Rational points and automorphic forms

J. Shalika, R. Takloo-Bighash, and Yu. Tschinkel

May 22, 2002

Abstract

We study the distribution of rational points of bounded height on certain equivariant compactifications of anisotropic inner forms of semi-simple groups.

Contents

1 Introduction

Let $\mathbf{x} \in \mathbb{P}^n(\mathbb{Q})$ be a $\mathbb{Q}$-rational point in the projective space of dimension n with coordinates $\mathbf{x} = (x_0 : x_1 : \cdots : x_n)$, such that

$$(x_0, x_1, \ldots, x_n) \in \mathbb{Z}^{n+1}_{\mathrm{prim}},$$

that is, the set of primitive $(n+1)$-tuples of integers. Define a height function

$$H(\mathbf{x}) := \max_j(|x_j|).$$

Of course, we could replace this norm by any other norm on $\mathbb{R}^{n+1}$, for example $\sqrt{x_0^2 + \cdots + x_n^2}$. Generally, for any number field F and $\mathbf{x} \in \mathbb{P}^n(F)$ we can define

$$H(\mathbf{x}) := \prod_{v \in \mathrm{Val}(F)} \max_j(|x_j|_v),$$

where the product is over all valuations of F. By the product formula, this does not depend on a particular choice of homogeneous coordinates for $\mathbf{x}$. Clearly, the number

$$N(\mathbb{P}^n, B) := \#\{\mathbf{x} \in \mathbb{P}^n(F) \mid H(\mathbf{x}) \leq B\}$$

is finite, for any $B > 0$. In 1964 Schanuel computed its asymptotic behavior, as $B \to \infty$,

$$N(\mathbb{P}^n, B) = \mathsf{c}(n, F, H) \cdot B^{n+1}(1 + o(1)),$$

where $\mathsf{c}(n, F, H)$ is an explicit constant (see [34]).

Let X be an algebraic variety over a F and $\mu : X \longrightarrow \mathbb{P}^n$ a projective embedding. Then $H \circ \mu$ defines a height function on the set of F-rational points $X(F)$ (more conceptually, the height is defined by means of an *adelic metrization* $\mathcal{L} = (L, \|\cdot\|_{\mathbb{A}})$ of the line bundle $L := \mu^*(\mathcal{O}(1))$). We obtain an induced counting function

$$N(X, \mathcal{L}, B) := \#\{\mathbf{x} \in X(F) \mid H \circ \mu(\mathbf{x}) \leq B\}.$$

One of the main themes of modern arithmetic geometry and number theory is the study of distribution properties of rational points on algebraic varieties. In particular, one is interested in understanding the asymptotic distribution of rational points of bounded height.

All theoretical and numerical evidence available so far indicates that one should expect an asymptotic expression of the form

$$N(X, \mathcal{L}, B) = \mathsf{c} \cdot B^{\mathsf{a}} \log(B)^{\mathsf{b}-1}(1 + o(1)),$$

for some $\mathsf{a} \in \mathbb{Q}, \mathsf{b} \in \frac{1}{2}\mathbb{Z}$ and a positive real c. In 1987 Manin had initiated a program aimed at interpreting the constants a, b and c in terms of intrinsic algebro-geometric and arithmetic invariants of X. The main observation was that a and b should depend only on the class of the embedding line bundle L in the Picard group $\mathrm{Pic}(X)$ of the variety X, more precisely its position with respect to the anticanonical class $[-K_X]$ and the cone of effective divisors $\Lambda_{\mathsf{eff}}(X) \subset \mathrm{Pic}(X)_{\mathbb{R}}$. The constant c, on the other hand, should reflect the dependence of the asymptotic expression on finer structures (like the choice of a norm in the definition of the height and p-adic densities).

