Monodromy of elliptic surfaces

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1 Introduction

Let $\mathcal{E} \to B$ be a non-isotrivial Jacobian elliptic fibration and $\tilde{\Gamma}$ its global monodromy group. It is a subgroup of finite index in $\mathrm{SL}(2,\mathbb{Z})$. We will assume that $\mathcal{E}$ is Jacobian. Denote by Γ the image of $\tilde{\Gamma}$ in $\mathrm{PSL}(2,\mathbb{Z})$ and by $\overline{\mathcal{H}}$ the upper half-plane completed by ∞ and by rational points in $\mathbb{R} \subset \mathbb{C}$. The j-map $B \to \mathbb{P}^1$ decomposes as $j_\Gamma \circ j_{\mathcal{E}}$, where

$$j_{\mathcal{E}} : B \to M_\Gamma = \overline{\mathcal{H}}/\Gamma$$

and $j_\Gamma : M_\Gamma \to \mathbb{P}^1 = \overline{\mathcal{H}}/\mathrm{PSL}(2,\mathbb{Z})$. In an algebraic family of elliptic fibrations the degree of j is bounded by the degree of the generic element. It follows that there is only a finite number of monodromy groups for each family.

The number of subgroups of bounded index in $\mathrm{SL}(2,\mathbb{Z})$ grows superexponentially [8], similarly to the case of a free group (since $\mathrm{SL}(2,\mathbb{Z})$ contains a free subgroup of finite index). For Γ of large index, the number of M_Γ-representations of the sphere $\mathbb{S}^2$ is substantially smaller, however, still superexponential (see 3.5).

Our goal is to introduce some combinatorial structure on the set of monodromy groups of elliptic fibrations which would help to answer some natural questions. Our original motivation was to describe the set of groups corresponding to rational or K3 elliptic surfaces, explain how to compute the dimensions of the spaces of moduli of surfaces in this class with given monodromy group etc. As a direct application of the methods developed in the present paper the authors and T. Petrov have obtained a proof of rationality and stable rationality of many classes of moduli spaces of elliptic fibrations with given monodromy, including all such moduli spaces of rational and elliptic K3 surfaces (see [4]). Our approach is based on a detailed study of the relation between special graphs on Riemann surfaces and subgroups of finite index in $\mathrm{PSL}(2,\mathbb{Z})$.

To determine $\tilde{\Gamma}$ we first describe all possible groups Γ. In order to classify possible Γ we consider the corresponding oriented Riemann surface M_Γ. The map $j_\Gamma : M_\Gamma \to \mathbb{P}^1$ provides a special triangulation of M_Γ (induced from the standard triangulation of $\mathbb{P}^1$ into two triangles with vertices in $0, 1, \infty$) (and vice versa). The preimages of $0, 1, \infty$ on M_Γ will be called A, B, I, respectively, and the triangulation will be called a j-triangulation. The barycentric subdivision of any triangulation of an oriented Riemann surface is a j-triangulation. However, not all j-triangulations arise in this way. (In particular, we consider more general triangulations than just barycentric subdivisions of "Belyi" triangulations.) Of course, the study of simplicial decompositions of oriented manifolds goes back at least to Alexander [1] (who proves that in *any* dimension n the barycentric subdivision of a given simplicial decomposition induces a simplicial map to the n-dimensional sphere, with its standard simplicial decomposition into two simplices). More recently, constructions of this type were rediscovered by many authors in connection with Belyi's theorem and Grothendieck's "Dessins d'enfants" program ([3], [9] and the references therein). A j-triangulation of a Riemann surface R induces a graph G_Γ on R, which is obtained by removing all AI- and BI-edges from the graph given by the 1-skeleton of the j-triangulation (see [12], for example). Thus we obtain a bijection between subgroups of $\mathrm{PSL}(2,\mathbb{Z})$ of finite index (modulo conjugation) and trivalent graphs G_Γ on a Riemann surface R with a coloring of the ends of G_Γ in two colors such that the complement to G_Γ is a (disjoint) union of (contractible) cells.

The plan of the paper is as follows. In section 2 we recall basic facts about the local and global monodromy groups of elliptic fibrations due to Kodaira. In section 3 we study j-modular curves M_Γ and their relationship with j-triangulations. In section 4 we give a modular construction of elliptic surfaces over M_Γ with prescribed monodromy groups. General elliptic fibrations over B are obtained as simple modifications of pullbacks of these elliptic fibrations from M_Γ. Our construction allows a relatively transparent description of a rather complicated set of global monodromy groups of elliptic surfaces. This transforms the general results of Kodaira theory to a concrete computational tool.

Conventions. We write $\mathbf{F}_n$ for the free group on n generators. Throughout the paper we work over $\mathbb{C}$.

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2 Generalities

In this section we give a brief summary of Kodaira's theory of elliptic fibrations. We refer to the papers by Kodaira [7] and to [2] and [6] for proofs and details.

