] = [O_6] + [O_2]$, we get that $\Lambda_{\text{eff}}(V)$ is generated by

$$\begin{gathered}
e_2, \quad e_3, \quad 3e_1 - 2e_3 - e_4, \quad 2e_1 - e_2 - e_3 - e_4, \quad e_1 - e_4,\\
4e_1 - 2e_2 - 2e_3 - e_4, \quad e_1 - e_2, \quad 2e_1 - 2e_2 - e_4, \quad 2e_1 - e_3.
\end{gathered}$$

The anticanonical class is given by

$$\omega_V^{-1} = 3e_1 - e_2 - e_3 - e_4.$$

Thus $\alpha(V)$ is the volume of the domain

$$\begin{cases}
3x - y - z - t = 1,\\
y > 0, \quad z > 0,\\
x - y > 0,\\
2x - z > 0,\\
x - t > 0,\\
3x - 2z - t > 0,\\
2x - y - z - t > 0,\\
4x - 2y - 2z - t > 0,\\
2x - 2y - t > 0,
\end{cases}$$

that is, of the domain P in $\mathbf{R}^3$ given by

$$\begin{cases} y > 0, \quad z > 0, \\ x - y > 0, \\ 2x - z > 0, \\ 1 - 2x + y + z > 0, \\ 1 + y - z > 0, \\ 1 - x > 0, \\ 1 + x - y - z > 0, \\ 1 - x - y + z > 0. \end{cases}$$

We compute its volume as follows: decompose P into cones with apex $(0,0,0)$ and supported by the faces not containing this point. Thus we consider the following faces of P:

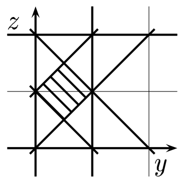
$$\begin{aligned} &F_1: \quad 1 - x = 0, && F_2: \quad 1 - 2x + y + z = 0, \\ &F_3: \quad 1 + y - z = 0, && F_4: \quad 1 + x - y - z = 0, \\ &F_5: \quad 1 - x - y + z = 0. \end{aligned}$$

One has

$$\alpha(V) = \mathrm{Vol}(P) = \frac{1}{3} \sum_{i=1}^{5} \mathrm{Area}(F_i).$$

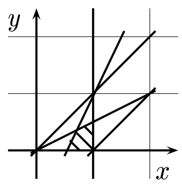
The area of F_1 is the volume of the domain

$$\begin{cases} y > 0, \quad z > 0, \\ 1 - y > 0, \\ 2 - z > 0, \\ -1 + y + z > 0, \\ 1 + y - z > 0, \\ 2 - y - z > 0, \\ z - y > 0, \end{cases}$$



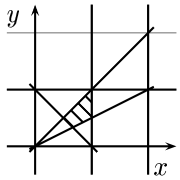
and we get $\mathrm{Area}(F_1) = \frac{1}{2}$. For F_2 we have the equations

$$\begin{cases} y > 0, \\ -1 + 2x - y > 0, \\ x - y > 0, \\ 1 + y > 0, \\ 2 - 2x + 2y > 0, \\ 1 - x > 0, \\ 2 - x > 0, \\ x - 2y > 0. \end{cases}$$



We get $\mathrm{Area}(F_2) = \frac{1}{6}$. For F_3 we have the same equations and the same area. For F_4 we have the equations

$$\begin{cases} y > 0, \\ 1 + x - y > 0, \\ x - y > 0, \\ -1 + x + y > 0, \\ 2 - x > 0, \\ -x + 2y > 0, \\ 1 - x > 0, \\ 2 - 2y > 0. \end{cases}$$



We find $\mathrm{Area}(F_4) = 1/8 + 1/24 = 1/6$. The face F_5 is given by the same equations and $\mathrm{Area}(F_5) = 1/6$. Finally

$$\alpha(V) = \frac{1}{3}\left(\frac{1}{2} + \frac{4}{6}\right) = \frac{7}{18}. \quad \square$$