Of course, it may happen that X has no rational points at all, or that $X(F)$ is entirely contained in a proper Zariski closed subset. In these cases, it is hopeless to try to read off the geometry of X from the asymptotics of rational points. We will therefore assume that $X(F)$ is Zariski dense. In general, it is not so easy to produce examples of interesting varieties with a Zariski dense set of rational points (unless, of course, X admits an action of an algebraic group with a Zariski dense orbit). For example, X could be a flag variety or an abelian variety. It is expected that the density of points (at least after a finite extension of the groundfield) holds for *Fano* varieties (that is, varieties with ample anticanonical class $[-K_X]$). This question is still open even in dimension 3 (see [20]). Here is a version of Manin's conjecture:

Conjecture 1.1 *Let X be an algebraic variety over a number field F such that its anticanonical class $[-K_X]$ is ample and $X(F)$ is Zariski dense. Then there exists a Zariski open subset $U \subset X$ such that*

$$N(U, -\mathcal{K}_X, B) = \mathsf{c}(\mathcal{K}_X) \cdot B(\log B)^{\mathsf{b}(X)-1}(1 + o(1))$$

for $B \to \infty$, where $-\mathcal{K}_X$ is a (metrized) anticanonical line bundle, $\mathsf{b}(X)$ is the rank of the Picard group $\mathrm{Pic}(X)$ *and* $\mathsf{c}(\mathcal{K}_X)$ *a non-zero constant.*

Remark 1.2 The restriction to Zariski open subsets is necessary since X may contain *accumulating* subvarieties (the asymptotics of rational points on these subvarieties will dominate the asymptotics of the complement). The constant $\mathsf{c}(\mathcal{K}_X)$ has an interpretation as a Tamagawa number (defined by Peyre in [32]). Finally, there is a similar description for arbitrary ample line bundles, proposed in [3], resp. [8]).

Conjecture 1.1 and its refinements have been proved for the following classes of varieties:

- smooth complete intersections of small degree in $\mathbb{P}^n$ (circle method);
- generalized flag varieties [17];
- toric varieties [5], [6];
- horospherical varieties [39];
- equivariant compactifications of $\mathbb{G}_a^n$ [11];

- bi-equivariant compactifications of unipotent groups [37], [38].

We expect that Manin's conjecture (and its refinements) should hold for equivariant compactifications of *all* linear algebraic groups G and their homogeneous spaces G/H. We provide further evidence for this expectation by outlining a proof of the above conjecture for certain smooth equivariant compactifications of $\mathbb{Q}$-anisotropic semi-simple $\mathbb{Q}$-groups of adjoint type.

This work focuses on the interplay between arithmetic geometry and automorphic forms. Though the main problem is inspired by Manin's conjecture in arithmetic geometry, our tools and techniques, which are naturally suited to the current context, are from the theory of automorphic forms and representations of p-adic groups. Our approach is inspired by the work of Batyrev and Tschinkel on compactifications of anisotropic tori [4] and the work of Godement and Jacquet on central simple algebras [19]. We are currently working on a generalization of our results to higher rank, where the presence of the Eisenstein series makes the problem even more interesting from the analytic point of view.

Finally, we would like to mention related work of Duke, Rudnick and Sarnak [14], Eskin, McMullen [15], Eskin, Mozes and Shah [16] on asymptotics of *integral* points of bounded height on homogeneous varieties. Their theorems neither imply our results nor follow from them.

Acknowledgements. We have greatly benefited from conversations with Arthur and Sarnak. The second author wishes to thank the Clay Mathematics Institute for partial support of this project. The third author was partially supported by the NSA, NSF and the Clay Mathematics Institute.

2 Methods and results

Let F be a number field and D a central simple algebra of rank m over F. Let Λ be an arbitrary lattice in D. Denote by $\mathrm{Val}(F)$ the set of all valuations and by S_∞ the subset of archimedean valuations of F. For each $v \in \mathrm{Val}(F)$, we put $\mathrm{D}_v = \mathrm{D} \otimes_F F_v$ and, for $v \notin S_\infty$, $\Lambda_v = \Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_v$. For almost all v, Λ_v is a maximal order in D_v. We proceed to define a family of norms $\|\cdot\|_{\Lambda_v}$ on D_v, one for each place v of F.

