

# Unramified correspondences

Fedor Bogomolov and Yuri Tschinkel

ABSTRACT. We study correspondences between algebraic curves defined over an algebraic closure of  $\mathbb{Q}$  or  $\mathbf{F}_p$ .

## 1. Introduction

A class  $\mathcal{C}(\overline{\mathbb{Q}})$  of complete algebraic curves over  $\overline{\mathbb{Q}}$  will be called *dominating* if for every algebraic curve  $C'$  over  $\overline{\mathbb{Q}}$  there exist a curve  $\tilde{C} \in \mathcal{C}(\overline{\mathbb{Q}})$  and a surjective map  $\tilde{C} \rightarrow C'$ . A curve  $C$  will be called *universal* if the class  $\mathcal{U}_C(\overline{\mathbb{Q}})$  of its unramified covers is dominating.

THEOREM 1.1 (Belyi). *Every algebraic curve  $C$  defined over a number field admits a surjective map onto  $\mathbb{P}^1$  which is unramified outside  $(0, 1, \infty)$ .*

In 1978 Manin pointed out that Belyi's theorem implies the following

PROPOSITION 1.2. *The class  $\mathcal{MU}(\overline{\mathbb{Q}})$  consisting of modular curves and their unramified covers is dominating.*

There are many other classes of curves with the same property, for example:

- (1) hyperelliptic curves and their unramified coverings;
- (2) the class  $\mathcal{CU}(\overline{\mathbb{Q}}) := \cup_{n \in \mathbb{N}} \mathcal{C}_n(\overline{\mathbb{Q}})$ , with  $\mathcal{C}_n(\overline{\mathbb{Q}})$  consisting of curves with function field  $\mathbb{Q}(z, \sqrt[n]{z(1-z)})$  and their unramified coverings.
- (3) the class  $\mathcal{CN}(\overline{\mathbb{Q}}) := \cup_{n \in \mathbb{N}} \mathcal{CN}_n(\overline{\mathbb{Q}})$  where  $\mathcal{CN}_n(\overline{\mathbb{Q}})$  consists of all unramified covers of any curve  $C_n$  with the property that  $C_n \rightarrow \mathbb{P}^1$  is ramified in  $(0, 1, \infty)$  only and all local ramification indices of  $C_n$  over 0 are divisible by 3, over 1 divisible by 2 and over  $\infty$  divisible by  $n$ . In particular, we could take  $C_n$  to be the modular curve  $X(n)$ .

PROOF. (Sketch) Let us consider the class of hyperelliptic curves and their unramified covers. Let  $C'$  be an arbitrary curve and  $\sigma : C' \rightarrow \mathbb{P}^1$  a *generic* map, branched over the points  $q_1, \dots, q_n$  (generic means that there is only one ramification point over each branch point and all local ramification indices are equal to 2). Denote by  $C$  a hyperelliptic curve whose ramification contains  $q_1, \dots, q_n$ . Then

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$\tilde{C} := C \times_{\mathbb{P}^1} C'$  is an unramified cover of  $C$  which surjects onto  $C'$ . For the classes  $\mathcal{CU}(\overline{\mathbb{Q}})$  and  $\mathcal{CN}(\overline{\mathbb{Q}})$  we use Belyi's theorem.  $\square$

QUESTION 1.3. Does there exist a universal algebraic curve  $C$  (over  $\overline{\mathbb{Q}}$ )?

QUESTION 1.4. Does there exist a number  $n \in \mathbb{N}$  such that every curve defined over  $\overline{\mathbb{Q}}$  admits a surjective map onto  $\mathbb{P}^1$  with ramification over  $(0, 1, \infty)$  such that all local ramification indices are  $\leq n$ ?

QUESTION 1.5. Is every curve  $C$  (over  $\overline{\mathbb{Q}}$ ) of genus  $g(C) \geq 2$  universal?

REMARK 1.6. It is clear that an affirmative answer to Question 1.4 implies a (constructive) affirmative answer to Question 1.3.

In this note we answer these questions in a simple model situation: instead of  $\overline{\mathbb{Q}}$  we consider the (separable) closure  $\overline{\mathbf{F}}_p$  of the finite field  $\mathbf{F}_p$ .

