On the effective cone of the moduli space of pointed rational curves

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Abstract. We compute the effective cone of the moduli space of stable curves of genus zero with six marked points.

1. Introduction

For a smooth projective variety, Kleiman's criterion for ample divisors states that the closed ample cone (i.e., the nef cone) is dual to the closed cone of effective curves. Since the work of Mori, it has been clear that *extremal rays* of the cone of effective curves play a special role in birational geometry. These correspond to certain distinguished supporting hyperplanes of the nef cone which are negative with respect to the canonical class. Contractions of extremal rays are the fundamental operations of the minimal model program.

Fujita [F] has initiated a dual theory, with the (closed) cone of effective divisors playing the central role. It is natural then to consider the dual cone and its generators. Those which are negative with respect to the canonical class are called *coextremal rays*, and have been studied by Batyrev [Ba]. They are expected to play a fundamental role in Fujita's program of classifying fiber-space structures on polarized varieties.

There are relatively few varieties for which the extremal and coextremal rays are fully understood. Recently, moduli spaces of pointed rational curves $\overline{M}_{0,n}$ have attracted considerable attention, especially in connection with mathematical physics and enumerative geometry. Keel and McKernan first considered the 'Fulton conjecture': The cone of effective curves of $\overline{M}_{0,n}$ is generated by one-dimensional boundary strata. This is proved for $n \leq 7$ [KeMc]. The analogous statement for divisors, namely, that the effective cone of $\overline{M}_{0,n}$ is generated by boundary divisors, is known to be false ([Ke] and [Ve]). The basic idea is to consider the map

$$r : \overline{M}_{0,2g} \hookrightarrow \overline{M}_g, \quad n = 2g,$$

identifying pairs $(i_1 i_2), (i_3 i_4), \dots, (i_{2g-1} i_{2g})$ of marked points to nodes. There exist effective divisors in $\overline{M}_g$ restricting to effective divisors not spanned by boundary

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divisors (see Remark 4.2). However, it is true that for each n the cones of $\mathfrak{S}_n$-invariant effective divisors are generated by boundary divisors [KeMc].

In recent years it has become apparent that various arithmetic questions about higher dimensional algebraic varieties defined over number fields are also closely related to the cone of effective divisors. For example, given a variety X over a number field F, a line bundle L in the interior of $\mathrm{NE}^1(X)$, an open $U \subset X$ over which $L^N (N \gg 0)$ is globally generated, and a height $H_{\mathcal{L}}$ associated to some adelic metrization $\mathcal{L}$ of L, we can consider the asymptotic behavior of the counting function

$$N(U, \mathcal{L}, B) = \#\{x \in U(F) \mid H_{\mathcal{L}}(x) \leq B\} \quad B > 0.$$

There is a heuristic principle that, after suitably restricting U,

$$N(U, \mathcal{L}, B) = c(\mathcal{L}) B^{a(L)} \log(B)^{b(L)-1}(1 + o(1)),$$

as $B \to \infty$ (see [BT]). Here

$$a(L) := \inf\{a \in \mathbb{R} \mid aL + K_X \in \mathrm{NE}^1(X)\},$$

$b(L)$ is the codimension of the face of $\mathrm{NE}^1(X)$ containing $a(L)L + K_X$ (provided that $\mathrm{NE}^1(X)$ is locally polyhedral at this point), and $c(\mathcal{L}) > 0$ is a constant depending on the chosen height (see [BM] and [BT] for more details). Notice that the explicit determination of the constant $c(\mathcal{L})$ also involves the knowledge of the effective cone.

