APPROXIMATION AT PLACES OF BAD REDUCTION FOR RATIONALLY CONNECTED VARIETIES

BRENDAN HASSETT AND YURI TSCHINKEL

Abstract. This paper addresses weak approximation for rationally connected varieties defined over the function field of a curve, especially at places of bad reduction. Our approach entails analyzing the rational connectivity of the smooth locus of singular reductions of the variety. As an application, we prove weak approximation for cubic surfaces with square-free discriminant.

Contents

1. Introduction

In number theory, many results and techniques rely on approximating adelic points by rational points. In this paper, we study geometric versions of these notions for rationally connected varieties over the function field of a curve. In this context, rational points correspond to sections of rationally-connected fibrations over the curve. We are looking for sections with prescribed jet data in finitely many fibers (see Section 2 for definitions).

Let k be an algebraically closed field of characteristic zero, B a smooth curve over k with function field $F = k(B)$. Let $\overline{B}$ be the smooth projective model of F and put $S := \overline{B} \setminus B$.

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Theorem 1. *Let X be a smooth proper rationally connected variety over F, and $\pi : \mathcal{X} \to B$ a model of X, i.e., $\mathcal{X}$ is an algebraic space flat and proper over B with generic fiber X. Let $\mathcal{X}^{sm}$ be the locus where π is smooth and $\mathcal{X}^{\bullet} \subset \mathcal{X}^{sm}$ be such that*

(1) *there exists a section $s : B \to \mathcal{X}^{\bullet}$;*
(2) *for each $b \in B$ and $x \in \mathcal{X}^{\bullet}_b$, there exists a rational curve $f : \mathbb{P}^1 \to \mathcal{X}^{\bullet}_b$ containing x and the generic point of $\mathcal{X}^{\bullet}_b$.*

Then sections of $\mathcal{X}^{\bullet} \to B$ satisfy approximation away from S.

Rationally-connected fibrations over curves have sections by [7]. The existence of a section through a finite set of prescribed points is addressed [12] 2.13 and [11] IV.6.10.1. Weak approximation is known in fibers of good reduction [8], so we take simultaneous resolutions of singular fibers of $\mathcal{X}$ whenever possible [2] [3]. Consequently, when $\mathcal{X} \to B$ admits a simultanteous resolution over some étale neighborhood of b, we replace $\mathcal{X}$ by this resolution. However, the resolved family may be an algebraic space, rather than a scheme, over B. This is why Theorem 1 is stated in this generality.

We shall actually prove a stronger result, Theorem 15, which is applicable in positive characteristic. In this context, Corollary 16 gives weak approximation at places of good reduction.

There are very few instances where weak approximation over function fields is known at all places [4]:

- stably rational varieties;
- connected linear algebraic groups and homogeneous spaces for these groups;
- homogeneous space fibrations over varieties that satisfy weak approximation, for example, conic bundles over rational varieties;
- Del Pezzo surfaces of degree at least four.

Even the case of cubic surfaces remains open, in general. Madore established weak approximation for cubic surfaces at places of good reduction [13]. His proof uses the abundance of distinct unirational parametrizations, and builds on ideas of Swinnerton-Dyer [15].

When is Theorem 1 applicable? Let X be a smooth projective rationally connected variety over $F = k(B)$, with B projective. There exists a regular model $\pi : \mathcal{X} \to B$, and any section $s : B \to \mathcal{X}$ is contained in $\mathcal{X}^{sm}$. For each singular fiber $\mathcal{X}_b$, fix an irreducible component $\mathcal{X}^{\bullet}_b \subset \mathcal{X}^{sm}_b$;

these determine an open subset $\mathcal{X}^\bullet \subset \mathcal{X}^{sm}$. To prove weak approximation for X, it suffices to prove approximation for each $\mathcal{X}^\bullet$ obtained in this way. We do not know how to verify (1) in general: Is there *any* section meeting a prescribed irreducible component of $\mathcal{X}_b^{sm}$? Further, there is no general result giving a regular model $\mathcal{X} \to B$ such that each irreducible component of $\mathcal{X}_b^{sm}$ has the property (2).

We give applications to cubic surfaces:

Theorem 2. *Let X be a smooth cubic surface over F and $\pi : \mathcal{X} \to B$ a model whose singular fibers are cubic surfaces with rational double points. Suppose there exists a section $s : B \to \mathcal{X}^{sm}$. Then sections of $\mathcal{X}^{sm} \to B$ satisfy approximation away from S.*

When the model is regular all sections are contained in the smooth locus, so we conclude:

Corollary 3. *Let X be a smooth cubic surface over F. Suppose X admits a regular model $\pi : \mathcal{X} \to B$ whose singular fibers are cubic surfaces with rational double points. Then weak approximation holds for X away from S.*

There exist cubic surfaces which do not admit models with at most rational double points in a given fiber, e.g., the isotrivial family

$$x^3 + y^3 + z^3 = tw^3$$

over the t-line. Nonetheless, Corollary 3 proves weak approximation for 'generic' cubic surfaces.