2.1 The setup

Let $f : \mathcal{E} \to B$ be a smooth relatively minimal non-isotrivial Jacobian elliptic fibration over a smooth curve B of genus $g(B)$. This means that

- $\mathcal{E}$ is a smooth compact surface and f is holomorphic,
- the generic fiber of f is a smooth curve of genus 1 (elliptic fibration),
- the fibers of $\mathcal{E}$ do not contain smooth rational curves of self-intersection -1 (relative minimality),
- we have a global zero section $e : B \to \mathcal{E}$ (Jacobian elliptic fibration),
- the j-function which to each smooth fiber $\mathcal{E}_b \subset \mathcal{E}$ assigns its j-invariant is a non-constant rational function on B (non-isotrivial).

2.2 Topology

Denote by $B^s = \{b_1, ..., b_k\} \subset B$ the set of points corresponding to singular fibers of $\mathcal{E}$, it is always non-empty. Let $B^0 = B \backslash B^s$ be the open subset of B where all fibers are smooth and $f^0 : \mathcal{E}^0 \to B^0$ the restriction of f. Topologically, f^0 is a smooth oriented fibration with fibers $\mathbb{S}^1 \times \mathbb{S}^1$, which is equipped with a section. The equivalence class of $\mathcal{E}^0$ under global diffeomorphisms inducing smooth isomorphisms on each fiber is determined by the topology of B^0 and by the homomorphism (representation) of the fundamental group $\pi_1(B^0)$ into the group of homotopy classes of orientation-preserving automorphisms of the torus $\mathbb{S}^1 \times \mathbb{S}^1$ (which is equal to $\mathrm{SL}(2,\mathbb{Z})$). Thus we have a homomorphism $\rho^c_{\mathcal{E}} : \pi_1(B^0) \to \mathrm{SL}(2,\mathbb{Z})$. This homomorphism - well defined modulo conjugation in $\mathrm{SL}(2,\mathbb{Z})$ - is called by Kodaira the *homological invariant* of the elliptic fibration $\mathcal{E}$.

Now we consider the local situation: according to Kodaira, the restriction of f to a small punctured analytic neighborhood Δ^*_b of a point $b \in B$ (disc Δ_b minus the point b) for every point $b \in B^s$ is also topologically nontrivial. Thus we have a homomorphism $\rho^c_b : \mathbb{Z} \to \mathrm{SL}(2,\mathbb{Z})$ (where $\mathbb{Z}$ is the fundamental group of the punctured disc with the standard generator t_b). Again, this homomorphism is defined modulo conjugation.

We can eliminate the ambiguity in the definitions above by the following procedure: choose a point $b_0 \in B^0$ and a set of non-intersecting paths connecting b_0 to the singular points $b_s \in B^s$. This set admits a natural cyclic order defined by the relative position of these paths in a small neighborhood of b_0.

A small neighborhood of this set is a disc inside B (with orientation). Now we can choose small oriented loops around each singular point b_s.

If we fix generators of $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1)$ for the fiber over b_0 then we obtain a system of *elements* $T_b \in \mathrm{SL}(2,\mathbb{Z})$ in the conjugacy class of t_b as well as a representation $\rho_{\mathcal{E}} : \pi_1(B^0) \to \mathrm{SL}(2,\mathbb{Z})$. We call the elements T_b local monodromies, the representation $\rho_{\mathcal{E}}$ the global monodromy representation and the group $\tilde{\Gamma} = \rho_{\mathcal{E}}(\pi_1(B^0)) \subset \mathrm{SL}(2,\mathbb{Z})$ the global monodromy group. The global monodromy representation depends only on the basis of $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1)$ at b_0. The local monodromy elements depend on the choice of the system of paths. Let Γ be the image of $\tilde{\Gamma}$ in $\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}/2$.

There is an important relation between local and global monodromy.

Lemma 2.1 *Let $\mathcal{E} \to \mathbb{P}^1$ be an elliptic fibration. Then the product*

$$P(\mathcal{E}) := \prod_{b \in B^s} T_b \in \mathrm{SL}(2,\mathbb{Z})$$

(taken in cyclic order) is equal to the identity.

Proof. The product $P(\mathcal{E})$ gives the monodromy transformation along the boundary of the disc Δ. Our fibration is smooth on the complement $B \setminus \Delta$. Therefore, it is a topologically trivial fibration over a disc in the case of $B = \mathbb{P}^1$ or a smooth $\mathbb{S}^1 \times \mathbb{S}^1$ fibration over the Riemannian surface B of genus $g(B)$ minus a disc. Now the relations follow from similar relations in $\pi_1(B \setminus \Delta)$.

Remark 2.2 Similarly, if the genus $g(B) \geq 1$ then $P(\mathcal{E})$ is a product of $g(B)$ commutators.