Such asymptotic formulas can be proved for smooth complete intersections in $\mathbb{P}^n$ of small degree using the classical circle method in analytic number theory and for varieties closely related to linear algebraic groups, like flag varieties, toric varieties etc., using adelic harmonic analysis ([BT] and references therein). No general techniques to treat arbitrary varieties with many rational points are currently available. To our knowledge, the only other variety for which such an asymptotic is known to hold is the moduli space $\overline{M}_{0,5}$ (Del Pezzo surface of degree 5) in its anticanonical embedding [dB]. Upper and lower bounds, with the expected $a(L)$ and $b(L)$, are known (see [VW]) for the Segre cubic threefold

$$\mathrm{Seg} = \{(x_0, \dots, x_5) : \sum_{j=0}^{5} x_j^3 = \sum_{j=0}^{5} x_j = 0\}.$$

This admits an explicit resolution by the moduli space $\overline{M}_{0,6}$ (Remark 3.1); see [Hu] for the relationship between the Segre cubic and moduli spaces.

Our main result (Theorem 5.1) is a computation of the effective cone of $\overline{M}_{0,6}$. Besides the boundary divisors, the generators are the loci in $\overline{M}_{0,6}$ fixed under

$$\sigma = (i_1 i_2)(i_3 i_4)(i_5 i_6) \in \mathfrak{S}_6, \quad \{i_1, i_2, i_3, i_4, i_5, i_6\} = \{1, 2, 3, 4, 5, 6\}.$$

This equals the closure of $r^*\mathfrak{h} \cap M_{0,6}$, where $\mathfrak{h}$ is the hyperelliptic locus in $\overline{M}_3$. The effective and moving cones of $\overline{M}_3$ are studied in detail by Rulla [Ru]. Rulla's inductive analysis of the moving cone is similar to the method outlined in Section 2. Results on the ample cone of $\overline{M}_{0,6}$ have been recently obtained by Farkas and Gibney [FG].

The arithmetic consequences of Theorem 5.1 will be addressed in a future paper.

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2. Generalities on effective cones

Let X be a nonsingular projective variety with Néron-Severi group $\mathrm{NS}(X)$ and group of one-cycles $\mathrm{N}_1(X)$. The closed effective cone of X is the closed convex cone

$$\mathrm{NE}^1(X) \subset \mathrm{NS}(X) \otimes \mathbb{R}$$

generated by effective divisors on X. Let $\mathrm{NM}_1(X)$ be the dual cone $\mathrm{NE}^1(X)^*$ in $\mathrm{N}_1(X) \otimes \mathbb{R}$. Similarly, let $\mathrm{NE}_1(X)$ be the cone of effective curves and $\mathrm{NM}^1(X)$ its dual, the nef cone.

We review one basic strategy, used in Section 5, for computing $\mathrm{NE}^1(X)$. Suppose we are given a collection $\Gamma = \{A_1, \ldots, A_m\}$ of effective divisors that we expect to generate the effective cone and a subset $\Sigma \subset \Gamma$. For any effective divisor E, we have a decomposition

$$E = M_\Sigma + B_\Sigma, \quad B_\Sigma = a_1 A_1 + \ldots + a_m A_m, \quad a_j \geq 0,$$

where B_Σ is the fixed part of $|E|$ supported in Σ. The divisor M_Σ may have fixed components, but they are not contained in Σ. Let $\mathrm{Mov}(X)_\Sigma$ denote the closed cone generated by effective divisors without fixed components in Σ. To show that Γ generates $\mathrm{NE}^1(X)$ it suffices to show that it generates $\mathrm{Mov}(X)_\Sigma$. Any divisor of $\mathrm{Mov}(X)_\Sigma$ restricts to an effective divisor on each $A_j \in \Sigma$. Consequently,

$$\mathrm{Mov}(X)_\Sigma \subset \mathrm{NM}_1(\Sigma, X)^*,$$

where $\mathrm{NM}_1(\Sigma, X) \subset \mathrm{N}_1(X)$ is generated by the images of the $\mathrm{NM}_1(A_i)$ and $A_i \in \Sigma$. To prove that Γ generates $\mathrm{NE}^1(X)$, it suffices then to check that

$$\{\text{cone generated by } \Gamma\}^* \subset \mathrm{NM}_1(\Sigma, X).$$

3. Geometry of $\overline{M}_{0,n}$

3.1. A concrete description of $\overline{M}_{0,n}$. In this section we give a basis for the Néron-Severi group of $\overline{M}_{0,n}$ and write down the boundary divisors and the symmetric group action.

