COMMUTING ELEMENTS IN GALOIS GROUPS OF FUNCTION FIELDS

by

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Abstract. — We study the structure of abelian subgroups of Galois groups of function fields.

Contents

Introduction

Setup. — We fix a prime number p. Let k be a field of characteristic $\neq p$ and 2. Assume that k does not admit finite extensions of degree divisible by p. Let $K = k(X)$ be the field of functions of an algebraic variety X defined over k and G_K the Galois group of a separable closure of K. The principal object of our study is the group $\Gamma = \Gamma_K$ - the (maximal) pro-p-quotient of the kernel $\mathrm{Ker}(\mathrm{G}_K \to \mathrm{G}_k)$.

Main theorem. — This paper contains a proof of the main theorem from [**2**] describing (topologically) noncyclic subgroups in the abelianization

$$\Gamma^a := \Gamma/[\Gamma,\Gamma]$$

which can be lifted to abelian subgroups of

$$\Gamma^c := \Gamma/[[\Gamma,\Gamma],\Gamma].$$

Let ν be a valuation of K. We denote by K_ν the completion of K with respect to ν, by Γ^a_ν the reduced valuation group and by I^a_ν the abelian inertia group of ν (see Sections 6.1 and 6.2 for the definitions).

THEOREM 1. — *Let F be a noncyclic subgroup of Γ^a. Suppose that it can be lifted to an abelian subgroup of Γ^c. Then there exists a nonarchimedean valuation ν of K such that*

- *F is contained in the abelian reduced valuation group $\Gamma^a_\nu \subset \Gamma^a$ (standard valuation group if the residue field of K_ν has characteristic $\neq p$);*
- *F contains a subgroup F' such that $F' \subset \mathrm{I}^a_v$ and F/F' is topologically cyclic.*

It is easy to see that abelian groups satisfying the conditions of the theorem can be lifted to abelian subgroups of G_K itself. Thus Theorem 1 shows that there are no obstructions to the lifting of abelian subgroups of G_K beyond the first nontrivial level.

Our paper is a contribution to the "anabelian geometry" program, initiated by Grothendieck. For other results in this direction we refer to [**8**], [**10**], [**7**].

Structure of the proof. — By Kummer theory, the elements of Γ^a can be interpreted as k^*-invariant $\mathbb{Z}_p$-valued logarithmic functions on K^*. The quotient space K^*/k^* has a natural structure of an infinite dimensional projective space over k, denoted by $\mathbb{P}(K)$. Consider a pair of (nonproportional) elements f_1, f_2 of Γ^a. They define a map

$$\begin{array}{rrl} \varphi: & \mathbb{P}(K) \to & \mathbb{A}^2(\mathbb{Z}_p) \\ & (v,v') \mapsto & (f_1(v), f_2(v')). \end{array} \tag{1.1}$$

If f_1, f_2 lift to a pair of commuting elements in Γ^c then the restrictions of the corresponding functions to any projective line $\mathbb{P}^1 \subset \mathbb{P}(K)$ are linearly dependent modulo constant functions (see Proposition 5.4.1). Thus every projective line in $\mathbb{P}(K)$ maps into an affine line in $\mathbb{A}^2$. This - together with the logarithmic property of f_1 and f_2 - imposes very strong conditions on the 2-dimensional subspace they span in the space of functions on $\mathbb{P}(K)$. Namely, this subspace contains a special nonzero function which we call an abelian flag function (an AF-function); it corresponds to an inertia element of some reduced valuation subgroup (see Section 6.2 and Section 2 for the definitions). The main problem is to prove the existence of this AF-function.

In Section 2 we define AF-functions and study them on abelian groups of ranks 2 and 3. Let f be a function on a vector space V with values in a set S, $V' \subset V$ a subspace and $f_{V'}$ the restriction of f to V'. A series of reductions leads to the following criterium: f is an (invariant) AF-function on V iff for all 3-dimensional subspaces V' the function $f_{V'}$ is an (invariant) AF-function on V' (3.2.1). Reduction to 2-dimensional subspaces is more problematic. For fields k is of characteristic $\mathrm{char}(k) > 2$ the reduction to dimension 2 can be established without the use of the logarithmic property leading to an easier proof of this case of the main theorem. A similar statement for fields k of characteristic zero requires the logarithmic property. The corresponding proofs are in Section 3.4.

The proof of the main theorem proceeds by contradiction. We assume that the $\mathbb{Z}_p$-span $\langle f_1, f_2\rangle_{\mathbb{Z}_p}$ does not contain an AF-function. The reductions and the logarithmic property imply that there exists a 3-dimensional $V \subset K$, two nonproportional functions $f_1', f_2' \in \langle f_1, f_2\rangle_{\mathbb{Z}_p}$ and a map $h' : \mathbb{Z}_p \to \mathbb{Z}/2$ such that each of the functions $h' \circ f_1', h' \circ f_2', h \circ f_1' + h \circ f_2'$ fails to be AF. In Section 4.2 we find a contradiction to this claim.

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2. Classes of functions

In this section we define certain classes of functions on abelian groups and vector spaces which will be used in the proofs.

2.1. Notations. — Denote by $\mathbb{Z}_{(0)} = \mathbb{Z} \setminus 0$ and by $\mathbb{Z}_{(q)}$ the set of all integers coprime to q. Let $\mathcal{A}_0$ (resp. $\mathcal{A}_q$) be the set of torsion free abelian groups (resp. vector spaces over the finite field $\mathbf{F}_q$, where q is a prime number $\neq 2$). We denote by $\mathrm{rk}(A) \in \{1, ..., \infty\}$ the minimal number of generators of A (as an abelian group). An element $a \in A$ (with $A \in \mathcal{A}_0$) is called primitive if there are no $a' \in A$ and $n \in \mathbb{N}$ such that $a = na'$. We denote by $\langle a_1, ..., a_n \rangle$ the subgroup generated by $a_1, ..., a_n$ and similarly by $\langle B \rangle$ the subgroup generated by elements of a subset $B \subset A$. We denote by $\mathcal{F}(A, S)$ the set of functions on A with values in a set S. We will say that a function $f \in \mathcal{F}(A, S)$ is induced from A/nA if for all primitive x and all $y \in A$ with $x - y = nz$ for some $z \in A$ one has $f(x) = f(y)$. For $B \subset A$ and $f \in \mathcal{F}(A, S)$ we denote by f_B the restriction to B (or simply f if the domain is clear from the context).

Assumption 2.1.1. — *Throughout, all abelian groups are either in $\mathcal{A}_0$ or in $\mathcal{A}_q$.*

2.2. Definitions. — We will work in the following setup:

$$A \subset V \subset K$$

where A is a $\mathbb{Z}$-(resp. $\mathbf{F}_q$-) sublattice of a k-vector space V which is embedded into K.

Definition 2.2.1. — *Let $A \in \mathcal{A}_0$ (resp. $A \in \mathcal{A}_q$) and $f \in \mathcal{F}(A, S)$. We say that f is invariant if*

$$f(na) = f(a)$$

for all $a \in A$ and all $n \in \mathbb{Z}_{(0)}$ (resp. all $n \in \mathbb{Z}_{(q)}$)

Let V be a vector space over k and $f \in \mathcal{F}(V, S)$. We say that f is invariant if

$$f(\kappa v) = f(v)$$

for all $\kappa \in k^*$ and $v \in V$.

An invariant function on $A = \mathbf{F}_q^n$ (minus 0_A) can be considered as a function on $\mathbb{P}^{n-1}(\mathbf{F}_q) = (A \setminus 0)/\mathbf{F}_q^*$. An invariant function on $V \setminus 0$ can be considered as a function on the projective space $\mathbb{P}(V)$ (over k) and we will denote by $\mathcal{F}(\mathbb{P}(V), S) \subset \mathcal{F}(V, S)$ the space of such functions.

Definition 2.2.2 (Filtration). — *Let A be finitely generated and $\mathcal{I}$ a totally ordered set. A (strict)* filtration *on A with respect to $\mathcal{I}$ is a set of subgroups $A_\iota \subset A$ (with $\iota \in \mathcal{I}$) such that*

- *$A = \cup_{\iota \in \mathcal{I}} A_\iota$;*
- *if $\iota < \iota'$ then $A_{\iota'}$ is a proper subgroup of A_ι.*

Notations 2.2.3. — Denote by

$$\overline{A}_\iota := A_\iota \setminus \cup_{\iota' > \iota} A_{\iota'}.$$

Notice that for all $\iota \in \mathcal{I}$ we have $\overline{A}_\iota \neq \emptyset$.

Definition 2.2.4 (AF-functions). — *A function f on a finitely generated group A (as in 2.1.1) is called an* abelian flag *function if*

- *f is invariant;*
- *A has a (strict) filtration by groups (with respect to an ordered set $\mathcal{I}$) such that f is constant on $\overline{A}_\iota$ for all $\iota \in \mathcal{I}$.*

If A is not finitely generated then f is an abelian flag function if $f_{A'}$ is an abelian flag function for every finitely generated subgroup $A' \subset A$.

We denote the set of abelian flag functions by $\mathcal{AF}(A, S)$. This property does not depend on the value of f on the neutral element 0_A. We will identify functions which differ only on 0_A.

Definition 2.2.5. — *Let V be a vector space over k. A function $f \in \mathcal{F}(V, S)$ is called an abelian flag function if*

- *f is invariant;*
- *for all (additive) sublattices $A \subset V$ the restriction $f_A \in \mathcal{AF}(A, S)$.*

Definition 2.2.6 (c-pairs). — *Let A be an abelian group as above and S a ring. We will say that $f_1, f_2 \in \mathcal{F}(A, S)$ form a c-pair if for every subgroup $C \subset A$ of rank 2 one has*

$$\mathrm{rk}(\langle f_1, f_2, 1\rangle_S) \leq 2.$$

Definition 2.2.7 (LF-functions). — *Let V be a unital algebra over k and S an abelian group. A function $f \in \mathcal{F}(V,S)$ is called a* logarithmic *function if*
- *f is invariant;*
- *$f(v \cdot v') = f(v) + f(v')$ for all $v, v' \in V \setminus 0$.*

The set of logarithmic functions will be denoted by $\mathcal{LF}(V,S)$. We shall refer to abelian flag (resp. logarithmic) functions as AF-functions (resp. LF-functions).

2.3. First properties. —

Remark 2.3.1. — Assume that A is *finitely generated.* Then for every $f \in \mathcal{AF}(A,S)$ and every subgroup $B \subset A$ there exists a proper subgroup $B^1_f \subset B^0_f = B$ such that f is constant on the complement $B \setminus B^1_f$. In particular, if $b_0 \in B \setminus B^1_f$ and $b_1 \in B^1_f$ then $f(b_0 + b_1) = f(b_0)$. Thus we can speak about generic elements of B and the generic value of f on B. We obtain a decreasing (possibly finite) $\mathbb{N}$-filtration (A^n_f) on A: A^n_f is the *subgroup* of nongeneric elements in A^{n-1}_f. Notice that an analogous statement for infinitely generated groups is not true, in general (for example, $\mathbb{Q}$ and valuation subgroup for a nonarchimedean valuation).

Lemma 2.3.2. — *Assume that $f \in \mathcal{AF}(A,S)$. Then*
- *for all subgroups $B \subset A$ one has $f_B \in \mathcal{AF}(B,S)$;*
- *if S is a ring then $sf + s' \in \mathcal{AF}(A,S)$ for all $s, s' \in S$;*
- *for every map $h : S \to S'$ one has $h \circ f \in \mathcal{AF}(A,S')$.*

Proof. — Evident from the definition. □

Lemma 2.3.3. — *Let $A \in \mathcal{A}_0$ and $f \in \mathcal{F}(A,S)$. Let $B \subset A$ be a subgroup of finite index. If $f_B \in \mathcal{AF}(B,S)$ then $f \in \mathcal{AF}(A,S)$.*

Proof. — Observe that $nA \subset B$ for some $n \in \mathbb{N}$. By Lemma 2.3.2, $f_{nA} \in \mathcal{AF}(nA,S)$. Now use the invariance of f. □

2.4. Orders. —

Remark 2.4.1. — Let $f \in \mathcal{F}(A, S)$ be such that for every subgroup $B \subset A$ with $\mathrm{rk}(B) \leq 2$ the restriction $f_B \in \mathcal{AF}(B, S)$. Then f defines a partial relation $\tilde{>}_f$ on A as follows: let $b, b' \in A$ with $f(b) \neq f(b')$ and consider the subgroup $B = \langle b, b' \rangle$. One of these elements, say b, is generic in B and the other is not. Then we define $b \tilde{>}_f b'$.

Lemma 2.4.2 (Definition). — *A function $f \in \mathcal{AF}(A, S)$ defines an* order $>_{f,A}$ *on A as follows:*

- *if $f(a) \neq f(a')$ then $a >_{f,A} a'$ iff $a \tilde{>}_f a'$ (that is, $f(a + a') = f(a)$);*
- *if $f(a) = f(a')$ then $a >_{f,A} a'$ iff there exists a $b \in A$ (a separator) such that $f(a) \neq f(b)$ and $a >_{f,A} b >_{f,A} a'$;*
- *finally, $a =_f a'$ if for all a'' we have $a'' >_{f,A} a$ iff $a'' >_{f,A} a'$.*

Notations 2.4.3. — We will write $>_f$ or $>$ and $\tilde{>}$ whenever A and f are clear from the context. We will also use the symbols $\geq$ and $\geq_f$.

Proof. — If $\mathrm{rk}(A) < \infty$, then for every element $a \neq 0$ there exists an $n(a)$ such that $a \in A_f^{n(a)} \setminus A_f^{n(a)+1}$. Then $a > b$ iff $n(a) < n(b)$. For general A, the correctness of the definition and the transitivity of $>$ are checked on finitely generated subgroups of A. For correctness, we assume that $f(a) = f(b)$ and consider the possibility that

$$a \,\tilde{>}\, c \,\tilde{>}\, b \,\tilde{>}\, c' \,\tilde{>}\, a$$

for two separators c, c' (leading to a contradiction). For transitivity, we may need to consider the possibility

$$a > b > c \geq_f a$$

(with separators, if necessary). □

Let A be an abelian group and $f \in \mathcal{AF}(A, S)$. Define the subsets:

$$A_f^\alpha = \{a \in A \,|\, a \geq_f \alpha\}$$

where α is (a representative of) the equivalence class in A with respect to the equivalence relation $=_f$.