2.3 The j-function

Lemma 2.3 *Let j be any nonconstant rational map $B \to \mathbb{P}^1$. Then there exists a unique subgroup of finite index $\Gamma \subset \mathrm{PSL}(2, \mathbb{Z})$ such that j decomposes as*

$$j : B \to \overline{\mathcal{H}}/\Gamma \to \overline{\mathcal{H}}/\mathrm{PSL}(2, \mathbb{Z}) = \mathbb{P}^1$$

(where $\overline{\mathcal{H}}$ is the completed upper half-plane) with the following property: for every $\Gamma' \subset \mathrm{PSL}(2, \mathbb{Z})$ with the same decomposition as above there exists an element $g \in \mathrm{PSL}(2, \mathbb{Z})$ such that $g\Gamma g^{-1} \subset \Gamma'$.

Proof. First of all, $\Gamma' = \mathrm{PSL}(2, \mathbb{Z})$ gives the required decomposition. Next, pick two subgroups $\Gamma, \Gamma' \subset \mathrm{PSL}(2, \mathbb{Z})$ such that the j-map decomposes as above. Consider the map

$$B \to \overline{\mathcal{H}}/\Gamma \times \overline{\mathcal{H}}/\Gamma' \to \overline{\mathcal{H}}/\mathrm{PSL}(2, \mathbb{Z}) \times \overline{\mathcal{H}}/\mathrm{PSL}(2, \mathbb{Z}).$$

Then the image of B lies in the diagonal

$$\overline{\mathcal{H}}/\mathrm{PSL}(2, \mathbb{Z}) \to \overline{\mathcal{H}}/\mathrm{PSL}(2, \mathbb{Z}) \times \overline{\mathcal{H}}/\mathrm{PSL}(2, \mathbb{Z}).$$

The preimage of the diagonal in

$$\overline{\mathcal{H}}/\Gamma \times \overline{\mathcal{H}}/\Gamma'$$

decomposes into the union

$$\cup_{g \in \mathrm{PSL}(2,\mathbb{Z})} \overline{\mathcal{H}}/g\Gamma g^{-1} \cap \Gamma'.$$

Since B is irreducible, it dominates exactly one of such curves. If for all g the group $g\Gamma g^{-1} \cap \Gamma' \neq \Gamma$ then Γ is not minimal. Finally, the index of Γ in $\mathrm{PSL}(2, \mathbb{Z})$ is bounded from above by the degree of j.

The elliptic fibration $\mathcal{E} \to B$ defines a rational function on B - the j-function. There is a relationship between the j-function and local (resp. global) monodromies. By Lemma 2.3 above, j determines the monodromy invariant $\rho^c_{\mathcal{E}}$ (modulo conjugation in $\mathrm{PSL}(2,\mathbb{Z})$).

Now consider the local situation: the restriction of j to the disc Δ_b is analytically equivalent to $j(b)+z^k$ if $j(b)$ is finite or z^{-k} if $j(b)$ is infinite ($k \in \mathbb{N}$). Here z is a local parameter. There are certain compatibility conditions between the local monodromy ρ_b and k. Kodaira classifies all pairs (ρ_b, k) which occur (see [2]). The types are labeled by I_n, II, III, IV, and I_n^*, II^*, III^*, IV^*). The local monodromy ρ_b around fibers of type I_n is unipotent. The local monodromy around the fibers of type II, III and IV is finite. For $*$-fibers the local monodromy is multiplied by -1 (I_0 is nonsingular, with trivial monodromy).

Theorem 2.4 *The pair (ρ_b, k) from Kodaira's list defines a unique (in the analytic category) relatively minimal Jacobian fibration over Δ_b. Any two Jacobian elliptic fibrations over an analytic disc Δ_b with the same (ρ_b, k) are fiberwise birationally isomorphic.*

Theorem 2.5 *For any nonconstant map $j : \mathbb{P}^1 \to \mathbb{P}^1$ there exists an elliptic fibration $\mathcal{E} \to \mathbb{P}^1$ with j-map j.*

If $\mathcal{E}$ and $\mathcal{E}'$ are elliptic fibrations over $\mathbb{P}^1$ such that $j = j'$ then there exists a function $\chi : \mathbb{P}^1 \to \mathbb{Z}/2 = \pm 1$ of finite support such that $\prod_{b\in\mathbb{P}^1} \chi(b) = 1$ and $\rho_b = \chi(b)\rho'_b$ for all $b \in \mathbb{P}^1$ (here ρ_b, resp. ρ'_b are the local monodromies for j, resp. j'). Conversely, for every such function χ there exists an elliptic fibration $\mathcal{E}'$ such that $\rho_b = \chi(b)\rho'_b$ (for all $b \in \mathbb{P}^1$) and $j' = j$.

Remark 2.6 The theorem says that if $\rho^c_{\mathcal{E}'} = \rho^c_{\mathcal{E}'}$ then

$$\rho_{\mathcal{E}} = \chi \cdot \rho_{\mathcal{E}'}.$$

In general, there are exactly $2^{g(B)+k-1}$ different liftings of the standard generators of $\pi_1(B^0)$ to $\mathrm{SL}(2,\mathbb{Z})$, (see part (a) of Theorem 11.1 p. 160 in [2]).