We recall the explicit iterated blow-up realization

$$\beta_n : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$$

from [**Has**] (see also a related construction in [**Kap**].) This construction involves choosing one of the marked points; we choose s_n. Fix points $p_1, \ldots, p_{n-1}$ in linear general position in $\mathbb{P}^{n-3} := X_0[n]$. Let $X_1[n]$ be the blow-up of $\mathbb{P}^{n-3}$ at $p_1, \ldots, p_{n-1}$, and let $E_1, \ldots, E_{n-1}$ denote the exceptional divisors (and their proper transforms in subsequent blow-ups). Consider the proper transforms $\ell_{ij} \subset X_1[n]$ of the lines joining p_i and p_j. Let $X_2[n]$ be the blow-up of $X_1[n]$ along the ℓ_{ij}, with exceptional divisors E_{ij}. In general, $X_k[n]$ is obtained from $X_{k-1}[n]$ by blowing-up along proper transforms of the linear spaces spanned by k-tuples of the points. The exceptional divisors are denoted

$$E_{i_1,\dots,i_k} \quad \{i_1,\dots,i_k\} \subset \{1,\dots,n-1\}.$$

This process terminates with a nonsingular variety $X_{n-4}[n]$ and a map

$$\beta_n : X_{n-4}[n] \to \mathbb{P}^{n-3}.$$

One can prove that $X_{n-4}[n]$ is isomorphic to $\overline{M}_{0,n}$. We remark that for a generic point $p_n \in \mathbb{P}^{n-3}$, we have an identification

$$\beta_n^{-1}(p_n) = (C, p_1, p_2, \dots, p_n),$$

where C is the unique rational normal curve of degree $n-3$ containing $p_1, \dots, p_n$ (see [**Kap**] for further information).

Let L be the pull-back of the hyperplane class on $\mathbb{P}^{n-3}$ by β_n. We obtain the following explicit basis for $\mathrm{NS}(\overline{M}_{0,n})$:

$$\{L, E_{i_1}, E_{i_1 i_2}, \dots, E_{i_1,\dots,i_k}, \dots, E_{i_1,\dots,i_{n-4}}\}.$$

We shall use the following dual basis for the one-cycles $\mathrm{N}_1(\overline{M}_{0,n})$:

$$\{L^{n-4}, (-E_{i_1})^{n-4}, \dots, (-E_{i_1,\dots,i_k})^{n-3-k} L^{k-1}, \dots, (-E_{i_1,\dots,i_{n-4}}) L^{n-5}\}. \quad (\dagger)$$

3.2. Boundary divisors. Our next task is to identify the boundary divisors of $\overline{M}_{0,n}$ in this basis. These are indexed by partitions

$$\{1, 2, \dots, n\} = S \cup S^c, \quad n \in S \text{ and } |S|, |S^c| \geq 2;$$

the generic point of the divisor D_S corresponds to a curve consisting of two copies of $\mathbb{P}^1$ intersecting at a node ν, with marked points from S on one component and from S^c on the other. Thus we have an isomorphism

$$\begin{array}{ccc} D_S & \simeq & \overline{M}_{0,|S|+1} \times \overline{M}_{0,|S^c|+1}, \\ (\mathbb{P}^1, S) \cup_\nu (\mathbb{P}^1, S^c) & \longrightarrow & (\mathbb{P}^1, S \cup \{\nu\}) \times (\mathbb{P}^1, S^c \cup \{\nu\}). \end{array} \quad (\ddagger)$$

The exceptional divisors are identified as follows:

$$E_{i_1,\dots,i_k} = D_{i_1,\dots,i_k,n}, \quad \{i_1,\dots,i_k\} \subset \{1,\dots,n-1\}, k \leq n-4.$$

The remaining divisors $D_{i_1,\dots,i_{n-3},n}$ are the proper transforms of the hyperplanes spanned by $(n-3)$-tuples of points; we have

$$[D_{i_1,\dots,i_{n-3},n}] = L - E_{i_1} - E_{i_2} - \dots - E_{i_1,\dots,i_{n-4}} - \dots - E_{i_2,\dots,i_{n-3}}.$$

REMARK 3.1. The explicit resolution of the Segre threefold

$$R : \overline{M}_{0,6} \to \mathrm{Seg}$$

alluded to in the introduction is given by the linear series

$$|2L - E_1 - E_2 - E_3 - E_4 - E_5|.$$

The image is a cubic threefold with ten ordinary double points, corresponding to the lines ℓ_{ij} contracted by R.