Lemma 2.4.4. — *Assume that* $\mathrm{rk}(A) < \infty$ *and $f \in \mathcal{AF}(A, S)$. Then each A_f^α is a subgroup of A and (A_f^α) is a filtration on A in the sense of Definition 2.2.2. Moreover, f is constant on $\overline{A}_f^\alpha$ for all α.*

Proof. — Evident. □

REMARK 2.4.5. — If $\mathrm{rk}(A) < \infty$ then (A_f^α) coincides with the filtration (A_f^n) introduced in Remark 2.3.1. Notice that the filtration (A_f^α) is not functorial under restrictions. In general, if $f \in \mathcal{AF}(A, S)$ and $B \subset A$ is a proper subgroup then $A_f^\alpha \cap B \neq B_f^\alpha$. In general, for maps $h : S \to S'$ the filtration A_f^α does not coincide with $A_{h\circ f}^\alpha$. However, $A_{h\circ f}^\alpha$ can be reconstructed starting from A_f^α. We will be interested in the case when $S' = \mathbb{Z}/2$.

LEMMA 2.4.6. — *Let $f \in \mathcal{F}(A, S)$ be such that:*

- *for every B with $\mathrm{rk}(B) \leq 2$ the restriction $f_B \in \mathcal{AF}(B, S)$ (by Remark 2.4.1, this defines a partial relation $\tilde{>}$ on A);*
- *$\tilde{>}$ extends to an order $>$ on $A \setminus 0$ (transitivity);*
- *for all $a, a', a'' \in A$ such that $a > a'$ and $a > a''$ one has*

$$a > a' + a''.$$

Then $f \in \mathcal{AF}(A, S)$.

Proof. — Evident. As in Lemma 2.4.4, we obtain a filtration by groups. □

2.5. Rank 2 case. —

EXAMPLE 2.5.1. — A typical AF-function is given as follows. Let p be a prime number and $A = \langle a, a' \rangle$ with

$$\begin{array}{rcl} A_f^{2n} & = & \mathbb{Z}p^n a \oplus \mathbb{Z}p^n a' \\ A_f^{2n+1} & = & \mathbb{Z}p^n a \oplus \mathbb{Z}p^{n+1} a' \end{array}$$

with f taking two values on A: one value on $\overline{A}_f^{2n}$ and a different value on $\overline{A}_f^{2n+1}$.

LEMMA 2.5.2. — *Let $A = \mathbb{Z} \oplus \mathbb{Z}$ and $f \in \mathcal{AF}(A, S)$. Then f is one of the following*

- *f is constant on $A \setminus 0$;*
- *f is constant on $A \setminus \mathbb{Z}a$, for some $a \in A$;*
- *there exists a prime number p and a subgroup C of index p^k (for some $k \geq 0$) such that f is constant on $A \setminus C$ and f_C is as in the Example 2.5.1.*

NOTATIONS 2.5.3. — In the second (resp. the third) case we put

$$p(A) = p(A, f) := 0,$$

$$p(A) = p(A, f) := p.$$

Proof. — Assume that f is nonconstant on $A \setminus 0$. Then there exist two primitive elements $a, a' \in A$ such that $f(a) \neq f(a')$ (and $B := \langle a, a' \rangle$ is a subgroup of finite index in A). Then one of these generators, say a', lies in the subgroup B^1_f. This means that B/B^1_f is a cyclic group. If it is a free cyclic group, then the function f is of the second type.

If it is a finite group then there is a proper subgroup C of finite index in B such that C contains B^1_f and $[C : B^1_f] = p$ (for some prime p). The function f is constant on $C \setminus B^1_f$. Hence $C^1_f = B^1_f$. We have the diagram

$$\begin{array}{ccccccccccc} A & \supset & B & \supset & C & \supset & B^1_f & \supset & B^2_f & & \supset \dots \\ & & & & & & \| & & \| & & \\ & & & & C & \supset & C^1_f & \supset & C^2_f = pC & & \end{array}$$

Indeed, since the generic value of f on pC is equal to the generic value of f on C it is not equal to the generic value of f on C^1_f. It follows that $pC \subset C^2_f$ and since $[C^1_f : pC] = p$ we have $pC = C^2_f$. By invariance, the function f_C is as in the Example 2.5.1. Again by invariance, the index $[A : C]$ is a p-power and f (on A) is of the third type.

□

COROLLARY 2.5.4. — *Let A and f be as in Lemma 2.5.2. Then*

- *f takes at most two values on $A \setminus 0$;*
- *if $na \in A^1_f$ for some n with $\gcd(n, p(A)) = 1$ then $a \in A^1_f$;*
- *if $A = \langle a_1, a_2 \rangle$ and $a_1 - a_2 = p(A)a_3$ ($a_3 \in A$) then $f(a_1) = f(a_2)$;*
- *if $|A/A^1_f| = 2$ and $a \in A$ is primitive then $f(a + 2a') = f(a)$ for all $a' \in A$;*
- *if $|A/A^1_f| > 2$ then A has a basis $\{a_1, a_2\}$ such that all three elements $a_1, a_2, a_1 + a_2$ are generic.*

LEMMA 2.5.5. — *Let $A = \mathbb{Z}/q \oplus \mathbb{Z}/q$ (with q prime) and $f \in \mathcal{AF}(A, S)$. Then f (considered as a function on $\mathbb{P}^1(\mathbf{F}_q)$) is constant on the complement to some point $P \in \mathbb{P}^1(\mathbf{F}_q)$.*

Proof. — See the proof of Lemma 2.5.2. □

LEMMA 2.5.6. — *Let A and $f \in \mathcal{F}(A, \mathbb{Z}/2)$ be such that:*
- *A has a basis (a, b) with $f(a) = f(a+b) \neq f(b)$;*
- *f is invariant (cf. Definition 2.2.4);*
- *f satisfies a functional equation: for all $a', b' \in (A \setminus 0)$ with*

$$f(a') = f(a),\ f(b') = f(b) \text{ and } f(a') = f(a' + b')$$

one has

$$f(ma' + nb') = f(ma' + (n + km)b') \tag{2.1}$$

for all $k, m, n \in \mathbb{Z}$.

Then $f \in \mathcal{AF}(A, \mathbb{Z}/2)$.

Proof. — We have a decomposition $A = A_a \cup A_b$ into two subsets (preimages of $0, 1$). We will generally use the letter a for elements in A_a and b for elements in A_b. Thus $f(a') = f(a) \neq f(b) = f(b')$ for all $a, a', b, b' \in A$.

First we consider the case $A \in \mathcal{A}_q$. By the functional equation,

$$f(a + nb) = f(a)$$

for all $n \in \mathbb{Z}$. By invariance,

$$f(ma + nb) = f(a)$$

for all $m, n \in \mathbb{Z}_{(q)}$. Thus f is constant on $A \setminus A^1_f$ (where $A^1_f = \mathbb{Z}b$).

Now we turn to the case $A \in \mathcal{A}_0$. Denote by $A^1_f = \langle A_b \rangle \subset A$ the subgroup in A generated by elements $b' \in A_b$. We claim that A^1_f is a *proper* subgroup of A (and clearly, $f(a_1) = f(a)$ for all $a_1 \in (A \setminus A^1_f)$).

Consider a pair of elements

$$\begin{aligned} b_1 &= m_1 a + n_1 b, \\ b_2 &= m_2 a + n_2 b. \end{aligned}$$

We can assume that $m_1, m_2 > 0$. Let

$$d_A := \min(\gcd(m_1, m_2))$$

be the minimum over all pairs $(b_1, b_2) \in A_b \times A_b$ (with positive m_1, m_2).

Assume first that there exists a b_1 such that $b_1 = d_A a + n_1 b$ (for some $n_1 \in \mathbb{Z}$). This is impossible for $d_A = 1$, by the functional equation. Now consider the case $d_A > 1$. In this case A^1_f is a proper subgroup of A (since for all $b_2 = m_2 a + n_2 b$ the coefficient m_2 is divisible by d_A and consequently for all $ma + nb \in A^1_f$ the coefficient m is divisible by d_A).

Now assume that there are no such b_1. Choose a pair

$$\begin{aligned} b_1 &= d_A m_1 a + n_1 b, \\ b_2 &= d_A m_2 a + n_2 b \end{aligned}$$

as above (such that $(m_1, m_2) = 1$) and integers $l_1, l_2 \in \mathbb{Z}$ such that $m_1 l_1 + 1 = m_2 l_2$. Then, (using invariance),

$$\begin{aligned} f(d_A r a + e_1 b) &= f(d_A(r+1)a + e_2 b) \\ &= f(b) \end{aligned}$$

for some $e_1, e_2 \in \mathbb{Z}$. Pick the smallest positive r_0 with this property. Then $r_0 > 1$ (since $f(d_A a + nb) = f(a)$ for all n, by assumption) and

$$f(d_A(r_0 - 1)a + nb) = f(a)$$

for all $n \in \mathbb{Z}$. Therefore, (using functional equation and invariance),

$$\begin{aligned} f(a) &= f(d_A(r_0 - 1)a + (2e_1 - e_2)b) \\ &= f((-d_A a + (e_1 - e_2)b) + (d_A r_0 a + e_1 b)) \\ &= f(d_A(r_0 + 1)a + e_2 b) \\ &= f(b), \end{aligned}$$

contradiction.

Thus A^1_f is a proper subgroup of A and $a_1 \in A_a$ for all $a_1 \notin A^1_f$. Now consider the subgroup

$$A^2_f := \langle A^1_f \cap A_a \rangle$$

(generated by $a_2 \in A^1_f$ with $a_2 \in A_a$). We claim that A^2_f is a proper subgroup of A^1_f. (Warning: the conditions of the Lemma are *not* symmetrical with respect to a and b.)

First observe that there exists a basis a', b' of A^1_f such that

$$f(a' + b') = f(b)$$

(otherwise, the subgroup $\langle A_b \rangle$ would be a proper subgroup in A^1_f, contradiction). We fix such a basis and claim that

$$f(ra' + b') = f(b)$$

for all $r \in \mathbb{Z}$. Indeed, if this is not the case then there exists a positive integer $r_0 > 1$ such that

$$\begin{array}{rcl} f(a) & = & f(r_0 a' + b'), \\ f((r_0 - 1)a' + b') & = & f(b). \end{array}$$

Then

$$\begin{aligned} f(a) &= f(a' + ((r_0 - 1)a' + b')) \\ &= f(a' - ((r_0 - 1)a' + b')) \\ &= f((r_0 - 2)a' + b') \end{aligned}$$

(here we used the functional equation, and the invariance). This contradicts the minimality of r_0. Similar argument works for $r < 0$.

Consider the set of pairs

$$\begin{array}{rcl} a_1 & = & m_1 a' + n_1 b' \\ a_2 & = & m_2 a' + n_2 b' \end{array}$$

with positive n_1, n_2 and denote by d_B the smallest $\gcd(n_1, n_2)$ on this set.

Assume that $d_B > 1$ and that there exists an $a_1 = m_1 a' + d_B b'$. Then for every $a_2 = m_2 a' + n_2 b' \in A^1_f$ the coefficient n_2 is divisible by d_B and A^2_f is a proper subgroup.

Now we can assume that

$$f(ma' + d_B b') = f(b)$$

for all $m \in \mathbb{Z}$. Choose a pair a_1, a_2 such that $\gcd(n_1, n_2) = d_B$. Then, (by invariance),

$$\begin{aligned} f(a) &= f(r_1 a' + e d_B b') \\ &= f(r_2 a' + (e+1) d_B b') \end{aligned}$$

for some $r_1, r_2 \in \mathbb{Z}$ and $e > 0$. Pick the smallest $e_0 > 1$ with this property. Then, (using the fact that

$$f((r_2 - r_1)a' + d_B b') = f(b)$$

and the functional equation), we get

$$\begin{aligned} f(a) &= f((r_1 a' + e_0 d_B b') + (r_2 - r_1)a' + d_B b') \\ &= f((2r_1 - r_2)a' + (e_0 - 1) d_B b'). \end{aligned}$$

This contradicts the minimality of e_0.

It follows that the subgroup $A_f^2 \subset A_f^1$ is a proper subgroup and f takes the value $f(b)$ on the complement $A_f^1 \setminus A_f^2$.

Since A_f^2 has a basis (a'', b'') with

$$f(a'') = f(a'' + b'') \neq f(b''),$$

we can apply the inductive step to A_f^2. □

2.6. Rank 3 case: $\mathcal{A}_q$. —

PROPOSITION 2.6.1. — *Let $q > 2$ be a prime number, $A = \mathbb{Z}/q \oplus \mathbb{Z}/q \oplus \mathbb{Z}/q$ and $f \in \mathcal{F}(A, \mathbb{Z}/2)$. Assume that for all subgroups $C \subset A$ with* $\mathrm{rk}(C) \leq 2$ *we have $f_C \in \mathcal{AF}(C, \mathbb{Z}/2)$. Then $f \in \mathcal{AF}(A, \mathbb{Z}/2)$.*

Proof. — We can consider f as a function on $\mathbb{P}(A)$. By Lemma 2.5.5, for every $C \subset A$ with $\mathrm{rk}(C) = 2$ the restriction f_C is either constant on $\mathbb{P}(C) \subset \mathbb{P}(A)$ or constant everywhere except one point. Let L_i be the set of lines $\ell \subset \mathbb{P}(A)$ such that the generic value of f_ℓ is i (for $i = 0, 1$). Assume that f is nonconstant on $\mathbb{P}(A)$. If L_0 is empty then there exists only one point $P \in \mathbb{P}(A)$ with $f(P) = 0$ (otherwise we can draw a line of type L_0 through two such points, and 0 must be the generic value on this line, contradiction). In this case $f \in \mathcal{AF}(A, \mathbb{Z}/2)$. Thus we can assume that both L_0 and L_1 are nonempty and that, for example, $|L_0| \geq |L_1|$. Then $|L_0| \geq (q^2 + q + 1)/2$. Choose an $\ell_1 \in L_1$.

There are two cases: f is constant on $\ell_1 \in L_1$ or f is nonconstant on some line ℓ_1. In both cases there exists at least one point $P \in \ell_1$ such that $f(P) = 1$ and such that there are two distinct lines ℓ_0, ℓ_0' passing through $P \subset \ell_1$.

Indeed, assume in both cases that through every generic point of ℓ_1 there passes only one line of type L_0. In the first case the total number of lines of type L_0 is bounded by $q+1$, contradiction to the assumption that $|L_0| \geq (q^2+q+1)/2$. In the second case, there are at most q lines of type L_0 passing through the nongeneric point and, by assumption, at most 1 line of type L_0 passing through each of the remaining q generic points of ℓ_1 (every line in L_0 intersects ℓ_1 in one point). Thus their number is bounded by $2q < (q^2 + q + 1)/2$, contradiction.