Corollary 4. *Let $\mathcal{H}ilb = \mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^3}(3))) \simeq \mathbb{P}^{19}$ denote the Hilbert scheme of cubic surfaces, $\mathcal{U} \to \mathcal{H}ilb$ the universal family, $D \subset \mathcal{H}ilb$ the discriminant divisor, and $B \subset \mathcal{H}ilb$ a smooth curve transverse to D (i.e., the discriminant is square-free along B). Then sections of*

$$\mathcal{X} = \mathcal{U} \times_{\mathcal{H}ilb} \operatorname{Spec}(F) \to B$$

satisfy approximation away from S.

Meeting the discriminant transversally is an open condition on the classifying map to the Hilbert scheme. The transversality implies that near singular points of $\mathcal{X}_b$, the model $\mathcal{X} := \mathcal{U} \times_{\mathcal{H}ilb} B$ has local analytic equation $x^2 + y^2 + z^2 = t$, where t is a local uniformizer for B at b. In particular, $\mathcal{X}$ is a regular model and Corollary 3 applies.

In our approach to approximation, we require precise control over proper rational curves in the smooth locus. One focus of this paper is to extend standard results on smooth proper rationally connected varieties to the non-proper case (see Section 4). The application to cubic surfaces entails a refinement of rational connectivity results of [9] (see Section 5).

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2. Notions of approximation

Let F be a global field, i.e., a number field or the function field of a curve B defined over an algebraically closed field k. Let S a finite set of places of F containing the archimedean places, $\mathfrak{o}_{F,S}$ the corresponding ring of integers, and $\mathbb{A}_{F,S}$ the restricted direct product over all places outside S.

Let X be an algebraic variety over F, $X(F)$ the set of F-rational points and $X(\mathbb{A}_{F,S}) \subset \prod_{v \notin S} X(F_v)$ the set of $\mathbb{A}_{F,S}$-points of X. The set $X(\mathbb{A}_{F,S})$ carries a natural direct product topology. One says that *weak approximation* holds for X away from S if $X(F)$ is dense in this topology.

The set $X(\mathbb{A}_{F,S})$ also carries a natural adelic topology: The basic open subsets are

$$\prod_{v \in S'} \mathfrak{u}_v \times \prod_{v \notin (S \cup S')} \mathcal{X}(\mathfrak{o}_v),$$

where S' is a finite set of nonarchimedean places disjoint from S, $\mathcal{X}$ is a model over $\mathrm{Spec}(\mathfrak{o}_{F,S})$, $\mathfrak{o}_v$ is the completion of $\mathfrak{o}_{F,S}$ at v, and $\mathfrak{u}_v \subset X(F_v)$ an open subset in the v-adic analytic topology on $X(F_v)$. This does not depend on the choice of model. *Strong approximation* holds for X away from S if $X(F)$ is dense in $X(\mathbb{A}_{F,S})$. Note that strong approximation implies weak approximation. Conversely, for $\mathcal{X}$ proper over the integers, weak approximation implies strong approximation, since $\mathcal{X}(\mathfrak{o}_v) = X(F_v)$; in these cases, we will use the term weak approximation for the sake of consistency.

Finally, there is a formulation which is sensitive to the choice of model $\mathcal{X}$. Consider the topology on $\prod_{v \notin S} \mathcal{X}(\mathfrak{o}_v)$ with basic open subsets

$$\prod_{v \in S'} \mathfrak{u}_v \times \prod_{v \notin (S \cup S')} \mathcal{X}(\mathfrak{o}_v),$$

with $\mathfrak{u}_v \subset \mathcal{X}(\mathfrak{o}_v)$ an open subset. We say that *approximation holds for S-integral points of $\mathcal{X}$* if $\mathcal{X}(\mathfrak{o}_{F,S})$ is dense in this product. This is a weak version of strong approximation.

We now focus on the function field case: Let $\overline{B}$ be a smooth projective model of B with $S = \overline{B} \setminus B$; place v correspond to points $b \in \overline{B}$. Let X be a smooth variety proper over $F = k(B)$, $\pi : \mathcal{X} \to B$ a model proper and flat over B (which exists by [14]), and $\mathcal{X}^\bullet \subset \mathcal{X}^{sm}$ a model for X surjecting onto B. Since π is proper, F-rational points of X correspond to sections $s : B \to \mathcal{X}$. If $\mathcal{X}$ is regular s factors through $\mathcal{X}^{sm}$.