THEOREM 1.7. *Let  $p \geq 5$  be a prime and  $C$  a hyperelliptic curve over  $\overline{\mathbf{F}}_p$  of genus  $g(C) \geq 2$ . Then  $C$  is universal: for any projective curve  $C'$  there exist a finite étale cover  $\tilde{C} \rightarrow C$  and a surjective regular map  $\tau : \tilde{C} \rightarrow C'$ .*

In Section 4 we prove the following geometric fact (over arbitrary algebraically closed fields of characteristic  $\neq 2, 3$ ):

PROPOSITION 1.8. *Every hyperelliptic curve  $C$  of genus  $\geq 2$  has a finite étale cover  $\tilde{C}$  which surjects onto the genus 2 curve  $C_0$  given by  $\sqrt[6]{z(1-z)}$ . In particular, if  $C_0$  is universal then every hyperelliptic curve of genus  $\geq 2$  is universal.*

REMARK 1.9. Applying the Chevalley-Weil theorem we conclude that the Mordell conjecture (Faltings' theorem) for  $C_0$  implies the Mordell conjecture for every hyperelliptic curve of genus  $\geq 2$ .

The fact that there is some interaction between the arithmetic of different curves has been noted previously. Moret-Bailly and Szpiro showed (see [6], [5]) that the proof of an *effective* Mordell conjecture for *one* (hyperbolic) curve (for example,  $C_0$ ) implies the ABC-conjecture, which in turn implies an effective Mordell conjecture for *all* (hyperbolic) curves (Elkies [4]). Here *effective* means an explicit bound on the height of a  $K$ -rational point on the curve for all number fields  $K$ . Here again, Belyi's theorem is used in an essential way.

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## 2. Main construction

NOTATION 2.1. Let  $\tau : C \rightarrow C'$  be a surjective map of algebraic curves. We denote by  $\text{Ram}(\tau) \subset C$  the ramification locus of  $\tau$  and by  $\text{Bran}(\tau) = \tau(\text{Ram}(\tau)) \subset C'$  the branch locus of  $\tau$ . For a point  $q \in C$  we denote by  $e_q(\tau)$  the local ramification index at  $q$ . We denote by

$$e(\tau) := \max_{q \in C} e_q(\tau)$$

the maximum local ramification index of  $\tau$ . We say that  $\tau$  has *simple* ramification if  $e(\tau) \leq 2$  and that  $\tau$  is *generic* if in addition there is only one ramification point over each branch point.

REMARK 2.2. Every curve admits a generic map onto  $\mathbb{P}^1$ , at least after a separable extension of the ground field.

Let  $p \geq 5$  be a prime number. In this section we work over an algebraic closure  $\overline{\mathbf{F}}_p$  of the finite field  $\mathbf{F}_p$ . First we show that there exists at least one curve satisfying the conclusion of Theorem 1.7.

Let  $\pi_0 : E_0 \rightarrow \mathbb{P}^1$  be the elliptic curve given by

$$\sqrt[3]{z(z-1)}.$$

Let  $\sigma_0 : C_0 \rightarrow \mathbb{P}^1$  be the genus 2 curve given by

$$\sqrt[6]{z(z-1)},$$

and  $\iota_0 : C_0 \rightarrow E_0$  the corresponding 2-cover. Clearly,  $\iota_0$  has simple ramifications over the preimages of 0, 1. Let  $C$  be an arbitrary curve. Choosing a generic function on  $C$  we get a generic covering  $\sigma : C \rightarrow \mathbb{P}^1$  (such covering is defined over  $\overline{\mathbf{F}}_p$ ). Assume further that  $\text{Bran}(\sigma) \subset \mathbb{P}^1$  does not contain  $(0, 1, \infty)$ .