For any pair of points $Q \in \ell_0 \backslash P$, $Q' \in \ell_0' \backslash P$ we have $f(Q) = f(Q') = 0$. The lines through Q, Q' are all of type L_0. Pick a point $P' \in \ell_1$ such that $P \neq P'$ and $f(P') = 1$ (such a point exists since ℓ_1 has at least 3 points and the generic value of f on ℓ_1 is 1). Every line through P' which does not pass through P is of type L_0 (since it intersects ℓ_0, ℓ_0' in distinct points). The family of such lines covers $\mathbb{P}(A) \setminus \ell_1$. It follows that the value of f on $\mathbb{P}(A) \setminus \ell_1$ is 0 and that $f \in \mathcal{AF}(A, \mathbb{Z}/2)$. $\square$

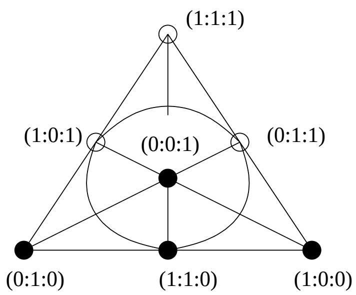
2.7. Exceptional lattices. —

EXAMPLE 2.7.1. — Let $\bar{A} = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $\bar{f} \in \mathcal{F}(\bar{A}, \mathbb{Z}/2)$. Assume that for all subgroups $\bar{C} \subset \bar{A}$ of rank 2 one has $f_{\bar{C}} \in \mathcal{AF}(\bar{C}, \mathbb{Z}/2)$ but $\bar{f} \notin \mathcal{AF}(A, \mathbb{Z}/2)$. Then A has a basis $\bar{e}_1, \bar{e}_2, \bar{e}_3$ such that

$$f(\bar{e}_1 + \bar{e}_3) = f(\bar{e}_2 + \bar{e}_3) = f(\bar{e}_1 + \bar{e}_2 + \bar{e}_3) = 0 \neq f(\bar{x})$$

for all other $\bar{x}$ (up to addition of 1 modulo 2).

Indeed, since $\mathbb{P}^2(\mathbb{Z}/2)$ has seven points it suffices to assume that f takes the same value on three of them and a different value on the remaining four. If the three points are on a line we have an AF-function. If not we get the claim. (In particular, such an f contradicts the conclusion of Lemma 2.6.1.)



Any function on $A = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ induced from the function on $\bar{A}$ considered above has the property that for every $C \subset A$ of rank ≤ 2 $f_C \in \mathcal{AF}(C, \mathbb{Z}/2)$.

We give another example of a function on $\mathbb{Z}^3$ with the same property.

EXAMPLE 2.7.2. — We keep the notations of Example 2.7.1. Choose a basis e_1, e_2, e_3 of $A = \mathbb{Z}^3$. Consider the projection $A \to \bar{A} = A/2A$, taking e_j to $\bar{e}_j$. The function f is defined by its values on primitive elements

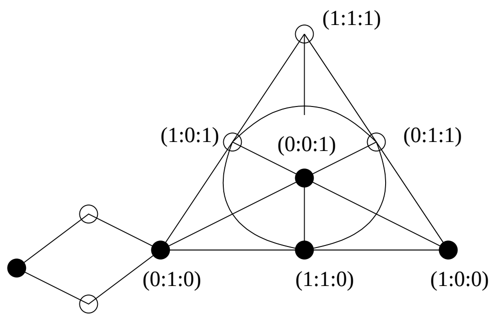
$$a = n_1e_1 + n_2e_2 + e_3n_3.$$

If $(n_1, n_2, n_3) \neq (0, 1, 0)$ modulo 2 then

$$f(a) = \bar{f}(\bar{a})$$

(where $\bar{f}$ was defined in Example 2.7.1). Otherwise,

$$f(n_1e_1 + n_2e_2 + n_3e_3) = \begin{cases} 0 & \text{if} \quad n_1 = 0 \mod 4 \\ 1 & \text{if} \quad n_1 = 2 \mod 4 \end{cases}$$



2.8. Rank 3 case: $\mathcal{A}_0$. —

Let A be an abelian group and $f \in \mathcal{F}(A, \mathbb{Z}/2)$. We have a decomposition of the set $A = A_a \cup A_b$ (preimages of 0 or 1, respectively). We will say that A has a *special basis* (with respect to f) if $A = \langle a_1, a_2, b_1 \rangle$ with

$$a_1, a_2, a_1 + b_1, a_2 + b_1 \in A_a, \quad b_1 \in A_b.$$

Proposition 2.8.1. — *Let $A = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and $f \in \mathcal{F}(A, \mathbb{Z}/2)$. Assume that A has a special basis (with respect to f) and that for every subgroup C with* $\mathrm{rk}(C) \leq 2$ *one has $f_C \in \mathcal{AF}(C, \mathbb{Z}/2)$. Then there is a proper subgroup $A^1 \subset A$ such that f is constant on $A \setminus A^1$.*

Proof. — The proof is subdivided into a sequence of Lemmas.

Lemma 2.8.2. — *Assume A has a special basis $\{a_1, a_2, b_1\}$. Then A does not have a basis $\{b_1, b_2, b_3\}$ with $b_1, b_2, b_3 \in A_b$.*

Proof. — Assume the contrary. Then

$$C := \langle a_1 + e_1 b_1, a_2 + e_1' b_1\rangle = \langle b_2, b_3\rangle$$

for some $e_1, e_2' \in \mathbb{Z}$. We know that

$$f(a_1 + e_1 b_1) = f(a_2 + e_1' b_1) = f(a)$$

for all e_1, e_1'. Contradiction to, for example, 2.5.2: C cannot have such pairs of generators. $\square$

LEMMA 2.8.3. — *Assume that A has a special basis $\{a_1, a_2, b_1\}$. Then*

$$\langle A_b\rangle \subset A$$

is a proper subgroup.

Proof. — Consider the projection

$$\mathrm{pr} \,:\, A \to \hat{A} := A/\langle b_1\rangle.$$

Assume that there exists an element

$$b = n_1 a_1 + n_2 a_2 + m_1 b_1 \in A_b$$

such that $\hat{b} = n_1 a_1 + n_2 a_2$ is primitive in $\hat{A} = \langle a_1, a_2\rangle$. Then it is part of a basis $\{\hat{x}, \hat{b}\}$ of $\hat{A}$. Take any x in the preimage $\mathrm{pr}^{-1}(\hat{x})$. Then

$$A = \langle x, b, b_1\rangle.$$

By Lemma 2.8.2, $x \notin A_b$, so we will denote it by a. Assume that

$$a + mb + m_1 b_1 \in A_b$$

for some $m, m_1 \in \mathbb{Z}$. This contradicts 2.8.2, since

$$A = \langle a + mb + m_1 b_1, b, b_1\rangle.$$

.

Consider the set R of all $r \in \mathbb{N}$ such that

$$b_r := ra + mb + m_1 b_1 \in A_b$$

(for some $m, m_1 \in \mathbb{Z}$). We have seen that $r > 1$. We claim that $r, r' \in R$ implies that $g := \gcd(r, r') \in R$. Indeed, assume the contrary and choose $l, l' \in \mathbb{Z}$ so that $g = lr - l'r'$. By invariance, $lb_r, l'b_{r'} \in A_b$. In the subgroup $B := \langle b_r, b_{r'}\rangle$ the element $a_g := lb_r - l'b_{r'} \in A_a \cap B$ is nongeneric. This implies that

$$b_r - na_g \in A_b \cap B$$

for all n. This leads to a contradiction and the claim follows. Thus we have proved that for all $b' \in A_b$ the corresponding coefficients n_1' are either zero or have a common divisor > 1. Consequently, $\langle A_b\rangle$ is a proper subgroup.

Now we assume that $\mathrm{pr}(A_b)$ does not contain primitive elements of $\langle a_1, a_2\rangle$, in other words: for all primitive $a \in \langle a_1, a_2\rangle$ and all $m_1 \in \mathbb{Z}$ one has

$$a + m_1 b_1 \in A_a.$$

For two pairs of (a, m_1) and (a', m_1') with primitive a, a' such that

$$\langle a_1, a_2\rangle = \langle a, a'\rangle$$

consider the subgroups

$$\begin{array}{rcl} D & := & \langle a, m_1 b_1\rangle \\ D' & := & \langle a', m_1' b_1\rangle \end{array}$$

and assume that both $p = p(D), p' = p(D') \neq 0$.

We claim that $p = p'$. Indeed, assume the contrary. By Lemma 2.5.2, there exist integers k, k' such that

$$f(qa + m_1 b_1) = f(q'a' + m_1' b_1) = f(b_1),$$

where $q = p^k, q' = p'^{k'}$ for some $k, k' \in \mathbb{N}$. Now consider the group

$$E := \langle qa + m_1 b_1, q'a' + m_1' b_1\rangle.$$

For all $n_1 \in \mathbb{Z}$ coprime to p' and all $n_1' \in \mathbb{Z}$ coprime to p the element

$$n_1(qa + m_1 b_1) + n_1'(q'a' + m_1' b_1) = n_1 qa + n_1' q'a' + (n_1 m_1 + n_1' m_1')b_1 \in A_a,$$

(since $\mathrm{pr}(A_b)$ does not contain primitive elements). The subset of such elements cannot be contained in a proper subgroup of E. On the other hand, it has to be: both generators of E are in A_b and $f \in \mathcal{AF}(E, \mathbb{Z}/2)$. Contradiction to the assumption that $p \neq p'$. Since for any pair of primitive a, a' generating a sublattice of finite index in $\langle a_1, a_2\rangle$ there exists a primitive element a_0 such that

$$\langle a, a_0\rangle = \langle a', a_0\rangle = \langle a_1, a_2\rangle$$

we conclude that for the corresponding D as above either $p(D) = 0$ or $p(D) = p$ for some fixed prime p.

To finish the proof of the lemma, consider an element

$$na + m_1 b_1 \in A_b$$

for some $n > 1$, some primitive $a \in \langle a_1, a_2 \rangle$ and some m_1 (coprime to $n > 1$). There are two possibilities: either n is zero or n is divisible by a fixed prime p (which is independent of the coefficients). It follows that $\langle A_b \rangle \subset A$ is a proper subgroup. $\square$

This concludes the proof of Proposition 2.8.1 $\square$

PROPOSITION 2.8.4. — *Let $A = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and $f \in \mathcal{F}(A, \mathbb{Z}/2)$. Assume that A does not have a special basis (with respect to f) and that for all subgroups $C \subset A$ of rank 2 one has $f_C \in \mathcal{AF}(C, \mathbb{Z}/2)$. Then f (up to addition of 1 modulo 2) is*

Case 1: as in Example 2.7.1 or

Case 2: as in Example 2.7.2.

Proof. — Assume first that for all subgroups $C \subset A$ of rank 2 the function f is either constant on $C \setminus 0$ or $|C/C_f^1| = 2$. Then f is induced from $A/2A$. Indeed, consider the subgroup $C := \langle x, y \rangle$ (with $x \in A$ primitive and y nonproportional to x). It suffices to consider the case when f is nonconstant and thus $|C/C_f^1| = 2$. By Corollary 2.5.4, if $x - y = 2z$ we have

$$f(x) = f(y),$$

so that f is induced from $A/2A$.

Now we assume that there exists a subgroup C of rank 2 such that $|C/C_f^1| > 2$ and f has generic value, say $f(a)$ on C. By Corollary 2.5.4, we can choose a basis $C = \langle a_1, a_2 \rangle$ such that all three $a_1, a_2, a_1 + a_2 \in A_a$. We will fix such a basis. For any $d \in A$ such that $\langle d, C \rangle = A$ consider the shift $d + C \subset A$.

1. We claim that both

$$(d + C) \cap A_a \neq \emptyset \text{ and } (d + C) \cap A_b \neq \emptyset.$$

Indeed, assume this is not so and choose, in the first case, some element $b \in d + C$. Then $\{b + a_1, b + a_2, a_1\}$ is a special basis of A. In the second case, for any $a \in d + C$ we get a special basis $\{a, a_1, b_1\}$, (where $b_1 \in A_b$ is some nongeneric element in C). Contradiction.

For any pair of generators $\{a_1', a_2'\}$ of C (without the assumption that $a_1' + a_2' \in A_a$) we have:

2. (Forbidden triangle.) There are no $b \in (d+C) \cap A_b$ such that both

$$b + a_1', b + a_2' \in A_a.$$

Indeed, $\{a_1', a_2', b\}$ would be a special basis for A.

3. (Forbidden square.) There are no $b \in (d+C) \cap A_b$ such that all three

$$b + a_1', b + a_2', b + a_1' + a_2' \in A_b.$$

Indeed, $\{b, b + a_1', a_2'\}$ would be a special basis for A.

4. Choose any element $b \in d + C$ and consider the subset $\{b + na_1\}$ (with $n \in \mathbb{N}$). Than there are two possibilities: either $b + na_1 \in A_b$ for all $n \in \mathbb{N}$ or there exists an $n_0 > 0$ such that

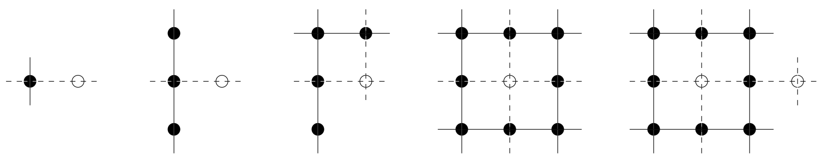
$$b + n_0 a_1 \in A_b \text{ and } b + (n_0 + 1)a_1 \in A_a.$$

Let us consider the second case: rename $b + n_0 a_1$ to b.

Then $b + a_1 \in A_a$. By 2, $b + a_2, b - a_2 \in A_b$. By 2 and by our assumption that $a_1 + a_2 \in A_a$, we have $b + a_1 + a_2 \in A_b$. By 2 (applied to $b - a_2$), we have $b + a_1 - a_2 \in A_b$; similarly,

$$b + 2a_1, b + 2a_1 - a_2, b + 2a_1 + a_2 \in A_b.$$

By 3, applied to $b + a_1 - a_2$, we have $b + 3a_1 \in A_a$.



Clearly, a pattern is emerging: we can rename $b + 2a_1$ to b and repeat the argument. Further, notice the symmetry with respect to a_1 and a_2, as well as the symmetry with respect to ± 1. In conclusion, we have:

For any $a_3 \in A_a$ such that $\langle a_1, a_2, a_3 \rangle = A$ we have

$$n_1 a_1 + n_2 a_2 + a_3 \in A_a$$

iff both n_1 and n_2 are divisible by 2.