7. Some statistical formulae

The most naive way to test the conjecture is to compute the quotient

$$N_{U,H}(B)/\boldsymbol{\theta}_H(V)B(\log B)^{t-1} \tag{7.1}$$

for large values of B. However, as explained in the introduction, the relative error term is expected to decrease slowly. Therefore it is natural to use the fact that we expect an asymptotic of the form

$$N_{U,H}(B) = BP(\log(B)) + o(B),$$

where P is a polynomial of degree $t-1$ with a dominant coefficient equal to $\boldsymbol{\theta}_H(V)$. With the program of D. J. Bernstein [Be], we can get a family of pairs $(B_i, N_{U,H}(B_i))_{1 \leqslant i \leqslant N}$. In the examples below we took for B_i successive powers of $6/5$ between 200 and 10^5. For any i between 1 and N, let

$$x_i = \log(B_i) \quad \text{and} \quad y_i = N_{U,H}(B_i)/B_i.$$

The simplest statistical tool in this setting is to look for a polynomial Q of degree $t-1$ such that

$$\sum_{i=1}^{N} (Q(x_i) - y_i)^2$$

is minimal and to compute its leading coefficient A_{t-1}. We then test the conjecture using the quotient

$$A_{t-1}/\boldsymbol{\theta}_H(V) \tag{7.2}$$

The advantage of this method is that, if the expected formula is correct, and if we take for B_i successive powers of a fixed real number λ between B_1 and B_N, then the relative error term for (7.2) should at least decrease as $C/(\log(B_N) - \log(B_1))^{t-1}$ for B_N/B_1 large enough with a constant C going to 0 as B_1 goes to infinity.

Of course, due to the arithmetic nature of $N_{U,H}(B)$, the errors are not as independent as one would need for a clean statistical treatment of the data. Also, since we do not have a good understanding of the difference

$$N_{U,H}(B) - BP(\log(B)),$$

and in order to limit the number of arbitrary parameters involved in the statistical computation, we prefered not to weight the points.

Notations 7.1. Let $R(X,Y)$ be a polynomial in $\mathbf{Q}[X,Y]$ and denote by $\langle R(X,Y)\rangle$ the mean value of $(R(x_i,y_i))_{1\leqslant i\leqslant N}$, that is,

$$\langle R(X,Y)\rangle = \frac{1}{N}\sum_{i=1}^{N} R(x_i,y_i).$$

If $t=2$ the leading coefficient of Q (if it is uniquely defined) is given by

$$A_1 = \frac{\langle XY\rangle - \langle Y\rangle\langle X\rangle}{\langle X^2\rangle - \langle X\rangle^2}.$$

If $t=3$ the leading coefficient is

$$A_2 = \frac{\langle YX^2\rangle - \langle Y\rangle\langle X^2\rangle - \frac{(\langle X^3\rangle - \langle X\rangle\langle X^2\rangle)(\langle YX\rangle - \langle Y\rangle\langle X\rangle)}{\langle X^2\rangle - \langle X\rangle^2}}{\langle X^4\rangle - \langle X^2\rangle^2 - \frac{(\langle X^3\rangle - \langle X\rangle\langle X^2\rangle)^2}{\langle X^2\rangle - \langle X\rangle^2}}.$$

If $t=4$, the leading coefficient is

$$A_3 = \frac{\langle YX^3\rangle - \langle Y\rangle\langle X^3\rangle - \frac{(\langle X^4\rangle - \langle X\rangle\langle X^3\rangle)(\langle YX\rangle - \langle Y\rangle\langle X\rangle)}{\langle X^2\rangle - \langle X\rangle^2} - \frac{\beta\delta}{\gamma}}{\langle X^6\rangle - \langle X^3\rangle^2 - \frac{(\langle X^4\rangle - \langle X\rangle\langle X^3\rangle)^2}{\langle X^2\rangle - \langle X\rangle^2} - \frac{\beta^2}{\gamma}},$$

with

$$\beta = \langle X^5\rangle - \langle X^3\rangle\langle X^2\rangle - \frac{\langle X^3\rangle - \langle X\rangle\langle X^2\rangle}{\langle X^2\rangle - \langle X\rangle^2}(\langle X^4\rangle - \langle X^3\rangle\langle X\rangle),$$
$$\gamma = \langle X^4\rangle - \langle X^2\rangle^2 - \frac{(\langle X^3\rangle - \langle X\rangle\langle X^2\rangle)^2}{\langle X^2\rangle - \langle X\rangle^2},$$
$$\delta = \langle YX^2\rangle - \langle Y\rangle\langle X^2\rangle - \frac{\langle X^3\rangle - \langle X\rangle\langle X^2\rangle}{\langle X^2\rangle - \langle X\rangle^2}(\langle YX\rangle - \langle Y\rangle\langle X\rangle).$$

In the next section, we denote by $\boldsymbol{\theta}_H^{\mathrm{stat}}(V)$ the leading coefficient A_{t-1}.