Definition 5. An *admissible section* of $\pi : \mathcal{X} \to B$ is a section $s : B \to \mathcal{X}^{sm}$. An *admissible N-jet* of π at b is a section of

$$\mathcal{X}^{sm} \times_B \mathrm{Spec}(\mathcal{O}_{B,b}/\mathfrak{m}_{B,b}^{N+1}) \to \mathrm{Spec}(\mathcal{O}_{B,b}/\mathfrak{m}_{B,b}^{N+1}).$$

An *approximable N-jet* of π at b is a section of

$$\mathcal{X} \times_B \mathrm{Spec}(\mathcal{O}_{B,b}/\mathfrak{m}_{B,b}^{N+1}) \to \mathrm{Spec}(\mathcal{O}_{B,b}/\mathfrak{m}_{B,b}^{N+1})$$

that may be lifted to a section of $\widehat{\mathcal{X}}_b \to \widehat{B}_b$, with $\widehat{B}_b = \mathrm{Spec}(\hat{\mathcal{O}}_{B,b})$ and $\widehat{\mathcal{X}}_b = X \times_B \widehat{B}_b$.

Hensel's lemma guarantees that every admissible N-jet is approximable. Let $\{b_i\}_{i \in I}$ be a finite set of points and j_i an admissible N-jet of π at b_i. We write $J = \{j_i\}_{i \in I}$ for the corresponding collection of admissible N-jets.

The notions of weak and strong approximation introduced above have geometric interpretations

- Weak and strong approximation hold for X away from S if any finite collection of approximable jets of π can be realized by a section $s : B \to \mathcal{X}$.
- This is equivalent to weak approximation holding for $X^\bullet$ away from S: Every jet in $\mathcal{X}$ at b can be realized by a section $\mathcal{X} \times_B \widehat{B}_b \to \widehat{B}_b$ meeting $\widehat{\mathcal{X}}_b^\bullet$.

- If $\mathcal{X}$ is regular these are equivalent to the condition that any collection of admissible jets of π can be realized by a section $s : B \to \mathcal{X}^{sm}$.

There is an analogous formulation of approximation for integral points:

- Approximation holds for sections of $\mathcal{X}^\bullet \to B$ away from S if each collection of jet data in $\mathcal{X}^\bullet$ can be realized by a section $s : B \to \mathcal{X}^\bullet$.
- If $\mathcal{X}$ is regular and $\mathcal{X}^\bullet = \mathcal{X}^{sm}$ this is equivalent to weak approximation for X.

3. Curves, combs, and deformations

The *dual graph* associated with a nodal curve C has vertices are indexed by the irreducible components of C and its edges indexed by the intersections of these components. A projective nodal curve C is *tree-like* if

- each irreducible component of C is smooth;
- the dual graph of C is a tree.

Definition 6. A *comb with m reducible teeth* is a projective nodal curve C with $m+1$ subcurves $D, T_1, \dots, T_m$ such that

- D is smooth and irreducible;
- $T_l \cap T_{l'} = \emptyset$, for all $l \neq l'$;
- each T_l meets D transversally in a single point; and
- each T_l is a chain of $\mathbb{P}^1$'s.

Here D is called the *handle* and the T_l the *reducible teeth*.

We will use the following lemma, which has the same proof as Proposition 24 of [8]:

Lemma 7. *Let C be a tree-like curve, W a smooth algebraic space, $h : C \to W$ an immersion with nodal image. Suppose that for each irreducible component C_l of C, $H^1(C_l, \mathcal{N}_h \otimes \mathcal{O}_{C_l}) = 0$ and $\mathcal{N}_h \otimes \mathcal{O}_{C_l}$ is globally generated. Then h deforms to an immersion.*

Suppose furthermore that $\mathfrak{w} = \{w_1, \dots, w_M\} \subset C$ is a collection of smooth points such that for each component C_l, $H^1(\mathcal{N}_h \otimes \mathcal{O}_{C_l}(-\mathfrak{w})) = 0$ and the sheaf $\mathcal{N}_h \otimes \mathcal{O}_{C_l}(-\mathfrak{w})$ admits a section nonzero at each point of the quotient

$$(\mathcal{N}_h \otimes \mathcal{O}_{C_l})/\mathcal{N}_{h|C_l}.$$

Then $h : C \to W$ deforms to an immersion of a smooth curve into W containing $h(\mathfrak{w})$.

4. Strong rational connectivity

Definition 8. A variety X is *rationally connected* (resp. *separably rationally connected*) if there is a family of proper irreducible rational curves $g : U \to Z$ (resp. $\pi_2 : U = \mathbb{P}^1 \times Z \to Z$) and a cycle morphism $u : U \to X$ such that

$$u^2 : U \times_Z U \to X \times X$$

is dominant (resp. smooth over the generic point)).

Intuitively, two generic points of X can be joined by an irreducible projective rational curve. Over fields of characteristic zero, rational connected varieties are also separably rationally connected [11] IV.3.3.1.

The notion of rational connectedness is a bit subtle over countable fields: For convenience, we work over an uncountable algebraically closed field. Over such a field, rational connectivity is equivalent to the condition that two very general points of X can be joined by such a rational curve.