We are interested in classifying global monodromies in some restricted class of surfaces, for example rational elliptic or elliptic K3 surfaces. For each of these classes the degree of j is bounded. This implies a bound on the index of the global monodromy group $\tilde{\Gamma}$ in $\mathrm{SL}(2,\mathbb{Z})$. Only a finite number of possible global monodromy groups $\tilde{\Gamma}$ and only few homological invariants can occur if we fix the image of $\tilde{\Gamma}$ in $\mathrm{PSL}(2,\mathbb{Z})$. Elliptic surfaces with the same j-invariant but different homological invariants are scattered through different topological classes. Our point of departure was that Kodaira's theory does not provide a sufficiently simple combinatorial control over the topology of the resulting surfaces. In the following sections we give some technical improvements of Kodaira's theory which lead to an effective algorithm.

3 j-modular curves

Let $\mathcal{E} \to B$ be an elliptic fibration as above. By Lemma 2.3 j-map decomposes as a product $j = j_\Gamma \circ j_\mathcal{E}$ where $j_\mathcal{E} : B \to \overline{\mathcal{H}}/\Gamma$ is a natural lifting of j onto the modular curve $M_\Gamma = \overline{\mathcal{H}}/\Gamma$ corresponding to Γ and

$$j_\Gamma : \overline{\mathcal{H}}/\Gamma \to \overline{\mathcal{H}}/\mathrm{PSL}(2,\mathbb{Z}) = \mathbb{P}^1. \tag{1}$$

The above decomposition shows that $\deg(j) = \deg(j_\mathcal{E}) \cdot \deg(j_\Gamma)$. In particular, for any non-isotrivial elliptic surface the group Γ is a subgroup of finite index in $\mathrm{PSL}(2,\mathbb{Z})$.

Definition 3.1 We call the pair (M_Γ, j_Γ) the j-modular curve corresponding to the monodromy group Γ.

Remark 3.2 Usual modular curves are $j-modular$. A $j-modular$ curve is simply any curve defined over a number field together with a special rational function on it (this follows from the theorem of Belyi [3], see 3.8). There is a countable number of such functions for each curve.

Let us give a combinatorial description of j-modular curves. They correspond to special triangulations of Riemann surfaces.

Definition 3.3 Let R be an oriented Riemann surface. A triangulation $\tau(R) = (\tau_0, \tau_1, \tau_2)$ of R is a decomposition of R into a finite union of open 2-cells τ_2 and a connected graph τ_1 with vertices τ_0 such that the complement $\tau_1 \setminus \tau_0$ is a disjoint union of open segments and the closure of any open 2-cell is isomorphic to the image of a triangle under a simplicial map.

The number of edges originating in a vertex x is called the valence at x and is denoted by $v(x)$.

Definition 3.4 A j-triangulation of R is a triangulation together with a coloring of vertices in three colors A, B and I such that

1. The colors of any two adjacent vertices are different.

2. There are 2 or 6 edges at vertices of color A and 2 or 4 edges at vertices of color B.

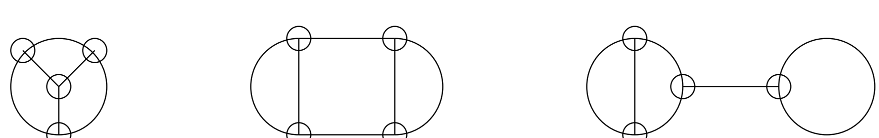
We will refer to vertices of color A (resp. B) with valence j as A_j (resp. B_j) vertices. If we delete the I-vertices from τ_0 and all edges AI and BI from τ_1 then the remaining connected 3-valent graph on R with A- and B-ends is called the j-graph associated to the j-triangulation. The complement to this graph is a disjoint union of a finite number of cells (neighborhoods of I-vertices). It might look as follows:

Here we use a small circle to indicate an A-vertex. The B-vertices are placed on the edges between two A-vertices. A "loose" end represents a B-vertex. A j-graph is called saturated if all A-vertices are A_6-vertices. Saturated graphs can be considered as arising from generalized triangulations of $\mathbb{P}^1$. An arbitrary graph can be obtained from a saturated graph by addition of trees.

Here are saturated graphs with $a_6 = 4$:

A j-triangulation on R can be reconstructed from a j-graph by placing one I-vertex into each connected component of R minus the j-graph and by connecting (cyclically) the I-vertex with vertices on the boundary of the corresponding connected component. The valences of A-vertices in an j-graph are 1 or 3, the valences of B-vertices are 1 or 2 and vertices of the same color are not connected by an edge.

Remark 3.5 The number of plane 3-valent graphs grows superexeponentially with the number of vertices.



The following well-known theorem forms the basis for our analysis of monodromy groups.