8. Presentation of the results

We consider only cubic surfaces of the form (6.1), (6.2), or (6.3). By [CTKS, Lemme 1], the corresponding surface V is $\mathbf{Q}$-rational and, in particular, $\mathrm{Br}(V) = \mathrm{Br}(\mathbf{Q})$. Thus the Brauer-Manin obstruction to weak approximation is void and

$$V(\boldsymbol{A}_{\mathbf{Q}})^{\mathrm{Br}} = V(\boldsymbol{A}_{\mathbf{Q}}) = \prod_{v\in\mathrm{Val}(\mathbf{Q})} V(\mathbf{Q}_v).$$

Moreover,

$$\beta(V) = \# H^1(\mathbf{Q}, \mathrm{Pic}(\overline{V})) = 1.$$

By (2.2) and (2.3), the constant $\boldsymbol{\theta}_H(V)$ may be written as

$$\boldsymbol{\theta}_H(V) = \alpha(V)\boldsymbol{\omega}_H(V(\boldsymbol{A}_{\mathbf{Q}}))$$

Using remark 4.4 we get

$$\boldsymbol{\theta}_H(V) = \alpha(V) \lim_{s\to 1}(s-1)^{t+2}\zeta_E(s) \times \boldsymbol{\omega}_{H,\infty}(V(\mathbf{R})) \\ \times \prod_{p|3abcd} \lambda_p \boldsymbol{\omega}_{H,p}(V(\mathbf{Q}_p)) \times \prod_{i=0}^{3} C_i,$$

where E is the étale algebra defined in 3.2. The residue of the zeta function could have been computed directly (see, for example, [Co, chapter 4]), but instead we used PARI. The volume at the real place is given by the formula

$$\frac{1}{2}\int_{\left\{(x,y,z,t)\,\middle|\,\left\{\substack{ax^3+by^3+cz^3+dt^3=0\\ \sup(|x|,|y|,|z|,|t|)\leqslant 1}\right.\right\}} \boldsymbol{\omega}_L(x,y,z,t),$$

where $\boldsymbol{\omega}_L$ is the Leray form

$$\boldsymbol{\omega}_L(x,y,z,t) = \frac{\sqrt[3]{d}^{\,-1}}{3(ax^3+by^3+cz^3)^{2/3}} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,.$$

Decomposing the domain of integration (and using the various expressions of the Leray form) it is possible to remove the singularities of this integral which is then easily estimated on a computer. The factors corresponding to the bad places have been described in section 5 and the constants C_0, C_1, C_2, and C_3 may be computed directly as in section 4.

We considered the following examples: for the cubic surfaces with a Picard group of rank 2 we used

$$(S_1) \qquad X^3+Y^3+2Z^3+4T^3=0,$$
$$(S_2) \qquad X^3+Y^3+5Z^3+25T^3=0,$$
$$(S_3) \qquad X^3+Y^3+3Z^3+9T^3=0.$$

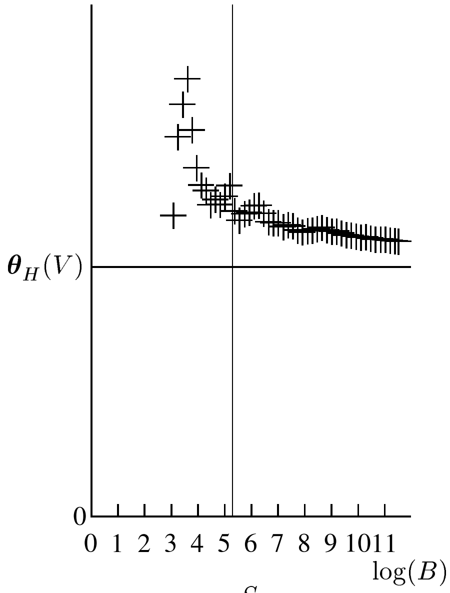
For the rank 3 case:

$$(S_4) \qquad X^3+Y^3+2Z^3+2T^3=0,$$
$$(S_5) \qquad X^3+Y^3+5Z^3+5T^3=0,$$
$$(S_6) \qquad X^3+Y^3+7Z^3+7T^3=0,$$
$$(S_7) \qquad 2X^3+2Y^3+3Z^3+3T^3=0,$$