Definition 9. Let X be a smooth algebraic space of dimension d and $f : \mathbb{P}^1 \to X$ a nonconstant morphism, so we have an isomorphism

$$f^*\mathcal{T}_X \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_d)$$

for suitable integers $a_1, \ldots, a_d$. Then f is *free* (resp. *very free*) if each $a_i \geq 0$ (resp. $a_i \geq 1$).

We refer the reader to [11] IV.3 for further facts about rationally connected varieties.

One technical result will play a prominent rôle in our analysis.

Proposition 10 ([11] IV.3.9.4)**.** *Let V be a smooth separably rationally connected (not necessarily proper) variety. Then there exists a nonempty subset $V^0 \subset V$ characterized as the largest open subset such that if $v_1, \ldots, v_m \in V^0$ are distinct closed points, then there is a very free curve in V^0 containing these as smooth points. Moreover, any rational curve $C \subset V$ that meets V^0 is contained in V^0.*

No example where $V^0 \neq V$ is known.

Remark 11. Let V_2 be a smooth variety, $V_1 \subset V_2$ a rationally connected dense open subvariety, and $V_2^0 \subset V_2$ the largest open set satisfying the conditions of Proposition 10 . Then $V_1^0 \subset V_2^0$. Thus a point $v \in V_2$ is in V_2^0 provided there is a rational curve $f : \mathbb{P}^1 \to V_2$ through v and meeting V_1^0.

Proposition 12. *Let V be a smooth separably rationally connected variety, and $\beta : W \to V$ an iterated blow-up of V along smooth subvarieties. Then $\beta^{-1}(V^0) = W^0$.*

Proof. The inclusion $W^0 \subset \beta^{-1}(V^0)$ is straightforward: Given points $w_1, \dots, w_m \in W^0$, there is a very free curve $g : \mathbb{P}^1 \to W^0$ containing them; we may choose this to be transversal to the exceptional divisor of β. The inclusion of sheaves

$$\mathcal{T}_W \hookrightarrow \beta^* \mathcal{T}_V$$

remains an inclusion after pull-back via g, as the support of the cokernel does not contain $g(\mathbb{P}^1)$. The positivity of $g^*\mathcal{T}_W$ implies the positivity of $(\beta \circ g)^*\mathcal{T}_V$, which means that $\beta \circ g : \mathbb{P}^1 \to V$ is also very free.

For the reverse direction, we may restrict to the case where W is the blow-up of V along a smooth subvariety Z of codimension $r > 1$, with exceptional divisor E. It is clear that $\beta^{-1}(V^0 \setminus Z) \subset W_0$, so consider some $w \in \beta^{-1}(z)$ with $z \in Z \cap V^0$. It suffices to construct a rational curve containing w and the generic point of W.

There exists a very free curve $f' : \mathbb{P}^1 \to V^0$ with the following properties:

(1) $f'(\mathbb{P}^1)$ meets Z only at z (we can always deform a very free curve so that it misses a codimension ≥ 2 subset);
(2) $f'(\mathbb{P}^1)$ is smooth at z and transverse to Z.

Let $g' : \mathbb{P}^1 \to W$ denote the lift to W, which is free in W, and $w' = g'(0)$. If $w' = w$ then we are done. Otherwise, let $\ell \subset \beta^{-1}(z) \simeq \mathbb{P}^{r-1}$ denote the line joining w and w'. Since g' is free, it admits a small deformation to a free curve $g'' : \mathbb{P}^1 \to W$ with $w'' := g''(0) \in \ell, w'' \neq w'$. (See Figure 1.)

We construct a comb $h : C \to W$ with handle $\ell \subset \mathbb{P}^{r-1} \subset W$ and two teeth $g', g'' : \mathbb{P}^1 \to W$. Using the exact sequence of normal bundles

$$0 \to \mathcal{N}_{\ell/E} \to \mathcal{N}_{\ell/W} \to \mathcal{N}_{E/X} \otimes \mathcal{O}_\ell \to 0$$

we find

$$\mathcal{N}_{\ell/W} \simeq \mathcal{O}_{\mathbb{P}^1}^{\dim(V)-r} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{r-2} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$$

where the negative summand is in the normal direction to E. Since $g'(\mathbb{P}^1)$ and $g''(\mathbb{P}^1)$ are transverse to E, applying Proposition 23 of [8] we see

$$\mathcal{N}_h \otimes \mathcal{O}_\ell \simeq \mathcal{O}_{\mathbb{P}^1}^{\dim(V)-r} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{r-2} \oplus \mathcal{O}_{\mathbb{P}^1}(1);$$

the quotient $(\mathcal{N}_h \otimes \mathcal{O}_\ell)/\mathcal{N}_{h|\ell}$ lies in the image of the positive summands.