Consider the diagram

$$\begin{array}{ccccc} C & \longleftarrow & C_1 & \longleftarrow & C_2 \\ \sigma \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}^1 & \longleftarrow & E_0 & & \\ & & \downarrow \varphi & & \\ & & E_0 & \longleftarrow & C_0 \end{array}$$

Here  $C_1 = C \times_{\mathbb{P}^1} E_0$  (it is irreducible since  $E_0 \rightarrow \mathbb{P}^1$  is a 2-cover). Then  $C_1 \rightarrow E_0$  has simple ramification over a finite number of points in  $E_0$ . Recall that  $E_0$  has a group scheme structure, and all  $\overline{\mathbf{F}}_p$ -points of  $E_0$  are torsion points. This implies that there exists an étale map  $E_0 \rightarrow E_0$  such that all ramification points of  $C_1$  over  $E_0$  are mapped to 0. More precisely, any finite set of  $\overline{\mathbf{F}}_p$ -points of  $E_0$  is contained in the group subscheme  $E_0^{et}[n] \subset E_0$  - the maximal étale subgroup of the multiplication by  $n$ -kernel  $E_0[n]$  (for some  $n \in \mathbb{N}$ ). For every positive integer  $n$  there exists a positive multiple  $m$  of  $n$  and an étale map  $E_0 \rightarrow E_0$  with kernel  $E_0^{et}[m]$ .

Taking the composition of  $C_1 \rightarrow E_0$  with the multiplication by a suitable  $m$ , we get a (possibly new) surjective regular map  $C_1 \rightarrow E_0$  which is ramified only over the zero point in  $E_0$  and has the property that all the local ramification indices are at most 2. Using this map let us define  $C_2 := C_0 \times_{E_0} C_1$ . Consequently, any component of  $C_2$  surjects onto  $C_1$  and is an étale covering of  $C_0$  (ramification cancels ramification). This component satisfies the conclusion of Theorem 1.7.

LEMMA 2.3. *Let  $C$  be any smooth complete algebraic curve and  $E$  any curve of genus 1. There exists a curve  $C_1$  which surjects onto  $C$  and  $E$  such that the ramification of the map  $C_1 \rightarrow E$  lies entirely over a single point of  $E$  and its local ramification indices are all equal to 2.*

PROOF. Consider a generic map  $\sigma : C \rightarrow \mathbb{P}^1$  with  $e(\sigma) \leq 2$ . Choose a double cover  $\pi : E \rightarrow \mathbb{P}^1$  such that the branch loci  $\text{Bran}(\sigma)$  and  $\text{Bran}(\pi)$  on  $\mathbb{P}^1$  are disjoint. Then the product  $C_1 := C \times_{\mathbb{P}^1} E$  is an irreducible curve which is a double cover of  $C$ . The curve admits a surjective map  $\iota_1 : C_1 \rightarrow E$  with  $e(\iota_1) \leq 2$ . Similarly to the previous construction we can find an unramified cover  $\varphi : E \rightarrow E$  such that the composition  $\varphi \circ \iota_1 : C_1 \rightarrow E$  is ramified only over one point in  $E$  and the local ramification indices are still equal to 2.  $\square$

COROLLARY 2.4. *Assume that some unramified covering  $\tilde{C}$  of  $C$  surjects onto an elliptic curve  $E$ . Assume further that there exists a point  $q$  on  $E$  such that all local ramification indices of the map  $\tilde{C} \rightarrow E$  over  $q$  are divisible by 2. Then  $C$  is universal.*

PROOF. It is sufficient to take the product of  $\tilde{C} \times_E C_1$ . Any irreducible component of the resulting curve will be an unramified covering of  $\tilde{C}$  (and hence  $C$ ) and will admit a surjective map onto  $C_1$  and  $C$ .  $\square$

COROLLARY 2.5 (Theorem 1.7). *Every hyperelliptic curve  $C$  over  $\overline{\mathbf{F}}_p$  (with  $p \geq 5$ ) of genus  $\geq 2$  is universal.*

PROOF. Consider the standard projection  $\sigma : C \rightarrow \mathbb{P}^1$  (of degree 2). Its branch locus  $\text{Bran}(\sigma)$  consists of  $2g + 2$  points. Let  $\pi : E \rightarrow \mathbb{P}^1$  be a double cover such that  $\text{Bran}(\pi)$  is contained in  $\text{Bran}(\sigma)$ . Then the product  $\tilde{C} = C \times_{\mathbb{P}^1} E$  is an unramified double cover of  $C$ . Moreover,  $\tilde{C}$  is a double cover of  $E$  with ramification at most over the preimages in  $E$  of the points in  $\text{Bran}(\sigma) \setminus \text{Bran}(\pi)$ . We now apply Corollary 2.4.  $\square$

In *finite* characteristic, there are many other (classes of) universal curves. For example, cyclic coverings with ramification in 3 points, hyperbolic modular curves, etc.