5. We claim that for any primitive $x \in C$ and $c \in C$ we have $f(x+4c) = f(x)$. Indeed, consider the lattice

$$E := \langle x, a_3 + 2c \rangle.$$

In E, we have two sublattices of index 2:

$$\begin{array}{rcl} E' & := & \langle a_3 + 2c, a_3 + 2c + 2x \rangle \\ E'' & := & \langle a_3 + 2c + x, a_3 + 2c - x \rangle. \end{array}$$

The generic values of f on these sublattices are different (by 4). It follows that one of them is equal E^1_f. By Corollary 2.5.4, we have $f(x + 2y) = f(x)$ for every $y \in E$, in particular

$$f(x + 2a_3 + 4c) = f(x).$$

Now consider the lattice

$$G := \langle x', a_3 \rangle,$$

where $x' = x + 4c$, and the sublattices

$$\begin{array}{rcl} G' & := & \langle a_3, a_3 + 2x' \rangle \\ G'' & := & \langle a_3 + x', a_3 - x' \rangle. \end{array}$$

Both have index 2 in G and have different generic values. If follows that $|G/G^1_f| = 2$. In particular,

$$f(x' + 2a_3) = f(x + 4c + 2a_3) = f(x + 4c).$$

Combining with the result for E we get our claim. It follows that for every sublattice $C \subset A$ of rank 2 $p(C) = 2$ and, moreover, that $|C/C^1_f|$ is equal to 2 or 4. If $|C/C^1_f| = 2$ for every subgroup C of rank 2 we get a contradiction to our assumption (this leads to Case 1).

6. We can assume that $|C/C^1_f| = 4$. We claim that f is as in Case 2. First of all,

$$f(n_1a_1 + n_2a_2 + m_3a_3) = f(n_1a_1 + n_2a_2 + a_3)$$

for all odd m_3 (equal to $f(a_3)$ iff $n_1 = n_2 = 0$ modulo 2, by 4). Next, $f(2m_3a_3 + c) = f(c)$ for all $m_3 \in \mathbb{Z}$ and all primitive $c \in C$. Since

$$f_C(x + 4c) = f(x)$$

for all primitive x and all $c \in C$ we conclude that either f_C is constant, or induced from $C/2C$ or as in Case 2. The first two possibilities contradict our assumptions on C. $\square$

3. Reductions

3.1. Reduction of S. —

LEMMA 3.1.1. — *Let $f \in \mathcal{F}(A, S)$. Assume that for all $h : S \to \mathbb{Z}/4$ the function $h \circ f \in \mathcal{AF}(A, \mathbb{Z}/4)$. Then $f \in \mathcal{AF}(A, S)$.*

Proof. — The invariance is obvious (if $f(na) \neq f(a)$ for some n, a then define h so that $h \circ f(na) \neq h \circ f(a)$, leading to a contradiction). Assume that there exist elements $a, b \in A$ such that $f(a), f(b)$ and $f(a+b)$ are pairwise distinct. Define h such that

$$\begin{array}{lcl} h \circ f(a) & = & 0, \\ h \circ f(b) & = & 1, \\ h \circ f(a+b) & = & 2. \end{array}$$

Then, by Lemma 2.4.2, $h \circ f \notin \mathcal{AF}(A, \mathbb{Z}/4)$, contradiction. We see that for any $a, b \in A$ with $f(a) \neq f(b)$ either $f(a+b) = f(a)$ or $f(a+b) = f(b)$. This defines a partial relation $\tilde{>}$ on A (as in Remark 2.4.1).

We need to check that $\tilde{>}$ can be extended to an order on A. Let $a, a' \in A$ be such that $f(a) = f(a')$. If there is no $b \in A$ such that $f(b) \neq f(a)$ then f is constant and thus $\in \mathcal{AF}(A, S)$. If for all such $b \in A$ we have $a \tilde{>} b$ and $a' \tilde{>} b$ then $a =_f a'$. Otherwise, b is a separator and we can assume that $a \tilde{>} b \tilde{>} a'$. Assume that for some other separator b' we have $a' \tilde{>} b' \tilde{>} a$. Let

$$\begin{array}{lcl} h \circ f(a) & = & 0, \\ h \circ f(b) & = & 1 \end{array}$$

and put (if $f(b) \neq f(b')$)

$$h \circ f(b') = 2.$$

By assumption, $h \circ f \in \mathcal{AF}(A, \mathbb{Z}/4)$, contradiction (we use that either $f(a) = f(a+b)$ or $f(b) = f(a+b)$, etc). Thus we have a correctly defined relation $>$ on A.

Now we check the transitivity of $>$. Assume that we have elements $a, b, c \in A$ such that $a >_f b > c$. Assume that $c \geq a$. If the values of f on a, b, c are pairwise distinct, put

$$\begin{aligned} h \circ f(a) &= 0, \\ h \circ f(b) &= 1, \\ h \circ f(c) &= 2. \end{aligned}$$

Since $h \circ f \in \mathcal{AF}(A, S)$ we get a contradiction. If $f(a) = f(b)$, let a' be their separator; if $f(b) = f(c)$ let b' be their separator and if $f(c) = f(a)$, let c' be their separator: $c \geq c' \geq a$. Then there is a map $h : S \to \mathbb{Z}/4$ such that $h \circ f \notin \mathcal{AF}(A, \mathbb{Z}/4)$, contradiction.

Finally, we need to check that if $a > b$ and $a > b$, then $a > b + c$. Again, we can introduce separators, if necessary, and proceed as above.

To conclude we apply Lemma 2.4.6. $\square$

Lemma 3.1.2. — *Let A be a finitely generated group, S a finite set and $f \in \mathcal{F}(A, S)$. Assume that for all $h : S \to \mathbb{Z}/2$ one has $h \circ f \in \mathcal{AF}(A, \mathbb{Z}/2)$. Then $f \in \mathcal{AF}(A, S)$.*

Proof. — As above, the invariance of f is obvious. Following the proof of Lemma 3.1.1, observe that for all $a, b \in A$ with $f(a) \neq f(b)$ either $f(a+b) = f(a)$ or $f(a+b) = f(b)$. Thus we have a partial relation $\tilde{>}$ on these pairs as in 2.4.1.

Let $h : S \to \mathbb{Z}/2$ be a nonconstant map and

$$S(h) := \{s \in S \mid \exists\, a \in A \setminus A^1_{h \circ f} \text{ with } f(a) = s\}.$$

Let h_0 be a map such that

$$|S(h_0)| = \min_h(|S(h)|).$$

We can assume that $S = \{1, ..., n\}$ and that $S(h_0) = \{1, ..., k_0\}$.

Assume that $1 < k_0 < n$. Let $a_1, ..., a_{k_0}$ be some elements in $A \setminus A^1_{h_0 \circ f}$ with $f(a_j) = j$. Then, for all $j \in S(h_0)$ and all $i \notin S(h_0)$ we have $a_j > x_i$ for all $x_i \in A$ with $f(x_i) = i$.

Let h' be the map sending each element in $\{n, 2, ..., k_0\}$ to 0 and each element in $\{1, k_0+1, ..., n-1\}$ to 1. One of the values is generic for $h' \circ f$. Assume that 0 is the generic value for $h' \circ f$. Then $a_n \notin A \setminus A^1_{h' \circ f}$ (indeed, if a_n were generic then $a_n > x_1$ for all x_1 with $f(x_1) = 1$, contradiction to the previous). But then $|S(h')| \leq k_0 - 1$, contradiction to the minimality of k_0.

Assume that 1 is the value of a generic element for $h' \circ f$. Similarly, the elements a with $f(a) \in \{k_0 + 1, ..., n-1\}$ cannot be generic for $h' \circ f$. It follows that generic elements for $h' \circ f$ are mapped to $1 \in S$. Contradiction to the assumption that $1 < k_0$.

If $k_0 = 1$ then

$$A^1 := \{a \in A \mid f(a) \neq 1\}$$

is a proper subgroup and f is constant on $A \setminus A^1$. Applying the same argument to A^1 we obtain a filtration (A^n) such that f is constant on $\overline{A}^n$ for all $n \in \mathbb{N}$. □

3.2. Reduction of the rank. —

LEMMA 3.2.1. — *Let A be an abelian group and $f \in \mathcal{F}(A, S)$. If for all subgroups $B \subset A$ with* $\mathrm{rk}(B) \leq 3$ *the restriction $f_B \in \mathcal{AF}(B, S)$ then $f \in \mathcal{AF}(A, S)$.*

Proof. — By definition, it suffices to consider finitely generated A. Invariance of f is clear. By Lemmas 3.1.1 and 3.1.2, it suffices to assume $S = \mathbb{Z}/2$. As in Remark 2.4.1 and in the proof of Lemma 3.1.1, we can define a partial relation $\tilde{>}$ on A, which by assumption and by Lemma 2.4.2 extends to an order on subgroups of rk ≤ 3 (see Lemma 2.4.2). We will denote the induced order on subgroups $C \subset A$ by $>_C$. We need to show that this order extends compatibly to A. Notice that for $C \subset D \subset A$ the order $>_D$ is stronger than the order $>_C$.

We have a decomposition of $A = A_a \cup A_b$ (preimages of $0, 1$). As in the proof of Lemma 2.5.6, we will use the letter a (resp. b) for elements in A_a (resp. A_b). The proof is subdivided into a sequence of Sublemmas.

SUBLEMMA 3.2.2. — *(Correctness) There are no a, a' and b, b' such that*

$$a \tilde{>} b \tilde{>} a' \tilde{>} b' \tilde{>} a.$$

Proof. — Introducing the subgroups

$$\begin{aligned} C &:= \langle a, b, a' \rangle \\ D &:= \langle a', b', a \rangle \\ M &:= \langle b, b', a + a' \rangle \\ N &:= \langle b', a' + b, a \rangle \end{aligned}$$

we obtain

$$a >_C b + a' >_C a' \text{ and } a' >_D b' + a >_D a.$$

It follows that

$$f(a+a') = f(a+a'+b) = f(a), \tag{3.1}$$

$$f(a'+a+b') = f(a'), \tag{3.2}$$

$$f(b+a') = f(b) \tag{3.3}$$

By Equations 3.1 and 3.2, neither b nor b' can be generic in M. Thus $a+a' >_M b, b'$ and

$$f(a+a'+b+b') = f(a+a') = f(a). \tag{3.4}$$

On the other hand, in N, the element a is not generic: $f(a+b') = f(b')$. Since $f(b+a') = f(b)$ and $f(b+a'+a) = f(a)$ (by 3.1) and since a is not generic, the element $b+a'$ cannot be generic. It follows that b' is generic in N and

$$f(b' + (b+a') + a) = f(b'),$$

contradiction to 3.4. □

The sublemma implies that we can extend $\tilde{>}$ to a relation $>$ on the whole A.

Sublemma 3.2.3. — *(Transitivity) If $x > y > z$ then $x > z$.*

Proof. — We have to consider 4 cases:

Case 1. $a > b > a'$. Transitivity follows from the definition.

Case 2. $a > b > b'$. Let a' be the separator. Assume $f(b'+a) = f(b)$. Then $b' > a$ and we have a contradiction to Sublemma 3.2.2. Thus $f(b'+a) = f(a)$ and $a > b'$.

Case 3. $a > a' > b'$. Let b be the separator. Again, if $b' > a$ we get a contradiction to Sublemma 3.2.2.

Case 4. $a > a' > a''$. Denote by b the separator between a and a'. We have $a > b > a' > a''$. Applying case 2 (with a's and b's interchanged), we get $b > a''$. Thus $a > a''$ (by the definition). □

Sublemma 3.2.4. — *(Additivity) If $x, y, z \in A$ and $x > y$ and $x > z$ then $x > y + z$.*

Proof. — We are looking at the following cases:

Case 1. $a > b, b'$. Neither b nor b' can be generic in the subgroup $\langle a, b, b'\rangle$. Thus a is generic and the claim follows.

Case 2. $a > b$ and $a > b' > a'$. Then

$$f(b' \pm a') = f(b') \text{ and } f(a + b' + a') = f(a).$$

Case 2.1. $f(b + a') = f(b)$. Consider the subgroup $\langle a, b + a', b\rangle$. The element b cannot be generic since $f(a + b) = f(a)$. It follows that $b + a'$ cannot be generic since

$$f(b + a' - b) = f(a') \neq f(b + a').$$

Thus a is the generic element and $a > b + a'$.

Case 2.2. $f(b + b') = f(b)$. Apply Case 1: $a > (b + b') - (b' - a')$.

Case 2.3. $f(b + a') = f(a)$, $f(b + b') = f(a)$. Consider the subgroup $\langle b, b', a'\rangle$. The element a' cannot be generic since $f(a' + b') = f(b')$. The element b cannot be generic since $f(b + a') = f(a)$. Finally, b' cannot be generic since $f(b + b') = f(a)$. Contradiction.

Case 3. $a > b' > a'$ and $a > b'' >_f a''$.

Case 3.1. $f(b' + a'') = f(b')$. Consider $\langle a, b', a''\rangle$: b' is nongeneric, therefore, a'' is also nongeneric - it follows that a is generic and that $a > b' + a''$. Consider $\langle a, b', a'\rangle$: again a is generic and b', a' are nongeneric, thus $a > (a' - b')$. Now we can apply case 1 or 2, depending on the value of $f(a' - b')$.

Case 3.2. By symmetry, we can assume that both

$$f(b' + a'') = f(a' + b'') = f(a).$$

Combining with the assumption of Case 3 we obtain

$$a'' > b' > a' > b'' > a'',$$

contradiction to Sublemma 3.2.2. □

Now we apply Lemma 2.4.6 to conclude the proof of Lemma 3.2.1. □

3.3. The exceptional case. —

LEMMA 3.3.1. — *Let $A = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and $f \in \mathcal{F}(A, S)$ be a function such that*

- *for every rank 2 sublattice $C \subset A$ we have $f_C \in \mathcal{AF}(C, S)$;*

- *$f \notin \mathcal{AF}(A, S)$;*
- *f does not have a special basis.*

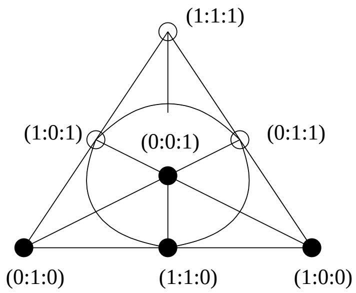
Then f takes exactly two values on A. Moreover, f is either as in Example 2.7.1 or as in Example 2.7.2.