Lemma 7 implies that $h : C \to W$ admits a deformation to a rational curve containing w. □

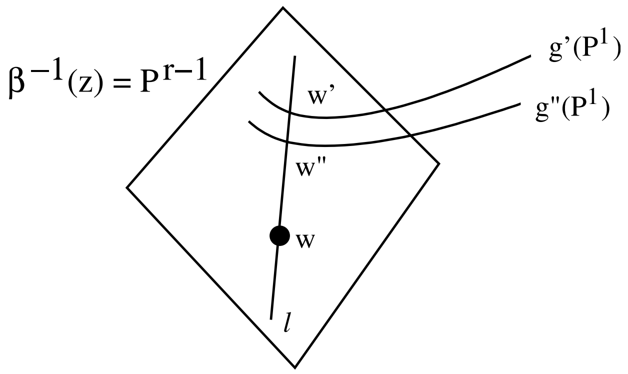

FIGURE 1. Constructing the comb

A similar argument gives the following strengthening of Proposition 10 (cf. Theorem 2.2 of [6])

Proposition 13. *Let V be a smooth separably rationally connected variety and $V^0 \subset V$ be the distinguished open subset characterized in Proposition 10. Then for any finite collection of jets*

$$j_i : \mathrm{Spec} k[\epsilon]/\left\langle \epsilon^{N+1} \right\rangle \hookrightarrow V^0, \quad i = 1, \ldots, m$$

supported at distinct points $v_1, \ldots, v_m$, there exists a very free rational curve smooth at $v_1, \ldots, v_m$ with the prescribed jets.

Proof. There is an iterated blow-up

$$\beta : W = W_N \to \ldots \to W_j \to \ldots \to W_1 \to V$$

and points $w_1, \ldots, w_m \in W$ so that if $g : C \to W$ is a morphism whose image contains $w_1, \ldots, w_m$ then the image of $f := \beta \circ g : C \to V$ contains the given collection of jets. Here is the description: Over each point v_i, we blow up V successively at N points. Given any smooth curve germ C with the prescribed N-jet at v_i, W_j is the blowup of W_{j-1} at the points of the proper transform of C lying over the v_i. Proposition 12 then implies there exists a very free curve $g : \mathbb{P}^1 \to W$ through $w_1, \ldots w_m$. However, the image of this curve in V will be singular at v_i if $g(\mathbb{P}^1)$ meets $\beta^{-1}(v_i)$ in more than one point.

We claim there exists a very free curve $g_i : \mathbb{P}^1 \to W$ meeting $\beta^{-1}(v_i)$ only at w_i, transversally. We choose this curve so that it is disjoint from $\beta^{-1}(v_j)$ when $j \neq i$. Fix generic points $x_i \in g_i(\mathbb{P}^1)$ and let $g_0 : \mathbb{P}^1 \to W$ be a very free curve intersecting $g_i(\mathbb{P}^1)$ transversely at x_i but not meeting any $\beta^{-1}(v_i)$. (For example, take $g_0 = (\beta^{-1} \circ f_0)$, where $f_0 : \mathbb{P}^1 \to V$ is a very free curve through $\beta(x_1), \ldots, \beta(x_m)$.) Consider the comb $h : C \to W$ with handle $g_0(\mathbb{P}^1)$ and m-teeth $g_i(\mathbb{P}^1)$. This deforms to a very free curve $h' : \mathbb{P}^1 \to W$ meeting each $\beta^{-1}(v_i)$ only at w_i, transversally.

The proof of the claim is a refinement of the argument for Proposition 12. We proceed by induction on N. The base case $N = 1$ is contained in the proof of Proposition 12, which gives a very free curve smooth at v_i with prescribed tangency. Let $E_{i,N} \simeq \mathbb{P}^{\dim(V)-1}$ be the last exceptional divisor of $\beta : W \to V$ over v_i, i.e., the exceptional divisor of the N-th blow-up. For $1 \leq j < N$, let $E_{i,j} \subset W_N$ denote the proper transform of the exceptional divisor of $W_j \to W_{j-1}$ over v_i; we have $E_{i,j} \simeq \mathrm{Bl}_{w_{i,j}} \mathbb{P}^{\dim(V)-1}$, where $w_{i,j}$ is the intersection of the proper transform of C with the exceptional divisor of $W_j \to W_{j-1}$.

Suppose that $g_i' : \mathbb{P}^1 \to W$ is a very free curve such that $\beta \circ g_i'$ is smooth with the desired $(N-1)$-jet at v_i. Let $w_i' = g_i'(\mathbb{P}^1) \cap \beta^{-1}(v_i)$ denote the unique point of intersection, which we assume is distinct from w_i. Let ℓ_N denote the line in $E_{i,N} \simeq \mathbb{P}^{\dim(V)-1}$ joining w_i and w_i', and z_{N-1} its point of intersection with $E_{i,N-1}$. Let $\ell_{N-1} \subset E_{i,N-1} \simeq \mathrm{Bl}_{w_{i,N-1}} \mathbb{P}^{\dim(V)-1}$ denote the proper transform of a line containing z_{N-1}, and z_{N-2} its point of intersection with $E_{i,N-2}$. Continue in this way, until we obtain $\ell_1 \subset E_{i,1}$, the proper transform of a line containing z_1. Finally, let $g_i'' : \mathbb{P}^1 \to W$ be a very free curve meeting the exceptional locus transversally at a generic point of ℓ_1. (See Figure 2.)