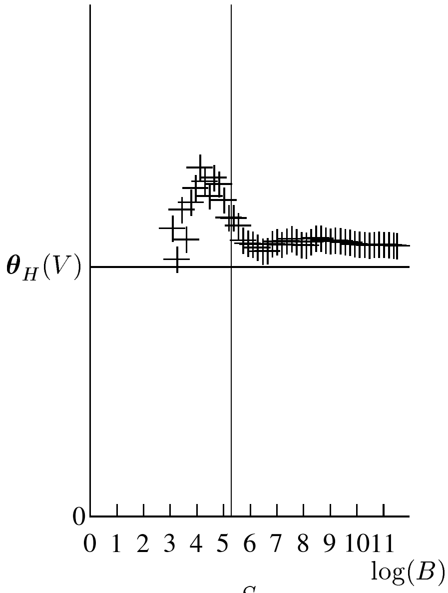
and for rank 4:

$$(S_8) \qquad X^3+Y^3+Z^3+T^3=0.$$

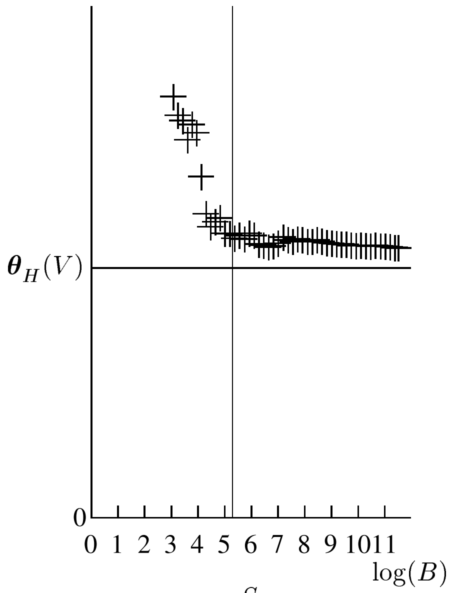
We draw below the corresponding experimental curves in which we compare the value of $N_{U,H}(B)/(B(\log B)^{t-1})$ with $\boldsymbol{\theta}_H(V)$. On each drawing, only the points on the right of the vertical line have been used for the computation of $\boldsymbol{\theta}_H^{\mathrm{stat}}(V)$.