CONJECTURE 2.6. *Every smooth complete curve  $C$  of genus  $g(C) \geq 2$  defined over  $\overline{\mathbf{F}}_p$  (for  $p \geq 2$ ) is universal.*

### 3. The case of characteristic 0

In this section we work over  $\overline{\mathbb{Q}}$ . We show that the method outlined in Section 2 can be employed in characteristic zero to produce natural infinite sets of algebraic points on  $\mathbb{P}^1$  which occur as ramification points of surjective maps from  $\mathbb{P}_2^1$  to  $\mathbb{P}_1^1$  branched over  $(0, 1, \infty) \in \mathbb{P}_1^1$  only and having an *a priori* bound on the ramification index (here  $\mathbb{P}_1^1$  and  $\mathbb{P}_2^1$  are two different copies of the projective line  $\mathbb{P}^1$ ).

Notice that, in principle, it is easy to produce *some* sets of points (of any finite cardinality) with this property: Take an  $n \geq 6$  and any triangulation of  $\mathbb{P}_2^1$  with vertices of index  $\leq n$ . A barycentric subdivision of each such triangulation defines a function from  $\mathbb{P}_2^1$  to  $\mathbb{P}_1^1$  with local ramification indices  $\leq 2n$  (for more details see [3]). Therefore, any curve with bounded ramification over this set of vertices will have bounded ramification over  $\mathbb{P}_1^1$ . However, we have no explicit control over the coordinates of the ramification points on  $\mathbb{P}_2^1$ .

An (obvious) analogous way to control ramification indices is to consider the following diagram

$$\begin{array}{ccc}
E & \xrightarrow{\pi} & \mathbb{P}_2^1 \\
\phi_n \downarrow & & \downarrow \varphi_{n,E} \\
E & \xrightarrow{\pi} & \mathbb{P}_1^1,
\end{array}$$

where the map  $\phi_n$  is the quotient by the subscheme of  $n$ -torsion points and the maps  $E \rightarrow \mathbb{P}^1$  are the standard double covers, ramified over  $(0, 1, \infty, \lambda)$ . Clearly, all the ramification points of  $\varphi_{n,E}$  (in  $\mathbb{P}_2^1$ ) are over  $0, 1, \infty$  and  $\lambda$  (in  $\mathbb{P}_1^1$ ) and  $e(\varphi_{n,E}) = 2$ . Belyi's theorem gives a map  $\beta : \mathbb{P}_1^1 \rightarrow \mathbb{P}_0^1$ , which ramifies only over the points  $(0, 1, \infty) \in \mathbb{P}_0^1$ , maps  $\{0, 1, \infty, \lambda\} \subset \mathbb{P}_1^1$  into  $\{0, 1, \infty\} \subset \mathbb{P}_0^1$  and has local ramification indices  $\leq n$ . Moreover, it provides an explicit bound on  $\deg(\beta)$  and, consequently, on  $e(\beta)$  (in terms of the absolute height of  $\lambda$ ). Let  $\beta_\lambda : \mathbb{P}_1^1 \rightarrow \mathbb{P}_0^1$  be a map such that

$$e(\beta_\lambda) = \inf_{\beta} \{e_{\beta}\}$$

over the set of all maps as above. Then the map  $\beta_\lambda \circ \varphi_{n,E} : \mathbb{P}_2^1 \rightarrow \mathbb{P}_0^1$  ramifies over three points only and has index  $e(\beta_\lambda \circ \varphi_{n,E}) \leq 2n$ . Let

$$R_E := \pi(E(\overline{\mathbb{Q}})_{\text{tors}}) \subset \mathbb{P}_2^1(\overline{\mathbb{Q}})$$

be the image of the torsion points of  $E$ . Let  $\sigma : C \rightarrow \mathbb{P}_2^1$  be any map ramified only in a subset of  $R_E$ . Let  $\pi := \beta_\lambda \circ \varphi_{n,E} \circ \sigma$ . Then

$$e(\pi) \leq 2e(\sigma) \cdot e(\beta_\lambda).$$

A natural application of the construction in Section 2 is as follows:

EXAMPLE 3.1. Let  $\pi : E \rightarrow \mathbb{P}^1$  be a triple cover with  $\text{Bran}(\pi) = \{0, 1, \infty\}$  ( $E$  is a CM elliptic curve with  $j$ -invariant 0). Consider the following diagram

$$\begin{array}{ccccc}
E & \xrightarrow{\pi} & \mathbb{P}_2^1 & & \\
\phi_n \downarrow & & \downarrow \varphi_{n,E} & & \\
C_0 & \longrightarrow & E & \xrightarrow{\pi} & \mathbb{P}_1^1,
\end{array}$$

where  $C_0$  is a curve of genus  $g(C_0) = 2$  given by  $\sqrt[6]{z(z-1)}$ ,  $\phi_n$  is the quotient map by the subscheme of torsion points of order  $n$ , and  $\varphi_{n,E}$  the corresponding map from  $\mathbb{P}_2^1$  to  $\mathbb{P}_1^1$  ramified only over  $(0, 1, \infty)$ . Let  $\mathcal{X}_g = \{X\}$  be the subset of curves of genus  $g$  admitting a map  $\sigma_X : X \rightarrow \mathbb{P}_2^1$  such that

- $e(\sigma_X) = 2$ ;
- $\text{Bran}(\sigma_X) \subseteq \pi(E(\overline{\mathbb{Q}})_{\text{tors}})$ .

Then, for any  $X \in \mathcal{X}_g$  the map

$$\varphi_{n,E} \circ \sigma_X : X \rightarrow \mathbb{P}_1^1$$

has index  $e(\varphi_{n,E} \circ \sigma_X) \leq 6$  and there exists an unramified cover  $\tilde{C} \rightarrow C_0$  surjecting onto  $X$ . Moreover,  $\mathcal{X}_g$  is *dense* (in real and  $p$ -adic topologies) in the natural Hurwitz scheme  $\mathcal{H}_g$  parametrizing curves of genus  $g$ .

The set of curves dominated by unramified covers of  $C_0$  is much larger than  $\mathcal{X}_g$ . Indeed, consider any 4-tuple of points in

$$\pi(E(\overline{\mathbb{Q}})_{\text{tors}}) \subseteq \mathbb{P}_2^1$$

and an elliptic curve  $E'$  obtained as a double cover of  $\mathbb{P}_2^1$  ramified in those 4 points. Then  $E'$  is also dominated by unramified covers of  $C_0$  and we can iterate the above construction for  $E'$ .

#### 4. Geometric constructions

Let  $(E, q_0)$  be an elliptic curve,  $q_1$  a torsion point of order two on  $E$  and  $\pi : E \rightarrow \mathbb{P}^1$  the quotient with respect to the involution induced by  $q_1$ . Let  $n$  be an odd positive integer and  $\varphi_{n,E} : \mathbb{P}_2^1 \rightarrow \mathbb{P}_1^1$  the map induced by

$$\begin{array}{ccc} E & \xrightarrow{\pi} & \mathbb{P}_2^1 \\ \phi_n \downarrow & & \downarrow \varphi_{n,E} \\ E & \xrightarrow{\pi} & \mathbb{P}_1^1. \end{array}$$

Any quadruple  $r = \{r_1, \dots, r_4\}$  of four distinct points in  $\varphi_{n,E}^{-1}(\pi(q_0))$  defines a genus 1 curve  $E_r$  (the double cover of  $\mathbb{P}^1$  ramified in these four points).

**PROPOSITION 4.1.** *Let  $\iota : C \rightarrow E$  be any finite cover such that all local ramification indices over  $q_0$  are even. Then there exists an unramified cover  $\tau_r : C_r \rightarrow C$  which dominates  $E_r$  and has only even local ramification indices over some point in  $E_r$ .*