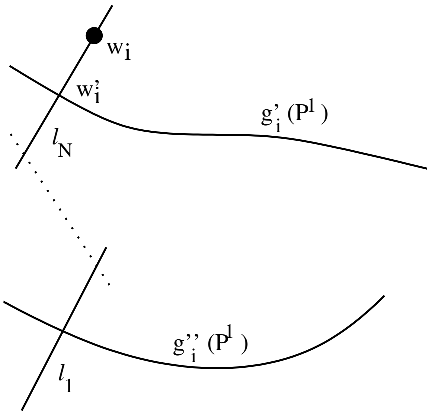

FIGURE 2. Constructing the comb with reducible teeth

Let $h : C \to W$ be the comb with handle ℓ_N and two reducible teeth:

(1) $g_i' : \mathbb{P}^1 \to W$;

(2) the union of the lines $\ell_{N-1}, \dots, \ell_1$ and the curve $g''_i : \mathbb{P}^1 \to W$;

By a normal bundle computation similar to that of Proposition 12, we find that $\mathcal{N}_h|\ell_N$ is ample and $\mathcal{N}_h$ is nonnegative on each of the remaining components: Again, Lemma 7 (or Proposition 24 of [8]) implies that h admits a deformation to an immersed rational curve containg w_i.

Here are the details of the computations (cf. [8] Section 5): The normal bundle of a line in projective space is

$$\mathcal{N}_{\ell_N/E_{i,N}} = \mathcal{N}_{\ell_N/\mathbb{P}^{\dim(V)-1}} \simeq \mathcal{O}_{\mathbb{P}^1}(+1)^{\dim(V)-2}$$

and the normal bundle for an exceptional divisor is

$$\mathcal{N}_{E_{i,N}/W} \simeq \mathcal{O}_{\mathbb{P}^{\dim(V)-1}}(-1).$$

For each j we have

$$0 \to \mathcal{N}_{\ell_j/E_{i,j}} \to \mathcal{N}_{\ell_j/W} \to \mathcal{N}_{E_{i,j}/W}|_{\ell_j} \to 0 \tag{4.1}$$

which for $j = N$ yields

$$\mathcal{N}_{\ell_N/W} \simeq \mathcal{O}_{\mathbb{P}^1}(+1)^{\dim(V)-2} \oplus \mathcal{O}_{\mathbb{P}^1}(-1),$$

with the negative component in the direction normal to $E_{i,N}$. We also have an extension

$$0 \to \mathcal{N}_{\ell_j/W} \to \mathcal{N}_h|_{\ell_j} \to Q(\ell_j) \to 0, \tag{4.2}$$

where $Q(\ell_j)$ is a torsion sheaf supported at the points where ℓ_j meets the adjacent components. For $j = N$ these are $g'_i(\mathbb{P}^1)$ and ℓ_{N-1}, and since the tangent vectors to these curves are normal to $E_{i,N}$, we find

$$\mathcal{N}_h|_{\ell_N} \simeq \mathcal{O}_{\mathbb{P}^1}(+1)^{\dim(V)-2} \oplus \mathcal{O}_{\mathbb{P}^1}(+1).$$

The normal bundle of the proper transform of a line in the blow-up of projective space at a point of the line is

$$\mathcal{N}_{\ell_j/E_{i,j}} = \mathcal{N}_{\ell_j/\mathrm{Bl}_{w_{i,j}}\mathbb{P}^{\dim(V)-1}} \simeq \mathcal{O}_{\mathbb{P}^1}^{\dim(V)-2}$$

for $j = 1, \dots, N-1$. Similarly, we can compute

$$\mathcal{N}_{E_{i,h}/W}|_{\ell_j} = \mathcal{O}_{\mathbb{P}^1}(-2)$$

so the exact sequence analogous to (4.1) yields

$$\mathcal{N}_{\ell_j/W} \simeq \mathcal{O}_{\mathbb{P}^1}^{\dim(V)-2} \oplus \mathcal{O}_{\mathbb{P}^1}(-2),$$

with the negative component in the direction normal to $E_{i,j}$. Using (4.2) and the fact that ℓ_j is adjacent to ℓ_{j+1} and ℓ_{j-1} (or $g''_i(\mathbb{P}^1)$ when $j = 1$), we find

$$\mathcal{N}_h|_{\ell_j} \simeq \mathcal{O}_{\mathbb{P}^1}^{\dim(V)-2} \oplus \mathcal{O}_{\mathbb{P}^1}.$$