3.3. The symmetric group action on $\overline{M}_{0,n}$. The symmetric group $\mathfrak{S}_n$ acts on $\overline{M}_{0,n}$ by the rule

$$\sigma(C, s_1, \dots, s_n) = (C, s_{\sigma(1)}, \dots, s_{\sigma(n)}).$$

Let $F_\sigma \subset \overline{M}_{0,n}$ denote the closure of the locus in $M_{0,n}$ fixed by an element $\sigma \in \mathfrak{S}_n$.

We make explicit the $\mathfrak{S}_n$-action in terms of our blow-up realization. Choose coordinates $(z_0, z_1, z_2, \dots, z_{n-3})$ on $\mathbb{P}^{n-3}$ so that

$$p_1 = (1, 0, \dots, 0), \dots, p_{n-2} = (0, \dots, 0, 1),\ p_{n-1} = (1, 1, \dots, 1, 1).$$

Each permutation of the first $(n-1)$ points can be realized by a unique element of PGL_{n-2}. For elements of $\mathfrak{S}_n$ fixing n, the action on $\overline{M}_{0,n}$ is induced by the corresponding linear transformation on $\mathbb{P}^{n-3}$. Now let $\sigma = (jn)$ and consider the commutative diagram

$$\begin{array}{ccc} \overline{M}_{0,n} & \stackrel{\sigma}{\to} & \overline{M}_{0,n} \\ \beta_n \downarrow & & \downarrow \beta_n \\ \mathbb{P}^{n-3} & \stackrel{\sigma'}{\dashrightarrow} & \mathbb{P}^{n-3} \end{array}.$$

The birational map σ' is the Cremona transformation based at the points $p_{i_1}, \dots, p_{i_{n-2}}$ where

$$\{i_1, \dots, i_{n-2}, j\} = \{1, 2, \dots, n-1\},$$

e.g., when $\sigma = (n-1, n)$ we have

$$\sigma(z_0, z_1, \dots, z_{n-3}) = (z_1 z_2 \dots z_{n-3}, z_0 z_2 \dots z_{n-3}, \dots, z_0 \dots z_{n-4}).$$

4. Analysis of surfaces in $\overline{M}_{0,6}$

4.1. The $\overline{M}_{0,5}$ case.

Proposition 4.1. $\mathrm{NE}^1(\overline{M}_{0,5})$ *is generated by the divisors* D_{ij}, *where* $\{ij\} \subset \{1, 2, 3, 4, 5\}$.

Sketch proof: This is well-known, but we sketch the basic ideas to introduce notation we will require later. As we saw in § 3.1, $\overline{M}_{0,5}$ is the blow-up of $\mathbb{P}^2$ at four points in general position. Consider the set of boundary divisors

$$\Sigma = \{D_{i5}, D_{ij}\} = \{E_i, L - E_i - E_j\}, \quad \{i, j\} \subset \{1, 2, 3, 4\}$$

and the set of semiample divisors

$$\Xi = \{L - E_i, 2L - E_1 - E_2 - E_3 - E_4, L, 2L - E_i - E_j - E_k\},\ \{i, j, k\} \subset \{1, 2, 3, 4\}.$$