Proof. — We can assume that $f : A \to S$ is surjective. By Lemma 3.1.2, there exists an

$$h : S \to \mathbb{Z}/2 = \{\circ, \bullet\}$$

such that $h \circ f \notin \mathcal{AF}(A, \mathbb{Z}/2)$. By Proposition 2.8.4, $h \circ f$ is either of the first or the second type.

Let us consider the first case. By Example 2.7.1, $h \circ f$ is given by



Let L be a line (=rank 2 lattice in C) reducing to the line through $(0:0:1)$ and $(0:1:1)$ modulo 2 and $P \in L$ be a point in

$$(h \circ f)^{-1}((0:0:1)).$$

By Corollary 2.5.4, an AF-function takes only two values on a lattice of rank 2. Since $f_L \in \mathcal{AF}(L, S)$ the value $f(P)$ is generic for L. Thus for every point Q on L in

$$(h \circ f)^{-1}((0:1:0)) \text{ and } (h \circ f)^{-1}(0:0:1)$$

we have

$$f(P) = f(Q).$$

Take any line L' which modulo 2 passes through $(0:0:1)$ and $(1:1:0)$. The value of f on the point of intersection $L \cap L'$ is $f(P)$. Since $f(P')$ is the generic value for L' for every point P' on L' in the preimage of $h \circ f((0:0:1))$ we have $f(P') = f(P)$. Moreover, $f(P') = f(Q')$ for every Q' in the preimage $h \circ f((1:1:0))$ in L'. Therefore, for *any* line L (resp. L'') such that the reduction modulo 2 passes through $(0:0:1)$ and $(0:1:1)$ (resp. $(0:0:1)$ and $(1:1:0)$) the generic value is $f(P)$. In particular, for every point R in the preimage of $(1:1:0)$ we have $f(R) = f(P)$.

Now consider the preimages of the points

$$(1:0:1),\ (1:1:1),\ (0:1:1).$$

Every one of those is generic for some rank 2 sublattice in A. Since these lattices intersect, we can apply the same reasoning as above. It follows that f can take only two values on A and, moreover, that f is induced from $A/2A$.

The second case is treated similarly. □

3.4. Checking the AF-property. — We summarize the discussion of the previous sections:

PROPOSITION 3.4.1. — *Let $A \in \mathcal{A}_q$, S a set and $f \in \mathcal{F}(A, S)$. Assume that $q > 2$ and that for all subgroups $C \subset A$ of rank ≤ 2 one has $f_C \in \mathcal{AF}(C, S)$. Then $f \in \mathcal{AF}(A, S)$.*

Proof. — By Lemma 3.2.1 it suffices to consider the case when $\mathrm{rk}(A) = 3$. Assume that $f \notin \mathcal{AF}(A, S)$. By Lemma 3.1.2, there exists a map $h : S \to \mathbb{Z}/2$ such that $h \circ f \notin \mathcal{AF}(A, \mathbb{Z}/2)$. By Lemma 2.6.1, there exists a subgroup $C \subset A$ of rank 2 such that $h \circ f_C \notin \mathcal{AF}(C, \mathbb{Z}/2)$. In particular, $f_C \notin \mathcal{AF}(C, S)$. □

PROPOSITION 3.4.2. — *Let k be a field of $\mathrm{char}(k) = 0$ and K/k an extension. Let S be a ring such that $2s \neq 0$ for all $s \in S$. Assume that $f \in \mathcal{LF}(K, S)$ and that for all $\mathbb{Z}$-submodules $C \subset K$ of rank ≤ 2 one has $f_C \in \mathcal{AF}(C, S)$. Then $f \in \mathcal{AF}(K, S)$.*

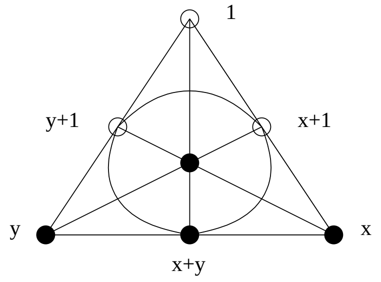
Proof. — Assume otherwise. Then, by Lemma 3.2.1, there exists a submodule $C \subset K$ of rank 3 such that $f_C \notin \mathcal{AF}(C, S)$. Moreover, there exists a map $h : S \to \mathbb{Z}/2$ such that $h \circ f_C \notin \mathcal{AF}(C, \mathbb{Z}/2)$. By Proposition 2.8.1, we can assume that C does not have a special basis (by restricting to a proper subgroup). Then, by Proposition 2.8.4, $h \circ f_C$ has the form described in Example 2.7.1 or 2.7.2. In both cases, f_C itself takes exactly two values.

Up to addition of a constant (and shifting C using the logarithmic property, if necessary), we can assume that f takes the values 0 and s (for some $s \in S \setminus 0$) and that

$$f(1) = 0.$$

Moreover, (in both cases!) there exist elements $x, y \in C \subset K$ such that

$$f(x) = f(y) = f(x+y) = f(x+y+1) = s,$$
$$f(x+1) = f(y+1) = 0.$$



Consider the sublattices in K

$$\begin{array}{rcl} D & := & \langle xy, y, 1\rangle; \\ E & := & \langle xy, 1, x+y\rangle. \end{array}$$

Using the logarithmic property we find that

$$f(xy) = 2s,\ f(y) = s,\ f(1) = 0.$$

By Lemma 3.3.1, f_D is an AF-function. Using the transitivity of the induced order on D we see that

$$1 >_D y >_D xy.$$

In particular, since $f(1) \neq f(xy)$, for any subgroup of K containing 1 and xy we have $1 > xy$. Similarly, on E the function f_E also takes 3 values. Therefore (by Lemma 3.3.1), $f_E \in \mathcal{AF}(E, S)$ and the induced order gives

$$x + y >_E 1 >_E xy$$

(by transitivity of the order relation on E). It follows that

$$f(x + y + xy + 1) = f(x + y) = s.$$

On the other hand, (using the logarithmic property),

$$f(x+y+xy+1) = f((x+1)(y+1)) = f(x+1) + f(y+1) = 0.$$

Contradiction. $\square$

4. AF-functions and geometry

ASSUMPTION 4.0.3. — *Throughout R is $\mathbb{Q}$, $\mathbb{Z}_p$ or $\mathbb{Z}/p$.*

4.1. Affine geometry. — Let k be a field and V a (possibly infinite dimensional) vector space over k, with an embedding of k as $k \cdot 1$. For every pair

$$f_1, f_2 \in \mathcal{F}(\mathbb{P}(V), R)$$

we have a map

$$\begin{array}{rccc} \varphi = \varphi_{f_1,f_2} : & \mathbb{P}(V) & \to & \mathbb{A}^2(R) \\ & v & \mapsto & (f_1(v), f_2(v)). \end{array}$$

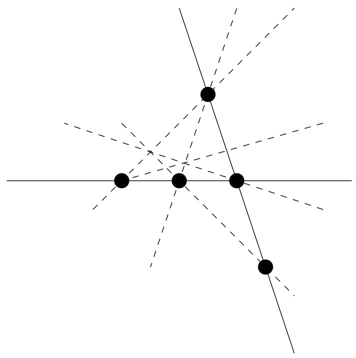
REMARK 4.1.1. — If f_1, f_2 form a c-pair (see Definition 2.2.6) then the image of every line in $\mathbb{P}(V)$ under φ_{f_1,f_2} is contained in a line in $\mathbb{A}^2(R)$.

PROPOSITION 4.1.2. — *If $f_1, f_2 \in \mathcal{F}(\mathbb{P}(V), R)$ form a c-pair then for every 3-dimensional k-vector space $V' \subset V$ there exists an affine line $\mathbb{L}_{V'} \subset \mathbb{A}^2$ and a point $d_{V'} \in \mathbb{A}^2(R)$ such that*

$$\varphi(\mathbb{P}(V')) \subset d_{V'} \cup \mathbb{L}_{V'}(R).$$

Proof. — Assume that $\varphi(\mathbb{P}(V'))$ contains 4 distinct points $\overline{p}_1, ..., \overline{p}_4$. Denote by $\overline{\ell}_{ij}$ the line through $\overline{p}_i$ and $\overline{p}_j$. Then the lines intersect in $\mathbb{A}^2(R)$ and the intersection point of these lines is contained in $\varphi(\mathbb{P}(V'))$. Indeed, denote by $\ell_{ij} \subset \mathbb{P}(V')$ the line passing through a pair of points p_i, p_j contained in $\varphi^{-1}(\overline{p}_i)$ (resp. $\varphi^{-1}(\overline{p}_j)$) and assume, for example, that $\overline{\ell}_{12} \cap \overline{\ell}_{34} = \emptyset$. The lines ℓ_{12} and ℓ_{34} intersect (or coincide) in $\mathbb{P}(V')$. The point of intersection is contained in the image. Thus the lines $\overline{\ell}_{12}, \overline{\ell}_{34}$ intersect and the image of $\mathbb{P}(V)$ contains the intersection point.

Now we can assume that $\varphi(\mathbb{P}(V'))$ contains at least 5 points $\overline{p}_j$, such that $\overline{p}_1, \overline{p}_2, \overline{p}_3 \in \overline{\ell} \in \mathbb{A}^2_R$ and $\overline{p}_4, \overline{p}_5 \notin \overline{\ell}$ and such that the line through $\overline{p}_4, \overline{p}_5$ intersects $\overline{\ell}$ in $\overline{p}_3$.



The goal is to show that drawing lines in $\mathbb{A}^2$ through the points of intersections of the existing lines one can produce 2 new pairs of points in $\mathbb{A}^2(R)$ such that the corresponding lines are parallel, leading to a contradiction.

First we assume that R is $\mathbb{Q}$ or $\mathbb{Z}/p$. Compactify $\mathbb{A}^2$ to $\mathbb{P}^2$ by adding a line ℓ_∞. After a projective transformation of $\mathbb{P}^2$ (with coefficients in R) we can assume that the points are given by

$$(1:0:1),\ (0:0:1),\ (0:1:1),\ (1:0:0),\ (0:1:0).$$

We use standard affine coordinates (z_1, z_2) on $\mathbb{A}^2$ and corresponding projective coordinates $(z_1 : z_2)$ on ℓ_∞. The set $\varphi(\mathbb{P}(V'))$ has the following properties:

- if $(z_1, z_2) \in \varphi(\mathbb{P}(V'))$ then $(z_1 : z_2) \in \ell_\infty$ is also contained in $\varphi(\mathbb{P}(V'))$.
- if $(z_1, z_2) \in \varphi(\mathbb{P}(V'))$ then $(z_1, 0)$, $(0, z_2)$ are also in $\varphi(\mathbb{P}(V'))$.
- if $(z_1, z_2) \in \varphi(\mathbb{P}(V'))$ then $(z_1 : z_2 : 0) \in \varphi(\mathbb{P}(V'))$ and $(z_1/z_2, 0) \in \varphi(\mathbb{P}(V'))$.
- if $(z_1, 0)$, $(z_2, 0) \in \varphi(\mathbb{P}(V'))$ then $(z_1 + z_2, 0) \in \varphi(\mathbb{P}(V'))$.
- if $(z, 0) \in \varphi(\mathbb{P}(V'))$ then $(0, z) \in \varphi(\mathbb{P}(V'))$.

To check the listed properties it suffices to compute the coordinates of the points of intersection of appropriate lines. For example, the first property follows from the fact that $(z_1 : z_2) \in \ell_\infty$ is the intersection of ℓ_∞ with the line through $(0, 0)$ and $(z_1, z_2) \in \mathbb{A}^2$. For the second, observe that $(z_1 : 0 : 1)$ can be obtained as the intersection of the line through $(0 : 0 : 1)$ and $(1 : 0 : 1)$ with the line through $(z_1 : z_2 : 1)$ and $(0 : 1 : 0)$, etc.

In particular, $\varphi(\mathbb{P}(V'))$ contains a subset of points (z_1, z_2), where z_1, z_2 are generated from coordinates of points in $\varphi(\mathbb{P}(V')) \cap (\mathbb{P}^2 \setminus \ell_\infty)$ by the above procedures. For $R = \mathbb{Q}$ or $R = \mathbb{Z}/p$, the set $\varphi(\mathbb{P}(V'))$ contains $\mathbb{A}^2(R) \cup \ell_\infty(R) = \mathbb{P}^2(R)$. In particular, one can find two lines in $\mathbb{P}(V')$ such that their images don't intersect in $\mathbb{A}^2$, contradiction.

Now we show how to extend this argument to $R = \mathbb{Z}_p$. As before, we assume that $\varphi(\mathbb{P}(V'))$ contains 5 points as in the picture above. One can choose a coordinate system such that $\varphi(\mathbb{P}(V'))$ contains the points $(1,0), (0,0), (0,1), (z_1, 0)$ and $(0, z_2)$ with $z_1, z_2 \in \mathbb{Q}_p$. Then it also contains some point (z_1', z_2') with *nonzero* coordinates $z_1', z_2' \in \mathbb{Q}_p$. The (projective) transformation $\mathcal{T}$ moving

$$(1,0), (0,0), (0,1) \ \mapsto (\infty, 0), (0,0), (0,\infty)$$

is given by

$$(w_1, w_2) = (\frac{z_1}{1-(z_1+z_2)}, \frac{z_2}{1-(z_1+z_2)})$$

and its inverse $\mathcal{T}^{-1}$ by

$$(z_1, z_2) = (\frac{w_1}{1+w_1+w_2}, \frac{w_2}{1+w_1+w_2}).$$

Apply the reasoning of the first part to $(w_1', w_2') := \mathcal{T}((z_1', z_2'))$. First we find that $\varphi(\mathbb{P}(V'))$ contains the points $(w_1', 0), (0, w_2')$, then that it contains all points of the form $(w_1'/2^{m_1}, w_2'/2^{m_2})$ for some $m_1, m_2 \in \mathbb{N}$, then that it contains all points $(r_1 w_1'/2^{m_1}, r_2 w_2'/2^{m_2})$ with $r_1, r_2 \in \mathbb{N}$, and, finally, that it contains all points with coordinates $(e_1 w_1', e_2 w_2')$ for arbitrary $e_1, e_2 \in \mathbb{Q}$.

To arrive at a contradiction it suffices to produce a pair of rational numbers e_1, e_2 such that

$$\mathcal{T}^{-1}((e_1 w_1', e_2 w_2')) \notin \mathbb{Z}_p \oplus \mathbb{Z}_p.$$

Clearly, (for any $w_1', w_2' \in \mathbb{Q}_p$) we can find $e_1, e_2 \in \mathbb{Q}$ such that

$$\frac{e_1 w_1'}{1 + e_1 w_1' + e_2 w_2'} \notin \mathbb{Z}_p.$$

This concludes the proof. $\square$

REMARK 4.1.3. — Proposition 4.1.2 is wrong for $R = \mathbb{Q}_p$.