**PROOF.** Assume that  $n \geq 3$  and consider the following diagram

$$\begin{array}{ccccc} C & \xleftarrow{\tau_2} & C_2 & \xleftarrow{\tau_r} & C_r \\ \iota \downarrow & & \downarrow \iota_2 & & \downarrow \iota_r \\ E & \xleftarrow{\varphi_n} & E & & E_r \\ \pi \downarrow & & \downarrow \pi & & \downarrow \pi_r \\ \mathbb{P}_1^1 & \xleftarrow{\phi_{n,E}} & \mathbb{P}_2^1 & & \mathbb{P}_2^1, \end{array}$$

where  $E_r$  is a double cover of  $\mathbb{P}_2^1$  ramified in any quadruple of points in the preimage  $\phi_{n,E}^{-1}(\pi(q_0))$  and  $C_r$  is any irreducible component of  $C_2 \times_{\mathbb{P}_2^1} E_r$ . Any point  $q_r \in E_r$  such that  $q_r \notin \text{Ram}(\pi_r)$  (that is, its image in  $\mathbb{P}_2^1$  is distinct from  $r_1, \dots, r_4$ ) has the claimed property.  $\square$

**REMARK 4.2.** Iterating this procedure (and adding isogenies) we obtain many elliptic curves  $E'$  which are dominated by curves having an unramified cover onto  $E$ . It would be interesting to know if for any two elliptic curves over  $\overline{\mathbb{Q}}$  there exists a (directed) cycle connecting them (at least modulo isogenies). We will now show that *any* elliptic curve can be connected in this way to  $E_0$ .

Let  $E_0 \subset \mathbb{P}^2 = \{(x : y : z)\}$  be the elliptic curve

$$x^3 + y^3 + z^3 = 0,$$

and

$$E_0[3] = \mathbf{T} := \left\{ \begin{array}{lll} (1 : 0 : 1), & (1 : 0 : -\zeta), & (1 : 0 : -\zeta^2), \\ (0 : 1 : 1), & (0 : 1 : -\zeta), & (0 : 1 : -\zeta^2), \\ (1 : 1 : 0), & (1 : -\zeta : 0), & (1 : -\zeta^2 : 0) \end{array} \right\}$$

its set of 3-torsion points (where  $\zeta$  is a primitive cubic root of 1). Denote by  $\mathcal{E}_\lambda = \{E_\lambda\}$  the family of elliptic curves on  $\mathbb{P}^2$  passing through  $\mathbf{T}$  given by

$$E_\lambda : x^3 + y^3 + z^3 + \lambda xyz = 0.$$

It is easy to see that for each  $\lambda$  the set  $E_\lambda[3]$  of 3-torsion points of  $E_\lambda$  is precisely  $\mathbf{T}$ . Let

$$\begin{array}{ccc} \pi & : & \mathbb{P}^2 \quad \rightarrow \quad \mathbb{P}^1 \\ & & (x : y : z) \mapsto (x + z : y) \end{array}$$

be the projection respecting the involution  $x \rightarrow z$  on  $\mathbb{P}^2$ . Denote by  $\pi_\lambda$  the restriction of  $\pi$  to  $E_\lambda$ . Clearly,  $\pi_\lambda$  exhibits each  $E_\lambda$  as a double cover of  $\mathbb{P}^1$  and  $\pi_\lambda$  has only simple double points for all  $\lambda$ . Moreover,

$$\pi(\mathbf{T}) = \{(0 : 1), (1 : -\zeta), (1 : -\zeta^2), (1 : -1), (1 : 0)\}$$

and for all  $\lambda$  there exists a (non-empty) set  $S_\lambda \subset \text{Bran}(\pi_\lambda) \subset \mathbb{P}^1$  such that  $\pi_\lambda^{-1}(S_\lambda) \subset \mathbf{T}$ . Let  $\pi'_0 : E'_0 \rightarrow \mathbb{P}^1$  be a double cover ramified in 4 points in  $\pi(\mathbf{T})$ .

**LEMMA 4.3.** *Let  $\iota : C \rightarrow E_\lambda$  be a double cover such that over at least one point in  $\text{Bran}(\iota)$  the local ramification indices are even. Then there exists an unramified cover  $\tilde{C} \rightarrow C$  and a surjective morphism  $\tilde{\iota} : \tilde{C} \rightarrow E'_0$  such that over at least one point in  $\text{Bran}(\tilde{\iota}) \subset E'_0$  all local ramification indices of  $\tilde{\iota}$  are even.*

**PROOF.** Consider the diagram

$$\begin{array}{ccc} E_\lambda & \xleftarrow{\iota} & C_1 \\ \varphi_3 \downarrow & & \downarrow \\ E_\lambda & \xleftarrow{\quad} & C \\ \pi_\lambda \downarrow & & \\ & & \mathbb{P}^1 \end{array}$$

