Rational connectedness over small fields

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(joint work with Fedor Bogomolov)

Let k be a field, $\bar{k}$ its separable closure, X an algebraic variety over k and $X(k)$ its set of rational points. We are interested in rational, resp. algebraic, points on X and in rational curves on X, defined over k or $\bar{k}$. For $k = \bar{k}$ of chacteristic zero we have (at least) two notions of "connectivity" via rational curves:

(1) For all $x_1, x_2 \in X(k)$ there exists a chain of rational curves $C_1 \cup \ldots \cup C_r \subset X$ connecting x_1 and x_2;
(2) For all $x_1, x_2 \in X(k)$ there exists a *free* rational curve $C \subset X$ connecting x_1, x_2.

For smooth projective X these two properties are equivalent. The situation is less clear for quasi-projective X. For example, we don't know whether (2) holds for the smooth locus of a singular Del Pezzo surface, or its partial desingularization.

There are versions involving arbitrary (or general) finite sets of points, prescribed local behavior at finitely many points, etc. (see [4]).

In arithmetic situations, when $k \neq \bar{k}$, there are more logical possibilities: one could ask for irreducible curves defined over the groundfield k, or for curves over k connecting points over $\bar{k}$. Of particular interest are *small* ground fields, such as finite fields $\mathbb{F}_q$ or the rationals $\mathbb{Q}$. A prototype result is the following theorem of Kollár and Szabó:

Theorem 1 ([5])**.** *Let X be a smooth projective separably rationally connected variety over $k = \mathbb{F}_q$. There is a function $\phi = \phi(\deg(X), \dim(X), n)$ such that for $q > \phi$ and for every set of n points $x_1, \ldots, x_n \in X(k)$ there exists a geometrically irreducible rational curve C, defined over k with $x_1, \ldots, x_n \in C(k)$.*

The theorem applies, e.g., to hypersurfaces $X \subset \mathbb{P}^N$ of low degree $d \leq N$. It turns out that rational connectivity holds sometimes even for $d = N + 1$:

Theorem 2 ([1])**.** *Let $X = \widetilde{A/G}$ be a Kummer surface over $k = \mathbb{F}_q$ (with q suffiently large), and $X^\circ \subset X$ the complement to exceptional curves. Then for every set $x_1, \ldots, x_n \in X^\circ(\bar{k})$ there exists a geometrically irreducible rational curve $C \subset X$, defined over k, such that $x_1, \ldots x_n \in C(\bar{k})$.*

This theorem applies, for example, to quartic Kummer surfaces. Choosing nonsupersingular A gives examples of nonuniruled K3 surfaces, which are "rationally connected". Using this we provide examples of nonuniruled surfaces of general type over finite fields with the same property.

The proof of Theorem 2 relies on a fact of independent interest

Theorem 3 ([2])**.** *Let C be a smooth projective curve of genus $\mathsf{g} > 2$ over $k = \mathbb{F}_q$ (with q sufficiently large) and J its Jacobian. Fix a point $c_0 \in C(k)$ and the* *embedding $C \hookrightarrow J$, via $c \mapsto c - c_0$. Then*

$$J(\bar{k}) = \cup_{m=1}^{\infty} m \cdot C(\bar{k}).$$

In fact, one can let m run through arithmetic progressions.

These results over finite fields have surprizing applications over number fields. Namely, let $X = \widetilde{J/G}$ be Kummer surface over a (sufficiently large) number field K, with J the Jacobian of a curve of genus 2. Choose models for X, J over the integers $\mathcal{O}_K$ and a finite set of nonarchimedian places of (sufficiently) good reduction S. Finally, for $v \in S$, choose $\bar{x}_v \in X(\mathbf{k}_v)$ - points in the reduction modulo v.

Theorem 4 ([3]). *There exists a K-rational point $x \in X(K)$ such that $x_v = \bar{x}_v$ modulo v, for all $v \in S$.*

Such a version of weak approximation, for first order jets, is unknown even for cubic surfaces over number fields.

References

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