Theorem 3.6 *Let R be an oriented compact Riemann surface with a j-triangulation. Then there exists a unique structure of a j-modular curve on R. Conversely, every structure of a j-modular curve on R corresponds to a j-triangulation.*

Proof. Let us first show how j_Γ defines a triangulation of M_Γ. The map $j : \overline{\mathcal{H}} \to \overline{\mathcal{H}}/\mathrm{PSL}(2,\mathbb{Z}) = \mathbb{P}^1$ is ramified over three points $0 = A, 1 = B, \infty = I$. The ramification index at 0 is equal to 3, the ramification index at 1 is 2 and the ramification index at ∞ is infinite. Similar result is true for

$$j_\Gamma : \overline{\mathcal{H}}/\Gamma = M_\Gamma \to \overline{\mathcal{H}}/\mathrm{PSL}(2,\mathbb{Z}) = \mathbb{P}^1.$$

Consider the standard triangulation $\tau_{st}(\mathbb{S}^2)$ of the sphere $\mathbb{S}^2 = \mathbb{P}^1$ into a union of two triangles with vertices $0, 1$ and ∞. The preimage of this triangulation provides a triangulation of M_Γ. If we color the preimages of the corresponding vertices in A, B and I then we obtain a j-triangulation as wanted.

Conversely, starting with a j-triangulation τ we construct an algebraic curve R together with a map $R \to \mathbb{S}^2$ ramified in $0, 1, \infty$ as follows. We have a map from the set of vertices to (A, B, I) (the color). Further, every edge will be mapped into the edges of the standard triangulation of $\mathbb{S}^2$, respecting the colors of the ends. This map is completed by the map of triangles, which maps the triangles ABI (with orientation inherited from R) to one of the triangles of $\tau_{st}(\mathbb{S}^2)$ and the triangles with the opposite R-orientation to the other.

Thus we have constructed a simplicial map which is locally an isomorphism except in the neighborhood of vertices. Since triangles in R sharing an edge are mapped into different triangles of $\mathbb{S}^2$ the above map is locally an isomorphism outside of vertices and is equivalent to a map z^n in the neighborhood of each vertex in R. Thus it corresponds to a unique algebraic curve R with a map $R \to \mathbb{P}^1$ which is ramified over the points A, B, I.

In general, such curves are described by subgroups of finite index in the free group on two generators $\mathbf{F}_2$. Our assumption on the ramification indices at points A, B implies that the curve R corresponds to a subgroup of finite index in the quotient $\mathbb{Z}/2 * \mathbb{Z}/3$ of $\mathbf{F}_2$. The group $\mathbb{Z}/2 * \mathbb{Z}/3$ equals $\mathrm{PSL}(2,\mathbb{Z})$. Thus local monodromy groups over A-vertices can be either 1 or $\mathbb{Z}/3$ and over B either 1 or $\mathbb{Z}/2$. This finishes the proof of the theorem.

Corollary 3.7 *The number of triangles in any j-triangulation is equal to $2\deg(j_\Gamma)$. Moreover, $2\deg(j_\Gamma) = \sum_i v(i)$, where the summation is over all vertices i with color I.*

Remark 3.8 It follows from Belyi's theorem that every arithmetic curve (an algebraic curve defined over a number field) can be realized as a j-modular curve. Moreover, the corresponding triangulation of the curve is a barycentric subdivision of an arbitrary triangulation the underlying Riemann surface. In this case, Γ is torsion free and a subgroup of $\mathbf{F}_2 \subset \mathrm{PSL}(2,\mathbb{Z})$. The corresponding j-graph is a trivalent graph *without ends*. These type of j-graphs are called *saturated*. They correspond to a relatively small fraction of possible monodromy groups.

Many properties of Γ as a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ can be easily recovered from the j-triangulation. For example, there is a bijection between the set of B_2-vertices and conjugacy classes of subgroups of order 2 in Γ. Similarly, there is a bijection between A_2-vertices and conjugacy classes of subgroups of order 3 in Γ. Finally, there is a bijection between the I-vertices and conjugacy classes of unipotent subgroups in $\Gamma \subset \mathrm{PSL}(2,\mathbb{Z})$. The generator of the unipotent subgroup is given by $\begin{pmatrix} 1 & v(i)/2 \\ 0 & 1 \end{pmatrix}$, where $v(i)$ is the valence of the corresponding I-vertex i.

4 j-modular surfaces

In this section we study Jacobian elliptic surfaces such that the map $j_{\mathcal{E}}$ has degree 1. Here $\tilde{\Gamma} \subset \mathrm{SL}(2,\mathbb{Z})$ is the global monodromy group of the elliptic fibration $\mathcal{E}$. We call such surfaces j-modular surfaces and denote them by $S_{\tilde{\Gamma}}$.

Consider the j-modular curve M_Γ where Γ is the image of $\tilde{\Gamma}$ in $\mathrm{PSL}(2,\mathbb{Z})$ under the natural projection. We want to solve the following problem: describe all surfaces $S_{\tilde{\Gamma}}$ together with the structure of a Jacobian elliptic fibration over the j-modular curve M_Γ such that the monodromy group $\tilde{\Gamma}$ surjects onto Γ. We want to give a complete answer to this question using the j-triangulation of M_Γ.