These semiample divisors come from the forgetting maps

$$\phi_i : \overline{M}_{0,5} \to \overline{M}_{0,4} \simeq \mathbb{P}^1, \quad i = 1, \dots, 5$$

and the blow-downs

$$\beta_i : \overline{M}_{0,5} \to \mathbb{P}^2, \quad i = 1 \dots, 5.$$

Kleiman's criterion yields

$$C(\Sigma) \subset \mathrm{NE}_1(\overline{M}_{0,5}) = \mathrm{NM}^1(\overline{M}_{0,5})^* \subset C(\Xi)^*.$$

All the inclusions are equalities because the cones generated by Ξ and Σ are dual; this can be verified by direct computation (e.g., using the computer program `PORTA` [PORTA]). □

4.2. Fixed points and the Cayley cubic. We identify the fixed-point *divisors* for the $\mathfrak{S}_6$-action on $\overline{M}_{0,6}$. When $\tau = (12)(34)(56)$ we have

$$\tau(z_0, z_1, z_2, z_3) = (z_0z_2z_3, z_1z_2z_3, z_0z_1z_2, z_0z_1z_3)$$

and F_τ is given by $z_0z_1 = z_2z_3$. It follows that

$$[F_\tau] = 2L - E_1 - E_2 - E_3 - E_4 - E_5 - E_{13} - E_{23} - E_{24} - E_{14}.$$

More generally, when $\tau = (ab)(cd)(j6)$ we have

$$[F_\tau] = 2L - E_1 - E_2 - E_3 - E_4 - E_5 - E_{ac} - E_{ad} - E_{bc} - E_{bd}.$$

Remark 4.2. Consider $(\mathbb{P}^1, s_1, \dots, s_6) \in F_\tau$ and the quotient under the corresponding involution

$$q : \mathbb{P}^1 \longrightarrow \mathbb{P}^1, \quad q(s_1) = q(s_2),\ q(s_3) = q(s_4),\ \text{etc.}$$

Consider the map $r : \overline{M}_{0,6} \to \overline{M}_3$ identifying the pairs (12), (34), and (56) and write $C = q(\mathbb{P}^1, s_1, \dots, s_6)$, so there is an induced $q' : C \to \mathbb{P}^1$. Thus C is hyperelliptic and F_τ corresponds to the closure of $r^*\mathfrak{h} \cap \overline{M}_{0,6}$, where $\mathfrak{h} \subset \overline{M}_3$ is the hyperelliptic locus.

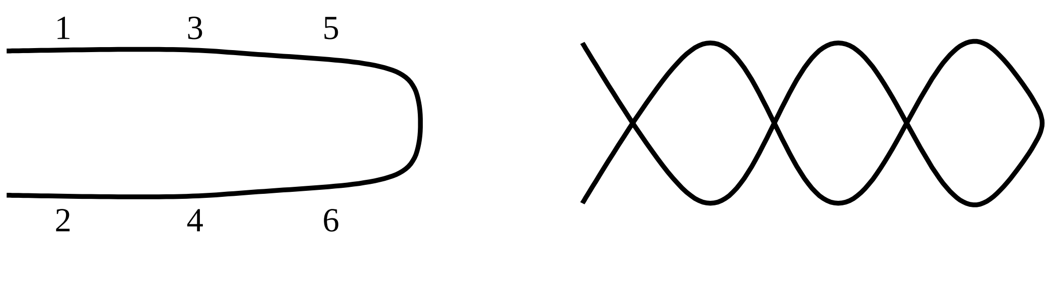

Figure 1. Trinodal hyperelliptic curves

In Section 5.3 we will use the description of the effective cone of the fixed point divisors F_σ. We have seen that these are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ blown-up at five points $p_1, \dots, p_5$. The projection from p_5

$$\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$$

induces a map $\varphi : F_\sigma \to \mathbb{P}^2$, realizing F_σ as a blow-up of $\mathbb{P}^2$: Take four general lines $\ell_1, \dots, \ell_4$ in $\mathbb{P}^2$ with intersections $q_{ij} = \ell_i \cup \ell_j$, and blow-up $\mathbb{P}^2$ along the q_{ij}. We write

$$\mathrm{NS}(F_\sigma) = \mathbb{Z}H + \mathbb{Z}G_{12} + \ldots + \mathbb{Z}G_{34},$$

where the G_{ij} are the exceptional divisors and H is the pull back of the hyperplane class from $\mathbb{P}^2$.