4.2. Projective geometry. —

PROPOSITION 4.2.1. — *Let k be any field and V a 3-dimensional vector space over k. Assume that $f_1, f_2 \in \mathcal{F}(\mathbb{P}(V), \mathbb{Z}/2)$ are such that*

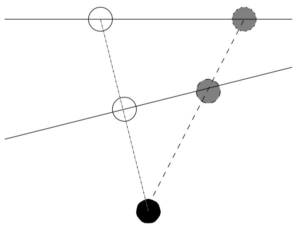
$$\varphi(\mathbb{P}(V)) \subset \{\overline{p}_{12}, \overline{p}_{13}, \overline{p}_{23}\} \subset \mathbb{A}^2(\mathbb{Z}/2),$$

where $\overline{p}_{12} = (0,0), \overline{p}_{13} = (1,0)$ and $\overline{p}_{23} = (0,1)$. Assume further that for all 2-dimensional vector spaces $V' \subset V$ the image $\varphi(\mathbb{P}(V'))$ is contained in at most two of these points. Then at least one of the functions f_1, f_2 or $f_3 = f_1 + f_2$ is an AF-function on every 2-dimensional subspace $V' \subset V$.

NOTATIONS 4.2.2. — Denote by $P_{ij} \subset \mathbb{P}(V)$ the preimage of $\overline{p}_{ij}$. Let T_i be the set of lines $t \subset \mathbb{P}(V)$ such that $\varphi(t) \subset \{\overline{p}_{ij}, \overline{p}_{ik}\}$.

The proof of Proposition 4.2.1 is subdivided into a sequence of lemmas.

LEMMA 4.2.3. — *Let t_i, t_i' be two lines in T_i. Every point p_{jk} defines a projective isomorphism between $t_i \cap P_{ij}$ and $t_i' \cap P_{ij}$.*



Proof. —

$\square$

LEMMA 4.2.4. — *If there exists a line $t_i \in T_i$ such that the number of points in $t_i \cap P_{ij}$ is ≤ 1 then either f_1, f_2 or f_3 is a GF-function on V.*

Proof. — First of all, if one of the sets P_{ij} is empty then one of the functions f_1, f_2, f_3 is the constant function, hence a flag function.

Assume that this is not the case and that there exists a line t_i such that $t_i \cap P_{ij} = \emptyset$. By assumption, there exist points of all three types. We can draw a line t_j through some points of type P_{ij} and P_{jk}. The line t_j intersects t_i in a point, which must be a point of type P_{ik}. Thus the line t_i contains points of all three types, contradiction.

Finally, assume that there exists a line t_i such that $t_i \cap P_{ij}$ consists of exactly one point. There are two possibilities: there are at least two lines of type T_j or there is exactly one line of type T_j. In the first case Lemma 4.2.3, shows that *all* lines of type T_j contain exactly one point of type P_{ij}. This means that there is only one point of type P_{ij} in $\mathbb{P}(V)$ (otherwise, we could draw a line through two of those points; this line cannot by of type T_i nor of type T_j). It follows that f_k is an AF-function on V (delta function).

Assume now that there exists exactly one line of type T_j. The complement to this line contains *only* points of type P_{ik}. It follows that f_j is an AF-function on $\mathbb{P}(V)$. □

Thus we can assume that there are at least 3 points of each type (which do not lie on a line) and that there are at least two lines of each type and that for every line $t_i \in T_i$ the set $t_i \cap P_{ij}$ has at least two elements.

Definition 4.2.5. — *We call the points $p_{ij}, p'_{ij} \in t_i$* related *if there exists a point p^0_{ij} such that the line joining p^0_{ij} and p_{ij} and the line joining p^0_{ij} and p'_{ij} are both of type T_j.*

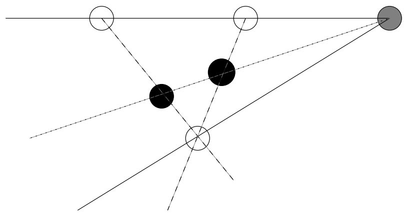
Lemma 4.2.6. — *If there exists a line $t_i \in T_i$ containing two nonrelated points $p_{ij}, p'_{ij} \in t_i$ then for every $t_j \in T_j$ passing through p_{ij} or p'_{ij} all points in $t_i \cap P_{ij}$ are related.*

Proof. — Consider this line t_i with two nonrelated points p_{ij}, p'_{ij}. Let $t_j \in T_j$ be any line passing through p_{ij}. Let p''_{ij} be an arbitrary point in $t_i \cap P_{ij}$, distinct from p_{ij}. Since p_{ij}, p'_{ij} are not related the line through p''_{ij} and p'_{ij} has to be of type T_i. It follows that all points of type P_{ij} on t_j are related through p'_{ij}. □

Lemma 4.2.7. — *Assume that $p_{ij}, p'_{ij} \in t_i$ are related. For every point $p_{ik} \in t_i$ there exists a projective automorphism $m_{ik} : t_i \to t_i$ such that*

- $m_{ik}(t_i \cap P_{ij}) = t_i \cap P_{ij}$;
- *the unique fixed point of m_{ik} is p_{ik};*
- $m_{ik}(p_{ij}) = p'_{ij}$.

Proof. — Consider the triangle spanned by $p_{ij}, p'_{ij} \subset t_i$ and p^0_{ij}. Draw a line t'_i through $p_{ik} \in t_i$ and p^0_{ij}. Pick a point p_{jk} on the line t_j through p_{ij} and p^0_{ij} and draw a line t_k through p_{jk} and p_{ik}. Denote by p'_{jk} the point of intersection of t_j with the line through p'_{ij} and p^0_{ij}. Using the points p_{jk} and p'_{jk} as centers for the projective isomorphism between the line joining t_i and the line t'_i, we obtain the projective automorphism m_{ik}.



□

Lemma 4.2.8. — *Assume that there exists a line t_i such that all points $p_{ij} \in t_i$ are related. Then all three functions f_1, f_2, f_2 are AF-functions on all lines of type T_i.*

Proof. — The function f_i is constant on lines of type T_i, hence an AF-function. The function $f_j + f_k$ is constant on (the fixed line) t_i. Let us show that f_j is an AF-function on t_i.

By Lemma 4.2.7, for any pair of related points p_{ij}, p'_{ij} and any point p_{ik} on a projective line of type t_i there exists a projective automorphism (transvection) with a single fixed point p_{ik} moving p_{ij} to p'_{ij}. Introduce coordinates on $\mathbb{P}^1$ such that $p_{ik} = (0:1)$, $p_{ij} = (1:0)$ and $p'_{ij} = (1:1)$. A unipotent lifting of the automorphism m_{ij} to $\mathrm{GL}(V) = \mathrm{GL}_2(k)$ can be written in the form

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

In this basis $f(e_1) = f(e_1 + e_2) = 1$ and $f(e_2) = 0$. Consequently, f satisfies the conditions of Lemma 2.5.6. (Notice that the points p_{ik} on t_i need not be related. This leads to the nonsymmetric shape of the functional equation 2.1.) Now we apply Lemma 2.5.6.

To conclude that f_1, f_2, f_3 are AF-functions on *all* lines of type T_i we use the projective isomorphism preserving both sets $t_i \cap P_{ij}$ and $t_i \cap P_{ik}$ (introduced in Lemma 4.2.3). □

Proof. — (of Proposition 4.2.1) We may assume that we are not in the situation of Lemma 4.2.4. If for all $i = 1, 2, 3$ and all lines t_i of type T_i

all points in $t_i \cap P_{ij}$ or all points in $t_i \cap P_{ik}$ are related then all three functions f_1, f_2, f_3 are AF-functions on t_i.

Assume there is a t_i and two points $p_{ij}, p'_{ij} \in t_i \cap P_{ij}$ which are not related. By Lemma 4.2.6 and Lemma 4.2.8, f_1, f_2, f_3 are AF-functions on all lines of type T_j. There are two cases: there exist two nonrelated points in $t_i \cap P_{ik}$ or not. In the first case, f_1, f_2, f_3 are AF-functions on all lines of type T_k. In the second case f_1, f_2, f_3 are AF-functions on lines of type T_i. If, for example, all three functions are AF-functions on lines of type T_j and T_k then the function f_i (being constant on lines of type T_i) is an AF-function on *all* lines in $\mathbb{P}(V)$. This concludes the proof. □

4.3. Logarithmic functions. — We keep the assumptions of 4.0.3.

PROPOSITION 4.3.1. — *Let k be a field of characteristic $\neq 2$ and K/k an extension. Assume that $f_1, f_2 \in \mathcal{LF}(K, R)$ form a c-pair (see 2.2.6 for the definition). Assume that the linear space $\langle f_1, f_2\rangle_R$ does not contain a (nonzero) AF-function. Then there exists a 3-dimensional $V \subset K$ such that for* every *(nonzero) $f' \in \langle f_{1,V}, f_{2,V}\rangle_R$ we have $f' \notin \mathcal{AF}(V, R)$.*

Proof. — For $\mathrm{char}(k) > 2$ we use Proposition 3.4.1 and for $\mathrm{char}(k) = 0$ Proposition 3.4.2. Since $f_1 \notin \mathcal{AF}(K, R)$ there exists a 2-dimensional subspace $V' \subset K$ such that $f_{1,V'} \notin \mathcal{AF}(V', R)$. Since

$$\mathrm{rk}\, \langle f_{1,V'}, f_{2,V'}, 1\rangle \leq 2$$

we have $f_{2,V'} - \mu_1 f_{1,V'} = \mu_2$ (for some $\mu_1, \mu_2 \in R$). Since

$$f_2 - \mu_1 f_1 \notin \mathcal{AF}(K, R),$$

by Section 3.4, there exists a 2-dimensional W' such that

$$f_{2,W'} - \mu_1 f_{1,W'} \notin \mathcal{AF}(W', R).$$

Choose some k-basis in $V' = \langle x_1, x_2\rangle$ and $W' = \langle y_1, y_2\rangle$ (with $x_j, y_j \in K^*$). Let $V = \langle x_1, x_2, y_2 y_1^{-1} x_1\rangle$. Then for every pair of $(\lambda_1, \lambda_2) \neq (0, 0)$

$$\lambda_1 f_1 + \lambda_2 (f_2 - \mu_1 f_1) \notin \mathcal{AF}(V, R).$$

Indeed, for pairs with $\lambda_1 \neq 0$ consider the restriction to V'. For pairs $(0, \lambda_2)$ with $\lambda_2 \neq 0$ consider the restriction to (a shift of) W' and use the invariance and the logarithmic property of f. □

LEMMA 4.3.2. — *Let k be a field of characteristic $\neq 2$, K/k an extension and $V \subset K$ a 3-dimensional vector space over k. Consider a c-pair $f_1, f_2 \in \mathcal{F}(\mathbb{P}(V), R)$. Assume that there are no nonzero AF-functions $f \in \langle f_1, f_2 \rangle_F$. Then there exist nonconstant (and nonproportional) functions $f_1', f_2' \in \langle f_1, f_2, 1 \rangle_R$ and a map $h' : R \to \mathbb{Z}/2$ such that neither of the three functions*

$$\begin{gathered} h' \circ f_1', \\ h' \circ f_2', \\ h' \circ f_1' + h' \circ f_2' \end{gathered}$$

is an AF-function on V.

Proof. — By 4.1.2, we know that

$$\varphi_{f_1,f_2}(\mathbb{P}(V)) \subset d_V \cup \mathbb{L}_V.$$

After a linear change of coordinates (over R) we can assume that

$$\varphi_{\tilde{f}_1,\tilde{f}_2}(\mathbb{P}(V)) = (0,1) \cup \{x - \text{axis}\},$$

where

$$\begin{aligned} \tilde{f}_1 &= \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3, \\ \tilde{f}_2 &= \mu_1 f_1 + \mu_2 f_2 + \mu_3 \end{aligned}$$

and

- $\tilde{f}_1(0) = \tilde{f}_2(0) = 0$;
- $\tilde{f}_2$ takes only two values;
- $\tilde{f}_2(v) = 0$ if $\tilde{f}_1(v) \neq 0$;
- $\tilde{f}_1(v) = 0$ if $\tilde{f}_2(v) \neq 0$.

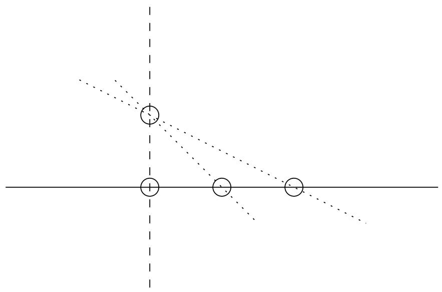
Let h be such that $h \circ \tilde{f}_1 \notin \mathcal{AF}(V, Z/2)$. We can assume that $h(0) = 0$. Let $v_1 \in V$ be such that $h \circ \tilde{f}_1(v_1) \neq 0$. After rescaling (and a corresponding rescaling of h), we can assume that $\tilde{f}_1(v_1) = 1$ and that $h(1) = 1$. Since $\tilde{f}_2$ takes only the values $0, 1$ we have

$$h \circ \tilde{f}_2 = \tilde{f}_2 \notin \mathcal{AF}(V, Z/2).$$

Notice that the last two properties imply that

$$h \circ (\tilde{f}_1 + \tilde{f}_2) = h \circ \tilde{f}_1 + h \circ \tilde{f}_2. \tag{4.1}$$

The lines in $\mathbb{P}(V)$ can be subdivided into 3 classes according to their image under $\varphi_{\tilde{f}_1,\tilde{f}_2}$.



The function $\tilde{f}_1$ is obviously constant on lines in $\mathbb{P}(V)$ whose image is contained in the y-axis, $\tilde{f}_2$ is constant on lines mapping to the x-axis and $\mu\tilde{f}_1 + \tilde{f}_2$ (with $\mu \neq 0$) is constant on the lines mapping to $\mu x + y = 1$. We call the lines of the first type T_1-lines, the lines of the second type T_2-lines and all other T_3-lines.