- *nonarchimedean v:* Choose a basis $\{\xi_1^v, \dots, \xi_k^v\}$ for Λ_v. For $g \in \mathrm{D}_v$, write $g = \sum_i c_i(g)\xi_i^v$ and set

$$\|g\|_v = \|g\|_{\Lambda_v} := \max_{i=1,\dots,k} \{|c_i(g)|_v\}.$$

  It is easy to see that this norm is independent of the choice of basis.

- *archimedean v:* Fix a Banach space norm $\|\cdot\|_v = \|\cdot\|_{\mathrm{D}_v}$ on the finite-dimensional real (or complex) vector space $\mathrm{D}_v = \mathrm{D} \otimes_F F_v$.

Clearly, for $c \in F_v$ and $g \in \mathrm{D}_v$, we have

$$\|cg\|_v = |c|_v \cdot \|g\|_v.$$

Consequently, for $c \in F^\times$ and $g \in \mathrm{D}$, we have

$$\prod_{v \in \mathrm{Val}(F)} \|cg\|_v = \prod_{v \in \mathrm{Val}(F)} \|g\|_v, \tag{1}$$

by the product formula. For an adelic point $g = (g_v)_v \in \mathrm{D}(\mathbb{A})$ define the *global height function*:

$$H(g) := \prod_{v \in \mathrm{Val}(F)} H_v(g) = \prod_{v \in \mathrm{Val}(F)} \|g_v\|_v.$$

By the product formula, H is well-defined on the projective group $\mathrm{D}(F)^\times/F^\times$. Moreover, H is invariant under the right and left action of a compact open subgroup

$$\mathrm{K}_0 = \prod_{v \notin S_\infty} \mathrm{K}_{0,v} \subset \mathrm{G}(\mathbb{A}_{\mathrm{fin}})$$

(if we fix an integral model for G then $\mathrm{K}_{0,v} = \mathrm{G}(\mathcal{O}_v)$, for almost all v). It will be convenient to assume that the Haar measure dg is such that $\mathrm{vol}(\mathrm{K}_0) = 1$.

From now on, we let G be an F-anisotropic inner form of a split semi-simple group $\tilde{\mathrm{G}}$ of adjoint type over a number field F. Let

$$\varrho_F : \mathrm{G} \longrightarrow \mathrm{D}^\times$$

be an F-group morphism from G to the multiplicative group of a central simple algebra over F of rank m. Extending scalars to a finite Galois extension E/F over which both G and D are split, we obtain a homomorphism

$$\varrho_E \ : \ \tilde{\mathrm{G}}(E) \longrightarrow \mathrm{GL}_m(E).$$

This homomorphism is obtained from an algebraic representation

$$\varrho \ : \ \tilde{\mathrm{G}} \longrightarrow \mathrm{GL}_m,$$

defined over F.

Remark 2.1 Conversely, from any algebraic representation $\varrho : \tilde{\mathrm{G}} \longrightarrow \mathrm{GL}_m$ over F we obtain a group homomorphism

$$\varrho_E : \tilde{\mathrm{G}}(E) \longrightarrow \mathrm{GL}_m(E),$$

which induces a map

$$\varrho_E^* : \mathrm{Z}^1(\mathrm{Gal}(E/F), \tilde{\mathrm{G}}(E)) \longrightarrow \mathrm{Z}^1(\mathrm{Gal}(E/F), \mathrm{PGL}_m(E)).$$