Proposition 4.3. $\mathrm{NE}^1(F_\sigma)$ *is generated by the* (-1)*-curves*

$$G_{12}, \dots, G_{34}, H - G_{ij} - G_{kl},$$

and the (-2)*-curves*

$$H - G_{ij} - G_{ik} - G_{il}, \quad \{i, j, k, l\} = \{1, 2, 3, 4\}.$$

Proof: Let Σ be the above collection of 13 curves. Consider also the following collection Ξ of 38 divisors, grouped as orbits under the $\mathfrak{S}_4$-action:

typical member	orbit size	induced morphism
H	1	blow-down $\varphi : F_\sigma \to \mathbb{P}^2$
$H - G_{12}$	6	conic bundle $F_\sigma \to \mathbb{P}^1$
$2H - G_{12} - G_{13} - G_{23}$	4	blow-down $F_\sigma \to \mathbb{P}^2$
$2H - G_{12} - G_{23} - G_{34}$	12	blow-down $F_\sigma \to \mathbb{P}^2$
$2H - G_{12} - G_{23} - G_{34} - G_{14}$	3	conic bundle $F_\sigma \to \mathbb{P}^1$
$3H - 2G_{12} - G_{13} - G_{23} - G_{34}$	12	blow-down $F_\sigma \to \mathbb{P}(1,1,2)$

Note that each of these divisors is semiample: the corresponding morphism is indicated in the table. In particular,

$$\begin{aligned} C(\Sigma) &:= \{\text{cone generated by } \Sigma\} \subset \mathrm{NE}_1(F_\sigma), \\ C(\Xi) &:= \{\text{cone generated by } \Xi\} \subset \mathrm{NM}^1(F_\sigma) \end{aligned}$$

and Kleiman's criterion yields

$$C(\Sigma) \subset \mathrm{NE}_1(F_\sigma) = \mathrm{NM}^1(F_\sigma)^* \subset C(\Xi)^*.$$

A direct verification using `PORTA` [**PORTA**] shows that the cones $C(\Sigma)$ and $C(\Xi)$ are dual, so all the inclusions are equalities. □

Remark 4.4. The image of F_τ under the resolution R of 3.1 is a cubic surface with four double points, classically called the *Cayley cubic* [**Hu**].

5. The effective cone of $\overline{M}_{0,6}$

We now state the main theorem:

Theorem 5.1. *The cone of effective divisors $\mathrm{NE}^1(\overline{M}_{0,6})$ is generated by the boundary divisors and the fixed-point divisors F_σ, where $\sigma \in \mathfrak{S}_6$ is a product of three disjoint transpositions.*

5.1. Proof of Main Theorem. We use the strategy outlined in § 2. Consider the collection of boundary and fixed-point loci

$$\Gamma = \{D_{ij}, D_{ijk}, F_\sigma, \quad \sigma = (ij)(kl)(ab), \quad \{i,j,k,l,a,b\} = \{1,2,3,4,5,6\}\}$$

and the subset of boundary divisors

$$\Sigma = \{D_{ij}, D_{ijk}\}.$$

We compute the cone $\mathrm{NM}_1(\Sigma, \overline{M}_{0,6})$, the convex hull of the union of the images of $\mathrm{NM}_1(D_{ij})$ and $\mathrm{NM}_1(D_{ijk})$ in $\mathrm{N}_1(\overline{M}_{0,6})$. Throughout, we use the dual basis for $\mathrm{N}_1(\overline{M}_{0,6})$ (cf. (†)):

$$\begin{aligned} &\{L^2, E_1^2, E_2^2, E_3^2, E_4^2, E_5^2, -LE_{12}, -LE_{13}, \\ &-LE_{14}, -LE_{15}, -LE_{23}, -LE_{24}, -LE_{25}, -LE_{34}, -LE_{35}, -LE_{45}\}. \end{aligned}$$