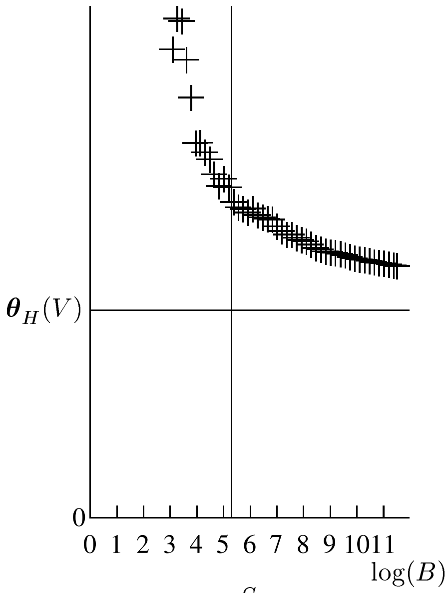
S_1



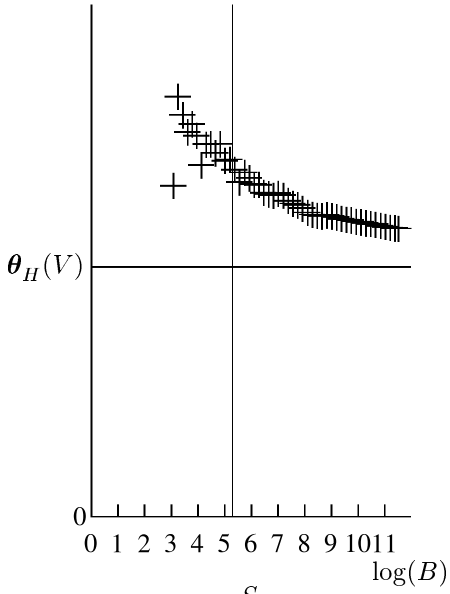
S_2



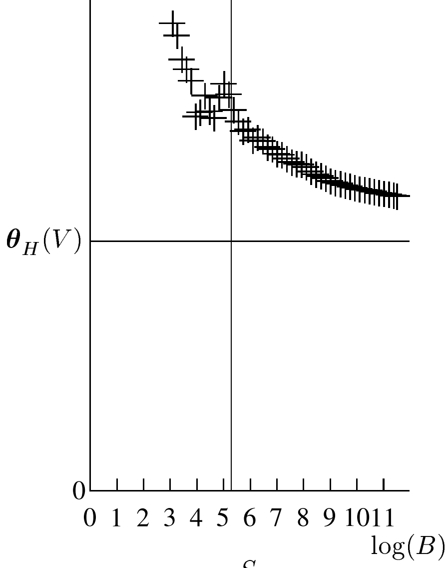
S_3



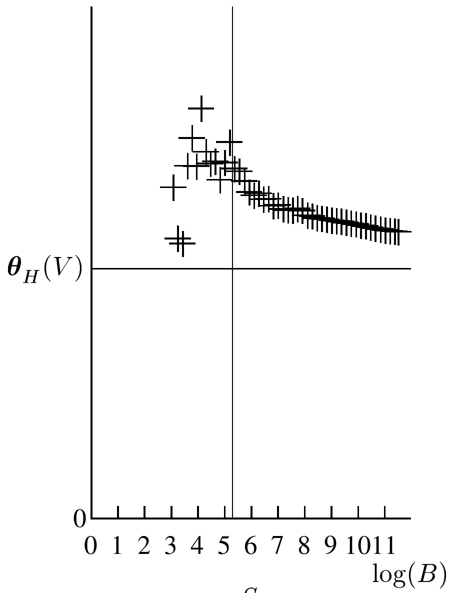
S_4


$\boldsymbol{\theta}_H(V)$
0
0 1 2 3 4 5 6 7 8 9 1011
$\log(B)$

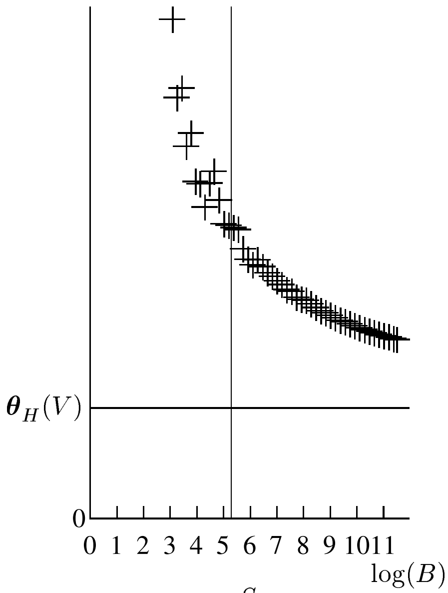
S_5


$\boldsymbol{\theta}_H(V)$
0
0 1 2 3 4 5 6 7 8 9 1011
$\log(B)$

S_6


$\boldsymbol{\theta}_H(V)$
0
0 1 2 3 4 5 6 7 8 9 1011
$\log(B)$

S_7


$\boldsymbol{\theta}_H(V)$
0
0 1 2 3 4 5 6 7 8 9 1011
$\log(B)$

S_8

We finish with tables of numerical results. The value of $\boldsymbol{\theta}_H^{\text{stat}}(V)$ is obtained from the pairs $(B_i, N_{U,H}(B_i))$ as described in section 7. We denote by $\zeta_{E_i}^*(1)$ the limit

$$\zeta_{E_i}^*(1) = \lim_{s\to 1}(s-1)^{t_i}\zeta_{E_i}^*(s),$$

where t_i is the number of components of E_i. Note that for the examples with a Picard group of rank 2, C_2 is equal to 1.