Let $c \in \mathrm{Z}^1(\mathrm{Gal}(E/F), \tilde{\mathrm{G}}(E))$ be the cocycle that defines the inner form G. Then $\varrho_E^*(c)$ defines a central simple algebra $\mathrm{D} \subset \mathrm{Mat}_m(E)$. It is easy to verify that ϱ_E descends to a morphism of F-groups

$$\varrho_F \ : \ \mathrm{G} \to \mathrm{D}^\times.$$

Thus we can use ϱ_F to pull back the height function from $\mathrm{D}^\times$ to G. We are interested in the asymptotics of

$$N(\varrho, B) := \#\{\gamma \in \mathrm{G}(F) \ | \ H(\varrho_F(\gamma)) \leq B\},$$

as $B \to \infty$. To put this in geometric perspective, the pair (G, ϱ_F) defines an equivariant compactification X of G and a G-linearized ample line bundle on X (and vice versa). Thus we are counting rational points on a Zariski open subset $\mathrm{G} \subset X$, with respect to some adelically metrized line bundle (depending on ϱ). Below we will see that for appropriate choices of ϱ the asymptotic formula for $N(\varrho, B)$ matches precisely Manin's prediction.

Our main technical assumption is the following:

Assumption 2.2 *The representation ϱ_F is absolutely irreducible.*

In order to state our theorem we need to introduce some notation. Fix a Borel subgroup B with maximal split torus T in $\tilde{\mathrm{G}}$ and denote by $X^*(\mathrm{T})$ the character group of T. Let Φ be the root system of $(\tilde{\mathrm{G}}, \mathrm{T})$, and $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ the set of simple roots. Also let $2\rho_{\mathrm{G}} = \sum_{\alpha \in \Phi^+} \alpha$. Since $\tilde{\mathrm{G}}$ is of adjoint type it is immediate that there are one-parameter subgroups $\{\hat{\alpha}_1, \ldots, \hat{\alpha}_r\}$ of T such that

$$< \hat{\alpha}_j, \alpha_i >= -\delta_{ij}.$$

Let $\varrho = \varrho_\lambda$ be the irreducible algebraic representation of $\tilde{\mathrm{G}}$ associated with a dominant weight λ. Let χ_λ be the character of T associated with λ. Since λ is dominant and $\tilde{\mathrm{G}}$ is of adjoint type, there exist non-negative integers $k_1(\varrho), \ldots, k_r(\varrho)$ such that

$$\chi_\lambda(t) = \prod_{i=1}^{r} \alpha_i(t)^{k_i(\varrho)}.$$

The numbers $k_i(\varrho)$, $1 \leq i \leq r$, are all non-zero if the representation ϱ is non-trivial. Set then

$$\mathsf{a}_\varrho := \max_{j=1,\ldots,r} \frac{1- < \hat{\alpha}_j, 2\rho_{\mathrm{G}} >}{k_j(\varrho)}, \quad \text{and } \mathsf{b}_\varrho := \#\{j \mid \frac{1- < \hat{\alpha}_j, 2\rho_{\mathrm{G}} >}{k_j(\varrho)} = \mathsf{a}_\varrho\}.$$

Also set

$$\mathsf{c}_\varrho := \lim_{s \to \mathsf{a}_\varrho} (s - \mathsf{a}_\varrho)^{\mathsf{b}_\varrho} \int_{\mathrm{G}(\mathbb{A})} H(\varrho_F(g))^{-s}\, dg,$$

(where dg is a suitably normalized Haar measure on $\mathrm{G}(\mathbb{A})$).

Remark 2.3 The anticanonical embedding of the *wonderful* compactification of G of de Concini-Procesi is associated with the weight $\kappa = 2\rho_{\mathrm{G}} + \sum_{i=1}^{r} \alpha_i$ (see [13],[9]).

It is not hard to see that if $\varrho = \varrho_\kappa$, then $\mathsf{a}_\varrho = 1$ and $\mathsf{b}_\varrho = r$. Our main theorem is the following:

Theorem 2.4 *For $\varrho = \varrho_\kappa$ we have*

$$N(\varrho, B) = \frac{\mathsf{c}_\varrho}{(r-1)!} \cdot B(\log B)^{r-1}(1 + o(1)),$$

as $B \to \infty$.