Then  $C_1 \rightarrow \mathbb{P}^1$  has even local ramification indices over all points in  $\pi(\mathbf{T})$ . It follows that

$$\tilde{C} := C_1 \times_{\mathbb{P}^1} E'_0 \rightarrow E'_0$$

has even local ramification indices over the preimages of the fifth point in  $\pi(\mathbf{T})$ , as claimed.  $\square$

**NOTATION 4.4.** Let  $\mathcal{C}$  be the class of curves such that there exists an elliptic curve  $E$ , a surjective map  $\iota : C \rightarrow E$  and a point  $q \in \text{Bran}(\iota)$  such that all local ramification indices at points in  $\iota^{-1}(q)$  are even.

**EXAMPLE 4.5.** Any hyperelliptic curve of genus  $\geq 2$  belongs to  $\mathcal{C}$ . More generally,  $\mathcal{C}$  contains any curve  $C$  admitting a map  $C \rightarrow \mathbb{P}^1$  with even local ramification indices over at least 5 points in  $\mathbb{P}^1$ .

PROPOSITION 4.6. *For any  $C \in \mathcal{C}$  there exists an unramified cover  $\tilde{C} \rightarrow C$  surjecting onto  $C_0$  (with  $C_0 \rightarrow \mathbb{P}^1$  given by  $\sqrt[6]{z(1-z)}$ ).*

PROOF. Consider  $C_1 = C \in \mathcal{C}$  with  $\iota_1 : C_1 \rightarrow E = E_\lambda$  as in 4.4. Define  $C_2$  as an irreducible component of  $C_1 \times_E E$ :

$$\begin{array}{ccc} C_1 & \xleftarrow{\tau_2} & C_2 \\ \downarrow \iota_1 & & \downarrow \iota_2 \\ E & \xleftarrow{\varphi_3} & E \\ & & \downarrow \pi_\lambda \\ & & \mathbb{P}^1 \end{array}$$

Define  $C_3 := C_2 \times_{\mathbb{P}^1} E_0$  by the diagram

$$\begin{array}{ccc} C_2 & \xleftarrow{\tau_3} & C_3 \\ \downarrow \sigma_2 & & \downarrow \iota_3 \\ \mathbb{P}^1 & \xleftarrow{\pi_0} & E_0. \end{array}$$

Observe that for  $q \in \text{Bran}(\pi_0)$  the local ramification indices in the preimage  $(\pi_\lambda \circ \iota_2)^{-1}(q)$  are all even. It follows that the map  $\tau_3 : C_3 \rightarrow C_2$  is *unramified* and that  $\iota_3 : C_3 \rightarrow E_0$  has even local ramification indices over (the preimage of)  $q_5 \in \{\pi(\mathbb{T}) \setminus \text{Bran}(\pi_0)\}$  (the 5th point). Define  $C_4$  as an irreducible component of  $C_3 \times_{E_0} E_0$  in the diagram

$$\begin{array}{ccc} C_3 & \xleftarrow{\tau_4} & C_4 \\ \downarrow \iota_3 & & \downarrow \iota_4 \\ E_0 & \xleftarrow{\varphi_3} & E_0. \end{array}$$

The map  $\iota_4$  is ramified over the preimages  $(\pi_0 \circ \varphi_3)^{-1}(q_5)$ , with even local ramification indices. Finally,  $C_5 = C_4 \times_{E_0} C_0$  from the diagram

$$\begin{array}{ccc} C_4 & \xleftarrow{\tau_5} & C_5 \\ \downarrow \iota_4 & & \downarrow \\ E_0 & \xleftarrow{\iota_0} & C_0. \end{array}$$

has a dominant map onto  $C_0$  and is unramified over  $C_4$  (and consequently,  $C_1$ ).  $\square$

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COURANT INSTITUTE OF MATHEMATICAL SCIENCES, N.Y.U., 251 MERCER STR., NEW YORK,  
NY 10012, U.S.A.

*E-mail address:* `bogomolo@cims.nyu.edu`

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ 08544-1000

*E-mail address:* `ytschink@math.princeton.edu`