Surface	S_1	S_2	S_3
B	100000	100000	100000
$N_{U,H}(B)$	433526	286040	455164
$\alpha(V)$	2	2	2
ab/cd	$1/8$	$1/125$	$1/27$
$\zeta_{E_0}^*(1)$	6.045998×10^{-1}	6.045998×10^{-1}	6.045998×10^{-1}
ac/bd	$1/2$	$1/5$	$1/3$
$\zeta_{E_1}^*(1)$	8.146241×10^{-1}	1.163730	1.017615
ad/bc	2	5	3
$\zeta_{E_2}^*(1)$	8.146241×10^{-1}	1.163730	1.017615
$\lambda_3'\boldsymbol{\omega}_H(V(\mathbf{Q}_3))$	$4/9$	$4/9$	$4/9$
p_0	2	5	
$\lambda_{p_0}'\boldsymbol{\omega}_H(V(\mathbf{Q}_{p_0}))$	$3/8$	$96/125$	
C_0	8.306815×10^{-1}	3.493824×10^{-1}	3.066383×10^{-1}
C_1	9.540383×10^{-1}	8.704106×10^{-1}	9.762028×10^{-1}
C_3	9.893865×10^{-1}	9.906098×10^{-1}	9.892790×10^{-1}
$\boldsymbol{\omega}_H(V(\mathbf{R}))$	3.255161	1.360417	2.221359
$\boldsymbol{\theta}_H(V)$	3.413500×10^{-1}	2.290769×10^{-1}	3.660885×10^{-1}
$N_{U,H}(B)/\boldsymbol{\theta}_H(V)B\log(B)$	1.103137	1.084575	1.079931
$\boldsymbol{\theta}_H^{\text{stat}}(V)/\boldsymbol{\theta}_H(V)$	0.988687	1.067208	1.051041

For the examples with a Picard group of rank 3, C_3 is equal to 1.

Surface	S_4	S_5	S_6	S_7
B	100000	100000	100000	100000
$N_{U,H}(B)$	3051198	1976482	3420784	1966160
$\alpha(V)$	1	1	1	1
ab/cd	$1/4$	$1/25$	$1/49$	$4/9$
$\zeta^*_{E_0}(1)$	8.146241×10^{-1}	1.163730	1.265025	1.028996
ac/bd	1	1	1	1
$\zeta^*_{E_1}(1)$	6.045998×10^{-1}	6.045998×10^{-1}	6.045998×10^{-1}	6.045998×10^{-1}
ad/bc	1	1	1	1
$\zeta^*_{E_2}(1)$	6.045998×10^{-1}	6.045998×10^{-1}	6.045998×10^{-1}	6.045998×10^{-1}
$\lambda'_3 \boldsymbol{\omega}_H(V(\mathbf{Q}_3))$	$16/27$	$16/27$	$16/27$	$16/27$
p_0	2	5	7	2
$\lambda'_{p_0} \boldsymbol{\omega}_H(V(\mathbf{Q}_{p_0}))$	$27/64$	13824/15625	$186624/117649$	$27/64$
C_0	8.306815×10^{-1}	3.493824×10^{-1}	3.066383×10^{-1}	8.306815×10^{-1}
C_1	9.540383×10^{-1}	8.704106×10^{-1}	9.297617×10^{-1}	8.196347×10^{-1}
C_2	7.827314×10^{-1}	8.112747×10^{-1}	9.228033×10^{-1}	8.294515×10^{-1}
$\boldsymbol{\omega}_H(V(\mathbf{R}))$	4.105301	2.347970	1.910125	2.430506
$\boldsymbol{\theta}_H(V)$	1.895795×10^{-1}	1.291945×10^{-1}	2.184437×10^{-1}	1.290720×10^{-1}
$N_{U,H}(B)/\boldsymbol{\theta}_H(V) B \log(B)^2$	1.214249	1.154191	1.181448	1.149252
$\boldsymbol{\theta}_H^{\mathrm{stat}}(V)/\boldsymbol{\theta}_H(V)$	0.981952	1.035070	0.999247	1.063376

For the last example we have $C_2 = C_3 = 1$ and $E_1 = E_2 = E_3$ and we get

Surface	S_8
B	100000
$N_{U,H}(B)$	12137664
$\alpha(V)$	$7/18$
$\zeta^*_{E_i}(1)$	6.045998×10^{-1}
$\lambda'_3 \boldsymbol{\omega}_H(V(\mathbf{Q}_3))$	$16/27$
C_0	3.066383×10^{-1}
C_1	5.129319×10^{-1}
$\boldsymbol{\omega}_H(V(\mathbf{R}))$	6.121864
$\boldsymbol{\theta}_H(V)$	4.904057×10^{-2}
$N_{U,H}(B)/\boldsymbol{\theta}_H(V) B \log(B)^3$	1.621894
$\boldsymbol{\theta}_H^{\mathrm{stat}}(V)/\boldsymbol{\theta}_H(V)$	1.012304

REFERENCES

[BM] V. V. Batyrev et Y. I. Manin, *Sur le nombre des points rationnels de hauteur bornée des variétés algébriques*, Math. Ann. **286** (1990), 27–43.