We note that this theorem implies Manin's conjecture for the wonderful compactification of G as above. We have also proved analogous results for arbitrary irreducible representations ϱ (in other words, for height functions associated with arbitrary ample line bundles on the wonderful compactification of G).

We will now sketch the proof (in the case $\varrho = \varrho_\kappa$). Using Tauberian theorems one deduces the asymptotic properties of $N(\varrho, B)$ from the analytic properties of the *height zeta function*

$$\mathcal{Z}(s, \varrho) = \sum_{\gamma \in \mathrm{G}(F)} H(\varrho_F(\gamma))^{-s}.$$

Actually, we will use the function

$$\mathcal{Z}(s, \varrho, g) = \sum_{\gamma \in \mathrm{G}(F)} H(\varrho_F(\gamma g))^{-s}.$$

For $\Re(s) \gg 0$, the right hand side converges (uniformly on compacts) to a function which is holomorphic in s and continuous in g on $\mathbb{C} \times \mathrm{G}(\mathbb{A})$. Since G is F-anisotropic, $\mathrm{G}(F)\backslash\mathrm{G}(\mathbb{A})$ is compact, and

$$\mathcal{Z} \in \mathsf{L}^2(\mathrm{G}(F)\backslash\mathrm{G}(\mathbb{A}))^{\mathrm{K}_0}$$

(recall that H is bi-invariant under K_0). Again since G is anisotropic, we have

$$\mathsf{L}^2(\mathrm{G}(F)\backslash\mathrm{G}(\mathbb{A})) = (\hat{\bigoplus_{\pi}}\mathcal{H}_\pi) \bigoplus (\bigoplus_{\chi} \mathbb{C}_\chi), \tag{2}$$

as a direct sum of irreducible subspaces. Here the first direct sum is over infinite-dimensional representations of $\mathrm{G}(\mathbb{A})$ and the second direct sum is a sum over all automorphic characters of $\mathrm{G}(\mathbb{A})$. Consequently,

$$\mathsf{L}^2(\mathrm{G}(F)\backslash\mathrm{G}(\mathbb{A}))^{\mathrm{K}_0} = (\hat{\bigoplus_{\pi}}\mathcal{H}_\pi^{\mathrm{K}_0}) \bigoplus (\bigoplus_{\chi} \mathbb{C}_\chi), \tag{3}$$

a sum over representations containing a K_0-fixed vector (in particular, the sum over characters is *finite*). For each infinite-dimensional π occurring in

(3) we choose an orthonormal basis $\mathcal{B}_\pi = \{\phi_\alpha^\pi\}_\alpha$ for $\mathcal{H}_\pi^{\mathrm{K}_0}$. We have next the "Poisson formula":

$$\mathcal{Z}(s,\varrho,g) = \sum_\pi \sum_{\phi\in\mathcal{B}_\pi} \langle \mathcal{Z}(s,\varrho,g),\phi(g)\rangle\phi(g) + \sum_\chi \langle \mathcal{Z}(s,\varrho,g),\chi(g)\rangle\chi(g). \quad (4)$$

Here the series on the right is seen to converge normally to $\mathcal{Z}(\varrho,g)$ for $\Re(s)\gg 0$. We will establish a meromorphic continuation of the right hand side of (4), leading to a proof of the main theorem.