We have an exact sequence

$$0 \to \mathbb{Z}/2 \to \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{PSL}(2,\mathbb{Z}) \to 1 \tag{2}$$

which induces a sequence

$$0 \to \mathbb{Z}/2 \to \Gamma' \to \Gamma \to 1, \tag{3}$$

where $\Gamma' \subset \mathrm{SL}(2,\mathbb{Z})$.

Lemma 4.1 *If Γ does not contain elements of order 2 then the exact sequence (3) splits. Equivalently, the j-triangulation of M_Γ does not contain B_2-vertices.*

Proof. The group $\mathrm{PSL}(2,\mathbb{Z}) = \mathbb{Z}/2 * \mathbb{Z}/3$. Any subgroup of finite index is a finite free product of groups isomorphic to $\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/3$. Assuming that Γ has no elements of order 2 we have a representation of Γ as a free product of groups $\mathbb{Z}, \mathbb{Z}/3$. If we lift the generators of these free generating subgroups to elements of the same order in Γ' we obtain a subgroup of Γ' which projects isomorphically onto Γ, in other words, a splitting of the exact sequence 3.

Remark 4.2 All splittings differ by $\mathbb{Z}/2$-characters of Γ $(H^1(\Gamma,\mathbb{Z}/2))$ and the one we obtain may not be the best (this will be specified in section 5). Namely, the preimages of unipotent generators can be products of unipotent elements by the central element in $\mathrm{SL}(2,\mathbb{Z})$. There may be no natural splitting.

We have to consider the following 3 cases:

Case 1. $\tilde{\Gamma} \simeq \Gamma$. There are finitely many such $\tilde{\Gamma}$ and they differ by a character of Γ. For each such character there exists a unique (up to birational morphisms) j-modular $S_{\tilde{\Gamma}}$. Indeed, take the quotient $V^o \to \mathcal{H}/\Gamma$ of the universal elliptic curve $\mathcal{E}^u \to \mathcal{H}$ by $\tilde{\Gamma}$. It has the structure of a fibration with a section and with generic fibers smooth elliptic curves. The monodromy of this fibration (over the open curve $B = \mathcal{H}/\Gamma$) is $\tilde{\Gamma} \simeq \Gamma$. Compactify V^o keeping the structure of an elliptic fibration (over $M_\Gamma = \overline{\mathcal{H}}/\Gamma$) and the zero section as above. It is clear that this construction is birationally universal. Indeed, if there is a Jacobian elliptic fibration V' with the given monodromy group $\tilde{\Gamma}$ then there is a rational fiberwise map $V \to V'$ which is regular on the grouplike parts of V and V'.

Case 2. There exists a lifting $\Gamma' \simeq \Gamma$ but $\Gamma' \not\simeq \tilde{\Gamma}$. The corresponding surfaces $S_{\tilde{\Gamma}}$ are obtained from surfaces in Case 1 by an even number of twists. Thus the set of such surfaces is parametrized by a symmetric power of $\mathbb{P}^1$

(modulo the action of the finite group of automorphisms of the embedded j-graph).

Case 3. The general case. Consider the universal elliptic curve $\mathcal{E}^u \to \mathcal{H}$ given as a quotient of $\mathbb{C} \times \mathcal{H}$ by $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$. The action of $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ on $\mathbb{C} \times \lambda$ is given by

$$\begin{aligned} e_1(z, \lambda) &= (z + 1, \lambda), \\ e_2(z, \lambda) &= (z + \lambda, \lambda) \end{aligned}$$

(here $(z, \lambda) \in \mathbb{C} \times \mathcal{H}$). The group $\mathrm{SL}(2, \mathbb{Z})$ acts on $\mathcal{E}^u$, stabilizing the section $(0, \lambda)$. Consider the quotient of the universal elliptic curve $\mathcal{E}^u \to \mathcal{H}$ by Γ'. We get an open surface V' admitting a fibration (with a section) over the open curve $B' = \mathcal{H}/\Gamma'$, whose generic fiber is a smooth rational curve. The map $\mathcal{E}^u \to V'$ is ramified over a divisor D which has at least two horizontal components: D_0 (which is a smooth zero-section of $V' \to B'$) and D_1 which projects to B' with degree 3 and is smooth and unramified over B' in the complement of singular fibers. Denote by V^o the open surface obtained by removing from V' the singular fibers. The surface V^o is fibered over an open curve B^o with fibers $\mathbb{P}^1$. The intersection of the divisor D with each fiber consists of exactly 4 points and D is unramified over B^o.

We want to define a double covering of V^o which is ramified on every component of D. There is a correspondence between such double coverings and special characters

$$\chi \in \mathrm{Hom}(\pi_1(V^o \setminus D), \mathbb{Z}/2).$$

The group $\pi_1(V^o \setminus D)$ has a quotient which is a central $\mathbb{Z}/2$-extension of the free group $\pi_1(B^o)$. This extension has a section (since the fibration $V^o \to B^o$ has a section) and therefore it splits into a product $\mathbb{Z}/2 \times \pi_1(B^o)$. A character χ defining a double cover of $V^o \setminus D$ is a character which is induced from $\mathbb{Z}/2 \times \pi_1(B^o)$ and which is an isomorphism on the central subgroup $\mathbb{Z}/2$ in $\mathbb{Z}/2 \times \pi_1(B^o)$.