[BT] V. V. Batyrev and Y. Tschinkel, *Tamagawa numbers of polarized algebraic varieties*, Nombre et répartition de points de hauteur bornée, Astérisque, vol. 251, SMF, Paris, 1998, pp. 299–340.

[Be] D. J. Bernstein, *Enumerating solutions to $p(a)+q(b) = r(c)+s(d)$*, to appear in Math. Comp. (1999).

[Co] H. Cohen, *A course in computational algebraic number theory*, Graduate Texts in Math., vol. 138, Springer-Verlag, Berlin, Heidelberg and New York, 1993.

[CT] J.-L. Colliot-Thélène, *The Hasse principle in a pencil of algebraic varieties*, Number theory (Tiruchirapalli, 1996), Contemp. Math., vol. 210, Amer. Math. Soc., Providence, 1998, pp. 19–39.

[CTKS] J.-L. Colliot-Thélène, D. Kanevsky, et J.-J. Sansuc, *Arithmétique des surfaces cubiques diagonales*, Diophantine approximation and transcendenc theory (Bonn, 1985), Lecture Notes in Math., vol. 1290, Springer-Verlag, Berlin, Heidelberg and New York, 1987, pp. 1–108.

[CTS] J.-L. Colliot-Thélène et J.-J. Sansuc, *La descente sur une variété rationnelle définie sur un corps de nombres*, C. R. Acad. Sci. Paris Sér. I Math. **284** (1977), 1215–1218.

[FMT] J. Franke, Y. I. Manin, and Y. Tschinkel, *Rational points of bounded height on Fano varieties*, Invent. Math. **95** (1989), 421–435.

[H-B] D. R. Heath-Brown, *The density of zeros of forms for which weak approximation fails*, Math. Comp. **59** (1992), 613–623.

[IR] K. Ireland and M. Rosen, *A classical introduction to modern number theory (second edition)*, Graduate texts in Math., vol. 84, Springer-Verlag, Berlin, Heidelberg and New York, 1990.

[Man1] Y. I. Manin, *Le groupe de Brauer-Grothendieck en géométrie diophantienne*, Actes Congrès intern. math., Tome 1 (Nice, 1970), Gauthiers-Villars, Paris, 1971, pp. 401–411.

[Man2] ———, *Cubic forms (second edition)*, North-Holland Math. Library, vol. 4, North-Holland, Amsterdam, New York and Oxford, 1986.

[Pe] E. Peyre, *Hauteurs et mesures de Tamagawa sur les variétés de Fano*, Duke Math. J. **79** (1995), n° 1, 101–218.

[PT] E. Peyre and Y. Tschinkel, *Tamagawa numbers of diagonal cubic surfaces, numerical evidence*, to appear in Math. Comp. (1999).

[SK] K. E. Smith, *Rational and non-rational algebraic varieties: lectures of János Kollár*, `http://xxx.lanl.gov/abs/alg-geom/9707013` (1997).

[S-D] P. Swinnerton-Dyer, *Counting rational points on cubic surfaces*, (L'Aquila, 1992) (C. Ciliberto, E. L. Livorni, and A. J. Sommese, eds.), Contemp. Math., vol. 162, AMS, Providence, 1994, pp. 371–379.

INSTITUT FOURIER, UFR DE MATHÉMATIQUES, UMR 5582, UNIVERSITÉ DE GRENOBLE I ET CNRS, BP 74, 38402 SAINT-MARTIN D'HÈRES CEDEX, FRANCE
URL: `http://www-fourier.ujf-grenoble.fr/~peyre`
E-mail address: `Emmanuel.Peyre@ujf-grenoble.fr`

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, WASHINGTON RD., PRINCETON, NJ 08544-1000, U.S.A.
E-mail address: `ytschink@math.princeton.edu`