A key result is the computation of the individual inner products $\langle\mathcal{Z},\phi\rangle$. After the usual unfolding it turns out that each of these is an Euler product with an explicit regularization. In particular, the pole of highest order of $\mathcal{Z}(s,\varrho,g)$ (or the main term in the asymptotic expression of $N(\varrho,B)$) is contributed by the trivial representation:

$$\int_{\mathrm{G}(\mathbb{A})} H(\varrho_F(g))^{-s}dg = \prod_{v\in \mathrm{Val}(F)} \int_{\mathrm{G}(F_v)} H_v(\varrho_F(g_v))^{-s}dg_v.$$

Local integrals of such type can be computed explicitly at almost all places (see [11]). They are reminiscent of Igusa's local zeta functions and their modern generalizations: "motivic" integrals of Batyrev, Kontsevich and Denef-Loeser (see [2], [12]). In our case, we have

$$\int_{\mathrm{G}(\mathbb{A})} H(\varrho_F(g))^{-s}dg = \prod_{j=1}^r \zeta_F(k_js+<\hat{\alpha}_j,2\rho_{\mathrm{G}}>)\cdot h(s,\varrho),$$

(where $h(s,\varrho)$ is a holomorphic function for $\Re(s)>1-\epsilon$ and some $\epsilon>0$).

Next we prove that each remaining term is holomorphic around $\Re(s)=1$. In general, we have

$$\begin{aligned}\langle\mathcal{Z},\phi\rangle &= \int_{\mathrm{G}(F)\backslash\mathrm{G}(\mathbb{A})} \mathcal{Z}(s,\varrho,g)\overline{\phi(g)}\,dg \\ &= \int_{\mathrm{G}(\mathbb{A})} H(\varrho_F(g))^{-s}\overline{\phi(g)}\,dg \\ &= \int_{\mathrm{G}(\mathbb{A})} H(\varrho_F(g))^{-s}\int_{\mathrm{K}_0}\overline{\phi(kg)}\,dk\,dg\end{aligned}$$

Next we follow an argument by Godement and Jacquet in [19]. Without loss of generality we can assume that

$$\mathrm{K}_0 = \prod_{v \notin S} \mathrm{K}_v \times \mathrm{K}_0^S,$$

for a finite set of places S. Here for $v \notin S$, K_v is a maximal special open compact subgroup in $\mathrm{G}(F_v)$. After enlarging S to contain all the places where G is not split, we can assume that $\mathrm{K}_v = \mathrm{G}(\mathcal{O}_v)$. In particular, for $v \notin S$ the local representations π_v are spherical. Thus we have a normalized local spherical function φ_v associated to π_v. We have assumed that each ϕ is right K_0-invariant. In conclusion,

$$\begin{aligned}\langle \mathcal{Z}, \phi \rangle = &\prod_{v \notin S} \int_{\mathrm{G}(F_v)} \varphi_v(g_v) H_v(\varrho_F(g_v))^{-s}\, dg_v \\ &\times \int_{\mathrm{G}(\mathbb{A}_S)} H(\varrho_F(\eta(g_S)))^{-s} \int_{\mathrm{K}_0^S} \phi(k\eta(g_S))\, dk\, dg_S.\end{aligned}$$

(Here $\eta : \mathrm{G}(\mathbb{A}_S) \to \mathrm{G}(\mathbb{A})$ is the natural inclusion map.) The second factor is relatively easy to deal with. Our main concern here is with the first factor. To proceed we need to invoke some fairly deep results from the representation theory of reductive groups to find non-trivial bounds on spherical functions. Depending on the semi-simple rank of G, there are two cases to consider:

Case 1: semi-simple rank 1. In this situation, G is an inner form of PGL_2 - that is, the projective group of a quaternion algebra. By the Jacquet-Langlands correspondence [21], there is an irreducible cuspidal automorphic representation $\pi' = \otimes_v \pi'_v$ of PGL_2 such that for $v \notin S$, we have $\pi_v = \pi'_v$. In particular, in order to obtain non-trivial bounds on spherical functions of infinite dimensional representations, we need to examine local components of cuspidal representations of GL_2 with trivial central character. Here the estimate we need follows from a classical result of Gelbart and Jacquet who established the symmetric square lifting from GL_2 to GL_3 [18], combined with a result of Jacquet and Shalika (see [22]). We also note the recent beautiful results of Kim and Shahidi towards sharper bounds in the Ramanujan-Petersson conjecture [23].