□

Definition 14. A smooth separably rationally connected variety Y is *strongly rationally connected* if any of the following conditions hold:

(1) for each point $y \in Y$, there exists a rational curve $f : \mathbb{P}^1 \to Y$ joining y and a generic point in Y;
(2) for each point $y \in Y$, there exists a free rational curve containing y;
(3) for any finite collection of points $y_1, \ldots, y_m \in Y$, there exists a very free rational curve containing the y_j as smooth points;
(4) for any finite collection of jets
$$\operatorname{Spec} k[\epsilon]/\left\langle \epsilon^{N+1} \right\rangle \subset Y, i = 1, \ldots, m$$
supported at distinct points $y_1, \ldots, y_m$, there exists a very free rational curve smooth at $y_1, \ldots, y_m$ and containing the prescribed jets.

The implications
$$(4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$$
are obvious. By Proposition 10, assertions (1)-(3) are each equivalent to the condition $Y = Y^0$. Property (4) is analogous to Theorem 2.2 of [6], which is stated for proper varieties. It follows from (1) by Proposition 13.

With basic properties of strongly rationally connected varieties established, Theorem 1 follows from the general result (cf. [11] IV.6.10.1):

Theorem 15. *Let $\pi : \mathcal{Y} \to B$ be a smooth morphism whose fibers are strongly rationally connected. Assume that π has a section. Then sections of $\mathcal{Y} \to B$ satisfy approximation away from S.*

Proof. Let $\overline{\pi} : \overline{\mathcal{Y}} \to \overline{B}$ be a proper flat model of $\mathcal{Y} \to B$, which exists by [14]. The section extends to a section $\overline{s}$ of $\overline{\pi}$. By a result of Artin and Néron [1] Corollary 4.6, there exists a blow-up with center supported in $\overline{\pi}^{-1}(S)$
$$\widetilde{\mathcal{Y}} \to \overline{\mathcal{Y}}$$
such that the proper transform of $\overline{s}(\overline{B})$ in $\widetilde{\mathcal{Y}}$ is contained in $\widetilde{\mathcal{Y}}^{sm}$.

Recall the proof of weak approximation at places of good reduction in Section 5 of [8]. This is a bootstrap argument, using the existence of a section in the smooth locus to construct sections with prescribed jets of successively higher order. *Properness* is used only to establish that the smooth fibers are strongly rationally connected, so we can produce very free curves with desired properties. In our situation, this is part of the hypotheses. □

Weak approximation at places of good reduction in positive characteristic was left unresolved in [8]. However, combining Theorem 15 with the main result of [5] yields:

Corollary 16. *Let $\pi : \mathcal{Y} \to B$ be a smooth proper morphism with separably rationally connected fibers. Then weak approximation holds away from $S = \overline{B} \setminus B$.*

5. Cubic surfaces

We work over an algebraically closed field of characteristic zero.

Definition 17. A *log Del Pezzo surface* is a pair (X, Δ) consisting of a normal projective surface X and an effective $\mathbb{Q}$-divisor $\Delta = \sum a_i \Delta_i, 0 < a_i \leq 1$ on X, with log terminal singularities, such that $-(K_X + \Delta)$ is ample. When Δ is empty, this is equivalent to saying that X has quotient singularities and ample anticanonical class.

Theorem 18 ([9] 1.6)**.** *The smooth locus of a log Del Pezzo surface (X, Δ) is rationally connected, i.e., two generic points in X^{sm} can be joined by an irreducible projective rational curve contained in X^{sm}.*

Example 19 ([16])**.** There exist projective rational surfaces with rational double points whose smooth locus is not rationally connected. Consider

$$\widetilde{X} = E \times \mathbb{P}^1$$

where $(E, 0)$ is an elliptic curve and the involution

$$\begin{array}{rcl} \iota : \widetilde{X} & \to & \widetilde{X} \\ (e, [x_0, x_1]) & \mapsto & (-e, [x_1, x_0]). \end{array}$$

The involution has eight isolated fixed points $\mathfrak{q} \subset \widetilde{X}$. The quotient $X = \widetilde{X} / \langle \iota \rangle$ has eight A_1 singularities and is rational: $X \to E / \langle \iota \rangle \simeq \mathbb{P}^1$ is a conic bundle. Since $\widetilde{X} - \mathfrak{q} \to X^{sm}$ is a covering space, $\pi_1(X^{sm}) \subset \pi_1(\widetilde{X} - \mathfrak{q})$ with index two. Thus

$$\pi(\widetilde{X} - \mathfrak{q}) \simeq \pi(\widetilde{X}) \simeq \pi(E) \simeq \mathbb{Z} \times \mathbb{Z}$$

and X^{sm} has infinite fundamental group. However, rationally connected varieties (even non-proper ones) have finite fundamental groups (see Lemma 7.8 of [9] and Proposition 2.10 of [10], for example).