Notice that $\tilde{f}_3$ restricted to the T_i-lines coincides with $\tilde{f}_i$ for $i = 1, 2$. By assumption, $\tilde{f}_3 \notin \mathcal{AF}(V, R)$, and by the results in Section 3.4, there exists a line $\ell_3 \subset \mathbb{P}(V)$ such that $\tilde{f}_3 \notin \mathcal{AF}(\ell_3, R)$. If ℓ_3 is of type T_1 or T_2 then h as above solves our problem. Thus we can assume that all three functions $\tilde{f}_1, \tilde{f}_2$ and $\tilde{f}_3$ are AF-functions on all lines of type T_1 and T_2. Consider a line ℓ_3 of type T_3 such that $\tilde{f}_3 \notin \mathcal{AF}(\ell_3, R)$. Then $\varphi(\ell_3) \subset \mu x + y = 1$, (with $\mu \neq 1$). That is, on ℓ_3 we have

$$\begin{array}{rcl} \mu\tilde{f}_1 + \tilde{f}_2 & = & 1 \\ \tilde{f}_1 + \tilde{f}_2 & = & \tilde{f}_3 \end{array}$$

It follows that all 3 functions $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3 \notin \mathcal{AF}(\ell_3, R)$. Now we change coordinates again, making ℓ_3 the new coordinate axis (in addition to the x-axis). We put

$$\begin{array}{rcl} f_1' & = & \mu\tilde{f}_1 + \tilde{f}_2 - 1 \\ f_2' & = & -\tilde{f}_2 \\ f_3' & = & f_1' + f_2'. \end{array}$$

Now we can apply the argument above: find an h' such that

$$h' \circ f_1' \notin \mathcal{AF}(V, \mathbb{Z}/2);$$

(after rescaling, if necessary) one can assume that

$$h' \circ f_2' = f_2',$$

(with $f'_2 = \lambda \tilde{f}_2$ for some nonzero λ). Since $(f'_1 + f'_2)$ restricted to $\ell_3 = \{f'_1 = 0\}$ is equal to f'_2, which is not AF on ℓ_3, (by our assumption that $\tilde{f}_2$ is not AF on ℓ_3), we conclude that

$$h' \circ (f'_1 + f'_2) \notin \mathcal{AF}(\ell_3, \mathbb{Z}/2).$$

By the same reasoning as above (see 4.1) we have

$$h' \circ (f'_1 + f'_2) = h' \circ f'_1 + h \circ f'_2.$$

This finishes the proof. □

4.4. Existence of AF-functions. —

PROPOSITION 4.4.1. — *Let K/k be an extension of fields. Assume that $f_1, f_2 \in \mathcal{LF}(K, R)$ form a c-pair. Then $\langle f_1, f_2 \rangle_R$ contains an AF-function.*

Proof. — We can assume that $f_1, f_2, 1$ are linearly independent on $\mathbb{P}(K)$. (Otherwise some linear combination is constant, hence an AF-function.)

Assume that $\langle f_1, f_2 \rangle_R$ does not contain an AF-function. By Proposition 4.3.1 combined with Lemma 4.3.2, there exist a 3-dimensional $V \subset K$, functions $f'_1, f'_2 \in \mathcal{F}(\mathbb{P}(V), R)$ and a map $h : R \to \mathbb{Z}/2$ such that

$$h \circ f'_1, h \circ f'_2, h \circ f'_3 = h \circ f'_1 + h \circ f'_2 \notin \mathcal{AF}(V, \mathbb{Z}/2).$$

These functions satisfy the assumptions of Proposition 4.2.1. We obtain a contradiction to its statement. □

5. Galois theory

5.1. Groups. — Let G be a (topological) group with unit $0 = 0_G$ and $g, g' \in G$. Denote by $[g, g']$ their commutator and by

$$G = G^{(0)} \supset G^{(1)} \supset ...$$

the lower central series: $G^{(i)}$ is the (closed) subgroup generated by $[g_i, g_0]$, where $g_i \in G^{(i)}$ and $g_0 \in G^{(0)}$. Denote by $G^a = G/G^{(1)}$ the abelianization of G, by $G^c = G/G^{(2)}$ the second quotient - it is a central extension of G^a - and by $G^{1,2} = G^{(1)}/G^{(2)}$. Let

$$\psi_c : G \to G^c, \qquad \psi_a : G \to G^a, \qquad \psi_{ca} : G^c \to G^a$$

be the quotient homomorphisms (we have $\psi_a = \psi_{ac} \circ \psi_c$).

LEMMA 5.1.1. — *Let G be a (profinite) group and $\psi : G \to A$ a (continuous) surjective homomorphism onto a finite group A. Assume that $\alpha \in H^2(A, \mathbb{Z}/p^n)$ is a class such that*

$$\psi^*(\alpha) = 0 \in H^2(G, \mathbb{Z}/p^n).$$

Then there exists a (continuous) homomorphism $\tilde{\psi} : G^c \to A$ such that $\tilde{\psi} \circ \psi_c = \psi$ and

$$\tilde{\psi}^*(\alpha) = 0 \in H^2(G^c, \mathbb{Z}/p^n).$$

We write $G(p^n)$ for the subgroup of G generated by g^{p^n} with $g \in G$ (for abelian groups and their central extensions, the product of p^n-th powers is a p^n-th power). For any profinite group G we denote by G_p its maximal pro-p-quotient. Notice that for any profinite group G we have $(G^a)_p = (G_p)^a$ and $(G^c)_p = (G_p)^c$.

5.2. Fields. —

ASSUMPTION 5.2.1. — *Fix a prime p. We assume that* $\mathrm{char}(k) \neq p$ *and that k does not admit finite separable extensions of degree divisible by p.*

Let K be a field over k. It has a structure of a vector space over k and therefore, K^*/k^* a structure of a projective space over k, though infinite-dimensional. We continue to denote this space by $\mathbb{P}(K)$ and by $\mathcal{F}(\mathbb{P}(K), \mathbb{Z}_p)$ the space of $\mathbb{Z}_p$-valued functions on $\mathbb{P}(K)$.

In Section 2.2 we defined the set $\mathcal{LF}(K, \mathbb{Z}_p)$. We now consider the topological space $\mathcal{LF}^{\mathrm{top}}(K, \mathbb{Z}_p)$. As a set it coincides with $\mathcal{LF}(K, \mathbb{Z}_p)$. The basis of the topology is given by $U_{E^*,n}$, where E^* is a finitely generated subgroup of K^*/k^* and $n \in \mathbb{N}$ - a function f is in $U_{E^*,n}$ if f is equal to 0 modulo p^n on E^*.

NOTATIONS 5.2.2. — For any k-vector space $V \subset K$ (not necessarily closed under multiplication in K) and $f \in \mathcal{F}(\mathbb{P}(K), \mathbb{Z}_p)$ we denote by f_V the restriction of f to $\mathbb{P}(V) = (V \setminus 0)/k^*$ (sometimes we will denote by the same symbol the restriction of f to V). For a finite set of functions $f_j \in \mathcal{F}(\mathbb{P}(K), \mathbb{Z}_p)$ we denote by $\langle f_1, ..., f_n \rangle$ the $\mathbb{Q}_p$-vector space they span in $\mathcal{F}(\mathbb{P}(K), \mathbb{Z}_p)_{\mathbb{Q}_p}$.

5.3. Galois groups. — Let K be a field as in Section 5.2. Denote by G_K the Galois group of a separable closure of K. It is a profinite compact topological group (we refer to [**11**] for basic facts concerning profinite groups). In general, the group G_K has a rather complicated structure. We will be interested in

$$\mathrm{G}_{K/k} := \mathrm{Ker}(\mathrm{G}_K \to \mathrm{G}_k),$$

more precisely, in the pro-p-group $\Gamma^c := (\mathrm{G}^c_{K/k})_p$ and its abelianization $\Gamma^a := (\mathrm{G}^a_{K/k})_p$. The group Γ^a is a torsion free abelian pro-p-group (by our assumptions on k).

Lemma 5.3.1. — *One has a (noncanonical) isomorphism of topological groups*

$$\Gamma^a \simeq \mathcal{LF}^{\mathrm{top}}(K, \mathbb{Z}_p).$$

Proof. — Since K contains all p-power roots of 1 we can choose an identification between $\mathbb{Z}_p$ and $\mathbb{Z}_p(1)$. Then, by Kummer theory, we have a nondegenerate pairing

$$\Gamma^a/\Gamma^a(p^n) \times K^*/(K^*)^{p^n} \to \mathbb{Z}/p^n,$$

given by

$$(\sigma, \kappa) \mapsto \sigma(\kappa)/\kappa$$

for $\sigma \in \Gamma^a/\Gamma^a(p^n)$ and $\kappa \in K^*$. We derive an isomorphism (of topological groups)

$$\Gamma^a = \mathrm{Hom}(\hat{K}^*, \mathbb{Z}_p)$$

(where $\hat{K}^*$ is the completion of K^* with respect to subgroups of p-power index). Moreover, every such homomorphism is trivial on k^*, since k^* does not admit finite extensions of degree divisible by p, by assumption. Thus, in our case, the latter group is isomorphic to $\mathcal{LF}^{\mathrm{top}}(K, \mathbb{Z}_p)$, (since K^* is dense in $\hat{K}^*$).

More explicitely, two elements of the Galois group Γ^a coincide if for all n their actions on all cyclic p^n-degree extensions of K coincide. Thus the resulting map

$$\Gamma^a/\Gamma^a(p^n) \to \mathcal{LF}^{\mathrm{top}}(K, \mathbb{Z}/p^n)$$

is a monomorphism.

Conversely, every element of $\mathcal{LF}^{\mathrm{top}}(K, \mathbb{Z}_p)$ defines an element of Γ^a. Any homomorphism $\chi : K^*/k^* \to \mathbb{Z}_p$ defines a compatible set of elements of $\Gamma^a/\Gamma^a(p^n)$ for all abelian extensions of K of degree p^n. Thus Γ^a and $\mathcal{LF}^{\mathrm{top}}(K, \mathbb{Z}_p)$ are isomorphic as topological groups.

(We use the fact that the group K^*/k^* has no torsion, by assumptions on k. Therefore, the map

$$\mathcal{LF}^{\mathrm{top}}(K, \mathbb{Z}/p^{n+1}) \to \mathcal{LF}^{\mathrm{top}}(K, \mathbb{Z}/p^n)$$

corresponding to the projection $\mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^n$ is surjective and we obtain an isomorphism between projective limits.) □

We consider K as a vector space over k, with a canonical embedding of k as a 1-dimensional subspace $k \cdot 1$. Every finite-dimensional subspace $V \subset K$ which contains k (as a subspace) defines a subfield K_V of K (generated by elements of a basis of V over k). Denote by G_V the Galois group of the (separable) closure of K_V. We have canonical maps

$$\Gamma^c_K \to \Gamma^c_V \text{ and } \Gamma^a_K \to \Gamma^a_V.$$

If V is 2-dimensional then K_V is isomorphic to $k(t)$, (where $1, t$ generate V over k).

Lemma 5.3.2. — *Let $V \subset K$ be 2-dimensional. Then*

$$H^2(\mathrm{G}_V, \mathbb{Z}/p^n) = 0$$

for all $n \geq 1$.

Proof. — One has

$$K_2(k(V)) = \sum_{\nu} k^*_\nu$$

where the sum is over all codimension 1 points of $k(V)$. By our assumptions on k, the group k^*_ν is p-divisible as well. Now we apply the theorem of Merkuriev-Suslin (see [**6**])

$$H^2(\mathrm{G}_V, \mathbb{Z}/p^n) = K_2(k(V))/p^n = 0.$$

□

Remark 5.3.3. — For $K = k(t)$ the irreducible divisors are parametrized by $\mathbb{P}^1_k = \mathbb{P}(V)$. In general, if $\dim V \geq 3$ then the transcendence degree of K_V over k is > 1 (and $\leq \dim V - 1$) and the description of Γ^a_V is complicated since there are many more irreducible divisors - they cannot be parametrized by an algebraic variety.

5.4. Commuting pairs. — If $[\tilde{f}_1, \tilde{f}_2] = 0$ for some lifts to Γ^c of elements $f_1, f_2 \in \Gamma^a$ then this commutator vanishes for all lifts. In this case we will call the pair f_1, f_2 a *commuting pair* (c-pair, since the following Proposition shows that it is indeed a c-pair in the sense of Definition 2.2.6). Natural c-pairs in Γ^a arise from valuations of fields.

PROPOSITION 5.4.1. — *If f_1, f_2 are a c-pair then for all 2-dimensional subspaces $V \subset K$ (not necessarily containing k) we have*

$$\dim\langle f_{1,V}, f_{2,V}, 1\rangle \leq 2.$$

Proof. — First assume that V contains k. We have the following diagram

$$\begin{array}{ccccccc} \mathrm{G}_K & \to & \mathrm{G}_K^c & \to & \Gamma_K^c & \to & \Gamma_K^a \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{G}_V & \to & \mathrm{G}_V^c & \to & \Gamma_V^c & \to & \Gamma_V^a \end{array}$$

Lemma 5.3.2 implies that for all surjective continuous homomorphisms

$$\Gamma_V^a \to A$$

onto a finite abelian group A and any cocycle $\alpha \in H^2(A, \mathbb{Z}/p^n)$ its image in $H^2(\mathrm{G}_V, \mathbb{Z}/p^n)$ is zero. By Lemma 5.1.1, it is already zero in $H^2(\mathrm{G}_V^c, \mathbb{Z}/p^n)$. If α is nonzero in $H^2(Z, \mathbb{Q}/\mathbb{Z})$ then we can conclude that its image in $H^2(\Gamma_V^c, \mathbb{Z}/p^n)$ is zero. This means that there exists a finite group B which is a central extension of A and a surjective continuous homomorphism

$$\Gamma_V^c \to B$$

such that

$$H^2(A, \mathbb{Z}/p^n) \to 0 \in H^2(B, Z/p^n).$$

Assuming that f_1, f_2 are nonproportional in Γ_V^a we construct an A with a nonzero cocycle $\alpha \in H^2(A, \mathbb{Q}/\mathbb{Z})$ as follows. Since f_1, f_2 are nonproportional there exists a sublattice of $k(V)$ of the form $\langle x, x+1\rangle$ such that f_1, f_2 remain nonproportional on this lattice. Thus the vectors $\hat{f}_1 = (f_1(x), f_1(x+1))$ and $\hat{f}_2 = (f_2(x), f_2(x+1))$ define a rank 2 lattice $\hat{A} \subset \mathbb{Z}_p \oplus \mathbb{Z}_p$ and we have a surjective homomorphism $\Gamma_V^a \to \hat{A}$. Then there exists an n such that the reduction A of $\hat{A}$ modulo p^n is a subgroup of index $< p^n$ in $\mathbb{Z}/p^n \oplus \mathbb{Z}/p^n$ This implies that there exists a nonzero cocycle $\alpha = \alpha(f_1, f_2) \in H^2(A, \mathbb{Z}/p^n)$ mapping to a nonzero element in $H^2(A, \mathbb{Q}/\mathbb{Z})$ (by the condition on $\det(\hat{f}_1, \hat{f}_2)$ modulo p^n). Thus we have surjective maps

$$\Gamma_V^c \to B \to A$$

where B is a finite group such that α maps to 0 in $H^2(B, \mathbb{Z}/p^n)$.