Case 2: semi-simple rank > 1. First we use a strong approximation argument to show that for $v \notin S$, the representation π_v is not one-dimensional, unless

π itself is one-dimensional (a similar argument appears in the work of Clozel and Ullmo [10]). Then we apply a recent result of Oh [31] which says that in the local situation, the one-dimensional representations are isolated in the the unitary dual of any semi-simple group of semi-simple rank greater than two, giving non-trivial bounds for spherical functions.

Putting everything together, we obtain that, for non-trivial representations, the inner product $\langle \mathcal{Z}, \phi \rangle$ is *holomorphic* for $\Re(s) > 1 - \epsilon$, (for some $\epsilon > 0$).

Finally, to prove the convergence of the right hand side (for appropriate s), we integrate by parts (with respect to the Laplacian Δ on the compact Riemannian manifold associated with $\mathrm{G}(\mathbb{A})$ and K_0), and combine L^∞-estimates for Δ-eigenfunctions with standard facts about spectral zeta functions of compact manifolds.

Remark 2.5 Similar arguments lead to a proof of *equidistribution* of rational points of bounded anticanonical height with respect to the Tamagawa measure associated with the metrization of $-K_X$.

References

[1] J. Arthur, L. Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Annals of Mathematics Studies, **120**, Princeton University Press, (1989).

[2] V. V. Batyrev, *Non-archimedean integrals and stringy Euler numbers of log terminal pairs*, Journ. of EMS **1**, 5–33, (1999).

[3] V. V. Batyrev, Yu. I. Manin, *Sur le nombre de points rationnels de hauteur bornée des variétés algébriques*, Math. Ann. **286**, 27–43, (1990).

[4] , V. V. Batyrev, Yu. Tschinkel, *Rational points of bounded height on compactifications of anisotropic tori*, Intern. Math. Res. Notices **12**, 591–635, (1995).

[5] V. V. Batyrev, Yu. Tschinkel, *Manin's conjecture for toric varieties*, Journ. Algebraic Geometry **7**, no. 1, 15–53, (1998).

[6] V. V. Batyrev, Yu. Tschinkel, *Height zeta functions of toric varieties,* Algebraic Geometry 5, (Manin's Festschrift), Journ. Math Sci, 82, no. 1, 3220–3239, (1998).

[7] V. V. Batyrev, Yu. Tschinkel, *Rational points on some Fano cubic bundles*, C.R. Acad. Sci., Paris **323**, Ser I, 41–46, (1996).

[8] V. V. Batyrev, Yu. Tschinkel, *Tamagawa numbers of polarized algebraic varieties*, in Nombre et répartition des points de hauteur bornée Asterisque **251**, 299–340, (1998).

[9] M. Brion, P. Polo, *Large Schubert varieties*, Journ. Repr. Th. **4**, 97–126, (2000).

[10] L. Clozel, E. Ullmo, *Equidistribution des points de Hecke*, to appear in Inventiones Math.

[11] A. Chambert-Loir, Yu. Tschinkel, *On the distribution of points of bounded height on equivariant compactifications of vector groups*, Inventiones Math. **48**, no. 2, 421–452, (2002).

[12] J. Denef, F. Loeser, *Germs of arcs on singular algebraic varieties and motivic integration*, Inventiones Math. **135**, 201–232, (1999).

[13] C. De Concini, C. Procesi, *Complete symmetric varieties*, LMN **996**, 1–44, (1983).

[14] W. Duke, Z. Rudnick, P. Sarnak, *Density of integer points on affine homogeneous spaces*, Duke Math. Journ. **71**, 181–209, (1993).

[15] A. Eskin, C. McMullen, *Mixing, counting and equidistribution in Lie group*, Duke Math. Journ. **71**, 143–180, (1993).

[16] A. Eskin, S. Mozes, N. Shah, *Unipotent flows and counting lattice points in homogeneous varieties*, Annals of Math. **143**, 253–299, (1996).