In other words, the restriction of χ to the subgroup $\pi_1(\mathbb{P}^1 \setminus 4 \text{ points})$ (for every fiber $\mathbb{P}^1$ of the fibration $V^o \to B^o$) is equal to the standard character of $\mathbf{F}_3$ (realized as $\pi_1(\mathbb{P}^1 \setminus 4 \text{ points})$) which sends the standard generators of $\mathbf{F}_3$ into the non-zero element of $\mathbb{Z}/2$.

We summarize this in the diagram:

$$\begin{array}{ccccc}
\mathbf{F}_3 & \longrightarrow & \pi_1(V^o \setminus D) & & \\
\downarrow & & \downarrow & & \\
\mathbb{Z}/2 & \times & \pi_1(B^o) & \to & \mathbb{Z}_2 \\
\uparrow & & \downarrow & & \\
\mathrm{Ker}(\chi) & \longrightarrow & \Gamma' & &
\end{array}$$

The group $\mathrm{Ker}(\chi)$ is a subgroup of $\mathbb{Z}/2\times\pi_1(B^o)$ of index 2 and it is isomorphic to $\pi_1(B^o)$. This induces a map $\mathrm{Ker}(\chi) \to \Gamma'$.

The character χ defines a double cover $W^o(\chi)$ of V^o. The preimage of every fiber $\mathbb{P}^1$ of $V^o \to B^o$ is an elliptic curve realized as a standard double cover of this $\mathbb{P}^1$. Thus we obtain an open surface $W^o(\chi)$ with a structure of an elliptic fibration over B^o. All fibers are smooth. The monodromy $\tilde{\Gamma}$ of this elliptic fibration coincides with the image of $\mathrm{Ker}(\chi)$ in Γ'. If $\tilde{\Gamma}$ is not equal to the whole of Γ' then the sequence 3 splits. This also means that the character χ is induced from Γ'.

The character χ completely defines the local monodromy around the points in $M_\Gamma \setminus B^o$. Now we compactify V^o keeping the structure of an elliptic fibration over M_Γ and keeping the zero section. Locally, in the neighborhood of $b \in M_\Gamma$ corresponding to singular fibers our elliptic fibration is birationally isomorphic to a standard fibration from the Kodaira list. The corresponding birational isomorphism is biregular on the complement to the singular fiber. The zero section is preserved under this birational isomorphism. Now we can modify our initial fibration via this fiberwise transformation along neighborhoods of singular fibers. The resulting surface V is smooth and it admits a structure of a Jacobian elliptic fibration with the same monodromy group $\tilde{\Gamma}$.

Now consider the diagram

$$\begin{array}{ccccc}
\mathcal{E} & & S_{\tilde{\Gamma}} & & \\
\downarrow & & \downarrow & & \\
B & \to & M_\Gamma & \to & \mathbb{P}^1
\end{array}$$

Let $B^o = B \setminus j^{-1}\{0,1,\infty\}$ and $M_\Gamma^o = M_\Gamma \setminus j_\Gamma^{-1}\{0,1,\infty\}$ (the points deleted from M_Γ are the A, B and I -vertices of the j-triangulation). There is a natural map $\pi_1(B^o) \to \pi_1(M_\Gamma^o)$ and a commutative diagram of monodromy homomorphisms:

$$\begin{array}{ccc} \pi_1(B^o) & \longrightarrow & \Gamma' \\ \downarrow & & \downarrow \\ \pi_1(M^o_\Gamma) & \longrightarrow & \Gamma \end{array}$$

and a monodromy homomorphism $\pi_1(M^o_\Gamma) \to \Gamma'$, compatible with the projection $\Gamma' \to \Gamma$.

Finally, we want to compare the lifting of the elliptic fibration $S_{\tilde{\Gamma}}$ to B and $\mathcal{E}$.

Case A. $\tilde{\Gamma} \simeq \Gamma$. Then $\mathcal{E}$ is (fiberwise birationally) isomorphic to one of the j-modular surfaces constructed in Case 1 above, namely to the surface corresponding to the section $\Gamma \to \mathrm{SL}(2, \mathbb{Z})$.

Case B. The general case. Then $\mathcal{E}$ is (fiberwise birationally) obtained as a composition of a pullback of a corresponding $S_{\tilde{\Gamma}}$ to B followed by an even number of twists.

5 The topological type of j-modular surfaces

In this section we determine j-modular surfaces of smallest possible $\chi(\mathcal{E})$ among all $S_{\tilde{\Gamma}}$ with fixed Γ. For simplicity we assume that $B = \mathbb{P}^1$. Similar techniques work for any base M_Γ.