The group B contains images of $\tilde{f}_1, \tilde{f}_2 \in \Gamma_K^c$ and

$$\tilde{A} = \langle \tilde{f}_1, \tilde{f}_2 \rangle \subset B$$

surjects onto A. The cocycle $\alpha \in H^2(A, \mathbb{Q}/\mathbb{Z})$ maps to a nonzero element in $H^2(\tilde{A}, \mathbb{Q}/\mathbb{Z})$ but to zero in $H^2(B, \mathbb{Q}/\mathbb{Z})$. Contradiction. It follows that the restrictions of f_1, f_2 to V are proportional.

Now we turn to the general case. For any 2-dimensional subspace $V \subset K$ and $x \in V \setminus 0$ consider the 2-dimensional space V' over k consisting of elements of the form $v' = x^{-1}v$ with $v \in V$. The space V' contains k and, therefore, $f_{1,V'} = \lambda f_{2,V'}$ for some $\lambda \in k$. Thus

$$f_1(v) + f_1(x^{-1}) = f_1(x^{-1}\, v) = \lambda f_2(x^{-1}\, v) = \lambda(f_2(x^{-1}) + f_2(v)),$$

for all $v \in V$, i.e., $\dim\langle f_{1,V}, f_{2,V}, 1\rangle \leq 2$. $\square$

6. Valuations

6.1. Notations. — A *scale* is a commutative totally ordered group (we will use the notations $>$ and $\geq$). Let K be a field and

$$\nu \,:\, K^* \to \mathcal{I}_\nu$$

a surjective homomorphism onto a scale $\mathcal{I}_\nu$. It is called a nonarchimedean valuation if

$$\nu(x+y) \geq \min(\nu(x), \nu(y)),$$

with an equality if $\nu(x) \neq \nu(y)$. The group $\mathcal{I}_\nu$ is called the scale of the valuation. We consider only nonarchimedean valuations and call them simply valuations.

A valuation ν can be extended to K by $\nu(0) = \infty > \iota$ for all $\iota \in \mathcal{I}_\nu$. It defines a topology on K. We denote by K_ν the completion of K with respect to this topology. The sets

$$\mathcal{O}_{\nu,\iota} = \{x \in K \mid \nu(x) \geq \iota\}$$

are additive subgroups. We call the subring $\mathcal{O}_\nu = \mathcal{O}_{\nu,\nu(1)}$ the *valuation ring* of ν. Denote by $\mathfrak{m}_\nu = \{x \mid \nu(x) > \nu(1)\}$ the *valuation ideal* (it is a maximal ideal in $\mathcal{O}_\nu$); by $\mathcal{O}_\nu^* = \mathcal{O}_\nu \setminus \mathfrak{m}_\nu$ the set of invertible elements and by $\mathrm{K}_\nu = \mathcal{O}_\nu/\mathfrak{m}_\nu$ the *residue field* of $\mathcal{O}_\nu$. We have a multiplicative decomposition $\mathcal{O}_\nu^*/(1+\mathfrak{m}_\nu) = \mathrm{K}_\nu^*$. Here $1+\mathfrak{m}_\nu$ is the multiplicative subgroup of $\mathcal{O}_\nu^*$ consisting of elements of the form $(1+m)$, $m \in \mathfrak{m}_\nu$.

6.2. Inertia group. — Let K and ν be as above. We have a natural embedding $\Gamma^a_{K_\nu} \to \Gamma^a_K$. Its image is called the *abelian valuation group*.

Definition 6.2.1. — *Denote by*

$$\Gamma^a_\nu \subset \Gamma^a_K = \mathcal{LF}^{\mathrm{top}}(K, \mathbb{Z}_p)$$

the subgroup of those functions which are trivial on $(1+\mathfrak{m}_\nu)$. *This group will be called the* abelian reduced valuation group.

Definition 6.2.2. — *Denote by*

$$\mathrm{I}^a_\nu \subset \Gamma^a_\nu \subset \mathcal{LF}^{\mathrm{top}}(K, \mathbb{Z}_p)$$

the subgroup of those functions $z^\chi_\nu \in \mathcal{LF}^{\mathrm{top}}(K, \mathbb{Z}_p)$ *such that*

$$z^\chi_\nu(\kappa) = \chi(\nu(\kappa))$$

for some homomorphism

$$\chi : \mathcal{I}_\nu \to \mathbb{Z}_p$$

and all $\kappa \in K^*$. *This group will be called the* abelian inertia group of ν. *The elements* z^χ_ν *are called* inertia elements *of* ν.

Of course, for all $\kappa, \kappa' \in K^*$ we have

$$z^\chi_\nu(\kappa \cdot \kappa') = z^\chi_\nu(\kappa) + z^\chi_\nu(\kappa')$$

and, since k contains all p-power roots, we have $\chi(\nu(k^*)) = 0$.

Remark 6.2.3. — If $\mathrm{char}(\mathrm{K}_\nu) \neq p$ then Γ^a_ν coincides with the abelian valuation group ([**13**]). Otherwise, Γ^a_ν is its proper subgroup.

6.3. Valuations and flag functions. —

EXAMPLE 6.3.1. — Let ν be a valuation on $K = k(X)$ which is trivial on k. Let z_ν^X be an inertia element of ν. It is a function on K^*, invariant under k^*. We can extend it arbitrarily to K, for example by $z_\nu^X(0) = 0$. Then z_ν^X is an abelian flag function on K (considered as a vector space over k).

EXAMPLE 6.3.2. — Let X be an algebraic variety defined over $k = \mathbb{Q}$, with good reduction X_p at p. Let ν be a divisorial valuation on the reduction $X_p \otimes \overline{\mathbf{F}}_p$. This valuation extends to a valuation on $K = \overline{\mathbb{Q}}(X)$, with values in $\mathbb{Q} \times \mathbb{Z}$. Any character of $\mathbb{Z}$ which is trivial on $\mathbb{Q}$ defines an inertia element z_ν^X, which can be considered as a function on K. This is an abelian flag function: for every finite dimensional subspace in K the corresponding filtration by groups consists of modules over p-integers in $\overline{\mathbb{Q}}$.

EXAMPLE 6.3.3. — Let K be field with a valuation ν, $\mathcal{O}_\nu$, $\mathfrak{m}_\nu$, K_ν as above. Let $\bar{V}$ be an n-dimensional vector space over K_ν and $\bar{f}$ an AF-function on $\mathbb{P}(\bar{V})$ (with respect to K_ν). Define f on $\mathcal{O}_\nu^n$, extending f trivially over the cosets $\mathcal{O}_\nu/\mathfrak{m}_\nu$. Consider the restriction of f to the subset

$$V_{\mathcal{O}} := \mathcal{O}_\nu^n \setminus (\mathfrak{m}_\nu \mathcal{O}_\nu)^n.$$

Put $V = K^n$ and consider the orbit space $(V \setminus 0)/K^*$. Every orbit has a representative in $V_{\mathcal{O}}$. Thus f defines an AF-function on $\mathbb{P}(V)$ (with respect to K).

THEOREM 6.3.4. — *Let $f \in \mathcal{LF}(K, \mathbb{Z}_p) \cap \mathcal{AF}(K, \mathbb{Z}_p)$. Then there exists a valuation ν on K with scale $\mathcal{I}_\nu$ and a map $\tilde{f} : \mathcal{I}_\nu \to \mathbb{Z}_p$ such that $f(\kappa) = \tilde{f} \circ \nu(\kappa)$ for all $\kappa \in K$.*

Proof. — An AF-function defines a filtration $(K_\alpha^f)_{\alpha \in \mathcal{A}}$. The logarithmic property of f implies that the ordered set $\mathcal{A}$ is an ordered group (a scale). The map $\nu : K \to \mathcal{A}$ is a homomorphism. Every K_α^f is a subgroup under addition. Since f is constant on $\overline{K}_\alpha^f$ it follows that ν is a nonarchimedean valuation and that f can be factored as claimed. □

Corollary 6.3.5. — *Assume that f satisfies the conditions of Theorem 6.3.4 and let v be the associated valuation. Consider the groups*

$$\mathcal{O}_v := \{\kappa \in K \mid f(\kappa) \geq f(1)\}$$

$$\mathfrak{m}_v := \{\kappa \in K \mid f(\kappa) > f(1)\}.$$

Then $\mathcal{O}_v/\mathfrak{m}_v$ is a field and $\mathcal{O}_v \setminus \mathfrak{m}_v$ consists of invertible elements in $\mathcal{O}_v$.

Proof. — For any $x \in K^*$ the sets $\overline{K}_f^{\alpha}$ are shifted (bijectively) under multiplication by x. In particular, if there is an element $y \in \overline{K}_f^{\alpha}$ such that $xy \in \overline{K}_f^{\alpha}$ then for all $y' \in \overline{K}_f^{\alpha}$ the element xy' is also in $\overline{K}_f^{\alpha}$. The set $\mathcal{O}_v \setminus \mathfrak{m}_v$ contains both x and $1 \cdot x$ (for any $x \in \mathcal{O}_v \setminus \mathfrak{m}_v$). Thus for all $x \in \mathcal{O}_v \setminus \mathfrak{m}_v$ there exists an inverse in $\mathcal{O}_v \setminus \mathfrak{m}_v$. □

6.4. AF-functions on Γ^a. —

Proposition 6.4.1. — *Assume that $f_1, f_2 \in \Gamma^a = \mathcal{LF}^{\mathrm{top}}(K, \mathbb{Z}_p)$ are linearly independent (over $\mathbb{Q}_p$) and that they form a c-pair. Then there exists a valuation ν of K such that the $\mathbb{Z}_p$-linear span of f_1, f_2 contains an inertial element $z_\nu^\chi \in \mathrm{I}_\nu^a$. Moreover, for all $\lambda_1, \lambda_2 \in \mathbb{Q}_p$ the restriction of $\lambda_1 f_1 + \lambda_2 f_2$ to $1 + \mathfrak{m}_v$ is identically 0.*

Proof. — By Proposition 4.4.1, the $\mathbb{Z}_p$-linear span of f_1, f_2 contains an AF-function. By Theorem 6.3.4 and definitions in Section 6.2, there exists a valuation ν on K such that this AF-function is equal to z_ν^χ for some inertia element $z_\nu^\chi \in \mathrm{I}_\nu^a$.

Let f be any function in the $\mathbb{Q}_p$-linear span of f_1, f_2. Since both f and z_ν^χ are multiplicative, it suffices to consider them on $\mathcal{O}_v$. By definition, $z_\nu^\chi = 0$ on

$$\mathcal{O}_\nu^* = \mathcal{O}_\nu \setminus \mathfrak{m}_\nu$$

First observe that for $m \in \mathfrak{m}$ with $z_\nu^\chi(m) \neq 0$ we have $f(1+m) = 0$. Indeed, consider the sublattice $C = \langle 1, m \rangle$. Since z_ν^χ is nonconstant on C and since it forms a c-pair with f, we conclude that f is proportional to z_ν^χ on this space. Thus $f(1+m) = z_\nu^\chi(1+m) = 0$ as claimed.

Now assume that $z_\nu^\chi(m) = 0 = z_\nu^\chi(1)$. Then (since z_ν^χ is an AF-function) there exists an $m_1 \in \mathfrak{m}$ with $z_\nu^\chi(m_1) \neq 0$ and $1 > m_1 > m$.

Consider the subgroup $\langle m_1, m\rangle$ with generic element m_1 and put $m_2 = m_1 - m$. Then $z_\nu^\chi(m') = z_\nu^\chi(m_2) \neq 0$ and

$$f(1+m_1) = f(1-m_2) = 0.$$

By Corolllary 6.3.5, we have

$$m_3 := \frac{1}{1+m_1-m_2} \in \mathcal{O}_v \setminus \mathfrak{m}_v.$$

Further, by the logarithmic property,

$$0 = f(1+m_1) + f(1-m_2) = f(1+m_1-m_2) + f(1-m_1m_2m_3).$$

Since

$$z_\nu^\chi(m_1) + z_\nu^\chi(m_2) = 2z_\nu^\chi(m_1) \neq 0$$

(as z_ν^χ takes values in $\mathbb{Z}_p$) and $z_\nu^\chi(m_3) = 0$ (as $m_3 \in \mathcal{O}_v^*$) we have that

$$f(1-m_1m_2m_3) = 0.$$

This concludes the proof. □

Corollary 6.4.2. — *If f_1, f_2 satisfy the conditions of 6.4.1 then there is a valuation ν such that $\langle f_1, f_2\rangle$ lies in the abelian reduced valuation group Γ_ν^a of ν.*

The subgroup of Γ^a generated by f_1, f_2 contains a cyclic subgroup generated by the inertial element z_ν^χ and the quotient of $\langle f_1, f_2\rangle$ by the subgroup generated by AF-elements has at most one topological generator. An analogous statement is true for liftable abelian groups of higher rank.

Lemma 6.4.3. — *Let $f_1, \dots f_n \in \mathcal{LF}(K, \mathbb{Z}_p)$ be linearly independent functions. Suppose that for every i, j the functions f_i, f_j form a c-pair. Then the group F (topologically) generated by $f_1, \dots f_n$ contains a closed subgroup F' consisting of AF-functions such that F/F' is topologically cyclic.*

Proof. — If all f_j are AF-functions then every $\mathbb{Z}_p$-linear combination of f_j is an AF-function. (Indeed, for every 2-dimensional $V \subset K$ and any pair f_i, f_j the restrictions $\mathrm{rk}\,\langle f_{i,V}, f_{j,V}\rangle \leq 2$. Now apply results from 3.4)

Assume that f_1 is not an AF-function. By Proposition 4.4.1, there exist $\lambda_{1j}, \lambda_j \in \mathbb{Z}_p$ such that $\lambda_{1j}f_1 + \lambda_j f_j$ is an AF-function. These functions generate a $\mathbb{Z}_p$-submodule F'' of corank 1, such that F/F'' is a direct sum of a torsion module and a rank one $\mathbb{Z}_p$-module. The torsion elements correspond to AF-functions. Denote by F' the module generated by F'' and (the preimages of) these torsion elements. Then F' consists of AF-functions and F/F' is (topologically) cyclic. $\square$

COROLLARY 6.4.4. — *The subgroup F' generated by AF-elements corresponds to the inertia subgroup of some valuation ν'. The lemma implies that every liftable noncyclic abelian group lies in some reduced valuation group and contains a group of corank one consisting of inertial elements.*

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