The following conjecture would allow us to apply Theorem 1 to prove weak approximation for many log Del Pezzo surfaces:

Conjecture 20. The smooth locus of a log Del Pezzo surface is strongly rationally connected.

We prove this for cubic surfaces:

Theorem 21. *Let $X \subset \mathbb{P}^3$ be a cubic surface with rational double points. Then X^{sm} is strongly rationally connected.*

Proof. Let $x_1 \in X^{sm}$ be a point. We produce a rational curve $R \subset X^{sm}$ joining x_1 and a generic point $x_2 \in X^{sm}$.

We will make explicit precisely how x_2 must be chosen. We assume:

(1) The tangent hyperplane section H_2 at x_2 is irreducible and nodal. In particular, $H_2 \subset X^{sm}$ and there are no lines $\ell \subset X$ containing x_2. Projection from x_2 then gives a double cover

$$\mathrm{Bl}_{x_2} X \to \mathbb{P}^2;$$

the covering transformation interchanges the exceptional divisor and the proper transform. We obtain a birational involution

$$\begin{array}{rcl} \iota_{x_2} : X & \dashrightarrow & X \\ x & \mapsto & x', \end{array}$$

where $\{x, x', x_2\}$ are collinear. This factors as the blow-up of x_2 followed by the blow-down of the proper transform of H_2. Note that ι_{x_2} fixes the singularities of X and thus takes X^{sm} to itself.

We also assume:

(2) H_2 does not contain x_1.

It follows that H_2 does not contain $x_1' = \iota_{x_2}(x_1)$. Moreover, x_1 and x_1' are in the open subset on which ι_{x_2} is an isomorphism.

We assume furthermore:

(3) x_2 is not contained in H_1.

It follows that $x_2 \notin H_1'$, the tangent hyperplane section at x_1'. Indeed, suppose that $x_2 \in H_1'$. We know that $x_2 \neq x_1'$ (because $x_1' \notin H_2$), so consider the line joining x_2 and x_1'. This meets X only at x_2 and x_1', so $x_1' = x_1$ and $x_2 \in H_1$, a contradiction.

Finally, we assume:

(4) H_1' is irreducible and nodal.

In particular, $H_1' \subset X^{sm}$.

Since $x_2 \notin H_1'$, ι_{x_2} is regular along H_1'. We verify that the rational curve $R = \iota_{x_2}(H_1')$ has the desired properties. Since $H_1' \subset X^{sm}$ and $\iota_{x_2}(X^{sm}) \subset X^{sm}$, we find $R \subset X^{sm}$. We have $x_1' \in H_1'$, so $x_1 = \iota_{x_2}(x_1') \in R$. Since H_1' meets H_2 in a point $y \neq x_2$, $x_2 = \iota_{x_2}(y) \in R$. $\square$

We now prove Theorem 2: For each singular fiber $\mathcal{X}_b$, $\mathcal{X}_b^{sm}$ is strongly rationally connected by Theorem 21. Approximation follows from Theorem 1.

Example 22. Here is another case where Conjecture 20 is easily verified. Let X be a partial resolution of a cubic surface Σ with at most A_1-singularities, i.e., we have a factorization of the minimal resolution

$$\widetilde{\Sigma} \to X \xrightarrow{\beta} \Sigma.$$

Then X^{sm} is strongly rationally connected.

Theorem 2 implies that Σ^{sm} is strongly rationally connected, hence $\beta^{-1}(\Sigma^{sm}) \subset (X^{sm})^0$. The locus $X^{sm} \setminus \beta^{-1}(\Sigma^{sm})$ is a union of (-2)-curves $\{E_i\}$, corresponding to the resolved singularities $\{p_i\}$ of Σ. If $(X^{sm})^0$ meets E_i, it must also contain E_i. Hence it suffices to show that for each E_i there exists a rational curve in X^{sm} meeting E_i and $\beta^{-1}(\Sigma^{sm})$ (see Remark 11).

To find this rational curve, consider the projection from p_i

$$\pi_i : \Sigma \dashrightarrow \mathbb{P}^2$$

which induces a morphism $\pi_i' : X \to \mathbb{P}^2$. The image of E_i is a plane conic and the image of the singularities of X has codimension two in $\mathbb{P}^2$, so there exists a rational curve

$$f : \mathbb{P}^1 \to \mathbb{P}^2 \setminus \pi_i'(\mathrm{Sing}(X))$$

meeting the image of E_i.

The same argument applies if X is obtained from a cubic surface Σ with A_1 and A_2 singularities by resolving some subset of $\mathrm{Sing}(\Sigma)$.

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Department of Mathematics, Rice University, MS 136, Houston, Texas 77251-1892, USA
E-mail address: hassett@rice.edu

Mathematisches Institut, Bunsenstr. 3-5, 37073 Göttingen, Germany, Courant Institute, NYU, 251 Mercer str., New York, NY 10012, USA
E-mail address: tschinkel@cims.nyu.edu