[17] J. Franke, Yu. I. Manin, Yu. Tschinkel *Rational points of bounded height on Fano varieties*, Inventiones Math. **95** no. 2, 421–435, (1989).

[18] S. Gelbart, H. Jacquet, *A relation between automorphic representations of* GL(2) *and* GL(3), Ann. Sci. École Norm. Sup. **(4) 11**, 471–542, (1978).

[19] R. Godement, H. Jacquet, *Zeta functions of simple algebras*, LNM **260**, (1972).

[20] J. Harris, Yu. Tschinkel, *Rational points on quartics*, Duke Math. Journ. **104**, no. 3, 477–500, (2000).

[21] H. Jacquet, R. Langlands *Automorphic forms on* GL(2), LNM **114**, Springer-Verlag, Berlin-New York, (1970)

[22] H. Jacquet, J. Shalika, *On Euler products and the classification of automorphic forms*, I and II, Amer. Journ. of Math., **103**, no. 3 and 4, 499–558 and 777-815, (1981).

[23] H. Kim, F. Shahidi, *Functorial products for* $\mathrm{GL}_2 \times \mathrm{GL}_3$ *and functorial symmetric cube for* GL_2, C. R. Acad. Sci. Paris Sér. I Math. **331**, no. 8, 599–604, (2000).

[24] H. Kim, F. Shahidi, *Cuspidality of symmetric powers with applications*, preprint (2000).

[25] J.-P. Labesse, J. Schwermer, *On liftings and cusp cohomology of arithmetic groups*, Inventiones Math. **83** no. 2, 383–401, (1986).

[26] R. Langlands, D. Ramakrishnan, *The zeta functions of Picard modular surfaces*, CRM Publications, (1992).

[27] W. Luo, Z. Rudnick, P. Sarnak, *On the generalized Ramanujan conjecture for* GL(n), Automorphic forms, automorphic representations, and arithmetic (Fort Worth, TX, 1996), 301–310, Proc. Sympos. Pure Math., **66**, Part 2, AMS, Providence, RI, (1999).

[28] I. G. Macdonald, *Special functions on a group of p-adic type*, Publ. Ramanujan Inst. for Adv. Studies, (1971).

[29] Moeglin, Waldspurger, *Le spectre résiduel de* GL(n), Ann. Sci. Ecole Norm. Sup. **(4) 22**, no. 4, 605–674, (1989).

[30] H. Oh, *Tempered subgroups and representations with minimal decay of matrix coefficients*, Bull. Soc. Math. France **126**, no. 3, 355–380, (1998).

[31] H. Oh, *Uniform pointwise bounds for matrix coefficients of unitary representations and applications to Kazhdan's constants*, preprint (1999).

[32] E. Peyre, *Hauteurs et mesures de Tamagawa sur les variétés de Fano*, Duke Math. Journ. **79** no. 1, 101–218, (1995).

[33] J. Rogawski, *Automorphic representations of unitary groups in three variables*, Ann. of Math. Studies **123**, Princeton Univ. Press, (1990).

[34] J.-P. Serre, *Lectures on the Mordell-Weil theorem*, Aspects of Mathematics, Vieweg, Braunschweig, (1997).

[35] S. Schanuel, *On heights in number fields*, Bull. Amer. Math. Soc. **70**, 262–263, (1964).

[36] J. Shalika, R. Takloo-Bighash, Yu. Tschinkel, *Distribution of rational points on equivariant compactifications of reductive groups I*, preprint (2001).

[37] J. Shalika, Yu. Tschinkel, *Rational points on equivariant compactifications of the Heisenberg group*, preprint (2001).

[38] J. Shalika, Yu. Tschinkel, *Height zeta functions of equivariant compactifications of unipotent groups*, in preparation.

[39] M. Strauch, Yu. Tschinkel, *Height zeta functions of toric bundles over flag varieties,* Selecta Math. **5(3)**, 325-396, (1999).