Jacobian elliptic fibrations over $\mathbb{P}^1$ arise in families, defined (in Weierstrass form) as follows. Denote by $U_0 = \mathbb{A}^1$ a chart of $\mathbb{P}^1$ obtained by deleting $(0:1)$ and by $U_\infty = \mathbb{A}^1$ the chart obtained by deleting $(1:0)$. On U_0 we use the coordinate t and on U_∞ the coordinate $s = 1/t$. Consider a hypersurface in $\mathbb{P}^2 \times U_0$ given by

$$zy^2 = x^3 + p_0(t)xz^2 + q_0(t)z^3$$

where p (resp. q) is a polynomial of degree $4r$ (resp. $6r$). In U_∞ the equation is similar, with $p_\infty(s) = p_0(1/s)s^{4r}$ and $q_\infty(s) = q_0(1/s)s^{6r}$. We get elliptic fibrations over U_0, U_∞ which we can glue to an elliptic surface $\mathcal{E} \to B$. The j-function (on U_0) is given by $p_0(t)^3/(4p_0(t)^3 + 27q_0(t)^2)$. The obtained fibration can be singular in fibers corresponding to $b \in B$ where $4p_0(t)^3 + 27q_0(t)^2 = 0$ and the singularities can be resolved by a sequence of blow-ups. The outcome is a (unique) smooth relatively minimal Jacobian elliptic fibration. Thus we get a family $\mathcal{F}_r$ of such elliptic fibrations. Notice that $12r = \chi(\mathcal{E})$. Conversely, a simply connected, compact, minimal Jacobian elliptic fibration with $\chi(\mathcal{E}) = 12r$ belongs to $\mathcal{F}_r$. The family $\mathcal{F}_r$ is parametrized by the coefficients of p_0, q_0 (subject to certain constrains) - it is a smooth irreducible variety. Every Jacobian elliptic fibration is birational to a minimal elliptic fibration and the j-map for both fibrations is the same.

On the other hand, the Euler characteristic $\chi(\mathcal{E})$ can be computed as a sum of contributions from singular fibers:

	χ_b		χ_b
I_0		I_0^*	6
I_n	n	I_n^*	$n+6$
II	2	IV^*	8
III	3	III^*	9
IV	4	II^*	10.

Here I_0 is a smooth fiber, I_n is a multiplicative fiber with n-irreducible components. The types $\mathrm{II}, \mathrm{III}$ and IV correspond to the case of potentially good reduction. More precisely, the neighborhood of such a fiber is a (desingularization of a) quotient of a local fibration with smooth fibers by an automorphism of finite order. The corresponding order is 4 for the case III and 3 in the cases II, IV. The fibers of type I_0^*, (resp. $\mathrm{I}_n^*, \mathrm{II}^*, \mathrm{III}^*, \mathrm{IV}^*$) are obtained from fibers I_0 (resp. $\mathrm{I}_n, \mathrm{IV}, \mathrm{III}, \mathrm{II}$) by twisting (changing the local automorphism by the involution $x \mapsto -x$ in the local group structure of the fibration). We shall call them $*$-*fibers* in the sequel.

Remark 5.1 The local invariant χ_b has a monodromy interpretation. Any element of a local monodromy at $b \in B^s$ has a minimal representation as a product of elements conjugated to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ in $\mathrm{SL}(2, \mathbb{Z})$. The length of this representation equals χ_b. This explains the equality $\chi_{b^*} = \chi_b + 6$ — the element $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ is a product of 6 elements conjugated to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ (elementary Dehn twists).

Proposition 5.2 *Fix Γ and assume that the number of vertices v_0 in the associated j-modular graph is divisible by 4. Then there exists a unique (up to birational transformations) j-modular surface with (minimal)*

$$\chi(S_{\tilde{\Gamma}}) = 3v_0.$$

If v_0 is not divisible by 4 then the set of j-modular surfaces with (minimal)

$$\chi(S_{\tilde{\Gamma}}) = 3(v_0 + 2)$$

forms a 1-parameter family.

Proof. First of all observe that v_0 is always even. Let $S_{\tilde{\Gamma}} \to M_\Gamma$ be a j-modular surface with given Γ and singular fibers exactly over the vertices of the j-modular triangulation. Then

$$\chi(S_{\tilde{\Gamma}}) = \sum_b \chi_b \geq 2a_2 + 3b_2 + \sum_I \chi_b.$$

At the same time

$$2\sum_I \chi_b = 2(3a_6 + a_2)$$

equals the number of triangles in the j-modular triangulation: every triangle contains exactly one A-vertex, A_2-vertices are contained in two triangles and A_6-vertices in 6 triangles. Therefore,

$$\chi(S_{\tilde{\Gamma}}) \geq 3a_6 + a_2 + 2a_2 + 3b_2 = 3v_0.$$

The difference

$$\chi(S_{\tilde{\Gamma}}) - 3v_0$$

is equal to 6 times the number of $*$-fibers. Twisting an even number of $*$-fibers we can diminish this difference – either to zero (when v_0 is divisible by 4) or to 6 (otherwise). In the latter case the $*$-fiber can be chosen freely on B^o.

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