Recall the isomorphism (‡)

$$(\pi_{ijk}, \pi_{lab}) : D_{ijk} \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1, \quad \{i,j,k,l,a,b,c\} = \{1,2,3,4,5,6\}$$

so that

$$\mathrm{N}_1(D_{ijk}) = \mathbb{Z}B_{ijk} \oplus \mathbb{Z}B_{lab}, \quad \mathrm{NM}_1(D_{ijk}) = \mathbb{R}_+B_{ijk} + \mathbb{R}_+B_{lab},$$

where B_{ijk} is the class of the fiber of π_{ijk} (and its image in $\mathrm{N}_1(\overline{M}_{0,6})$). For example, the inclusion $j_{345} : D_{345} \hookrightarrow \overline{M}_6$ induces

$$(j_{345})_* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$$

using the bases (†) for $\mathrm{N}_1(\overline{M}_{0,6})$ and $\{B_{126}, B_{345}\}$ for $\mathrm{N}_1(D_{345})$. In particular, we find

$$\mathrm{NM}_1(\{D_{ijk}\}, \overline{M}_{0,6}) = C(\{B_{ijk}\}), \quad \{i, j, k\} \subset \{1, 2, 3, 4, 5, 6\},$$

with $\binom{6}{3} = 20$ generators permuted transitively by $\mathfrak{S}_6$ (Table 1).

The boundary divisor D_{ij} is isomorphic to $\overline{M}_{0,5}$ with marked points $\{k, l, a, b, \nu\}$ where $\{i, j, k, l, a, b\} = \{1, 2, 3, 4, 5, 6\}$ and ν is the node (cf. formula (‡)). The proof of Proposition 4.1 gives generators for the nef cone of D_{ij}. Thus the cone $\mathrm{NM}_1(D_{ij}, \overline{M}_{0,6})$ is generated by the classes

$$\{A_{ij}, A_{ij;k}, A_{ij;l}, A_{ij;a}, A_{ij;b}, C_{ij}, C_{ij;k}, C_{ij;l}, C_{ij;a}, C_{ij;b}\} \subset \mathrm{N}_1(\overline{M}_{0,6})$$

corresponding to the forgetting and blow-down morphisms

$$\{\phi_\nu, \phi_k, \phi_l, \phi_a, \phi_b, \beta_\nu, \beta_k, \beta_l, \beta_a, \beta_b\}.$$

As an example, consider the inclusion $j_{45} : D_{45} \hookrightarrow \overline{M}_{0,6}$ with

$$j_{45*} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T.$$

Applying this to the nef divisors of D_{45} gives the generators for $\mathrm{NM}_1(D_{45}, \overline{M}_{0,6})$ (Table 2).

However, four of the (-1)-curves in D_{ij} are contained in $D_{ijk}, D_{ijl}, D_{ija}$, and D_{ijb}, with classes $B_{ijk}, B_{ijl}, B_{ija}$, and B_{ijb} respectively. Thus we have the relations

$$C_{ij} = A_{ij;k} + B_{ijk}, \quad C_{ij;k} = A_{ij} + B_{ijk}$$

which implies that the C_{ij} and $C_{ij;k}$ are redundant:

Proposition 5.2. *The cone* $\mathrm{NM}_1(\Sigma, \overline{M}_{0,6})$ *is generated by the* A_{ij}*, the* $A_{ij;k}$*, and the* B_{ijk}*.*

These are written out in Tables 1,3, and 4.

Our next task is to write out the generators for the dual cone $C(\Gamma)^*$, as computed by `PORTA` [PORTA]. Since Γ is stable under the $\mathfrak{S}_6$ action, so are $C(\Gamma)$ and its dual cone. For the sake of brevity, we only write $\mathfrak{S}_6$-representatives of the generators, ordered by anticanonical degree.

The discussion of Section 2 shows that Theorem 5.1 will follow from the inclusion

$$C(\Gamma)^* \subset \mathrm{NM}_1(\Sigma, \overline{M}_{0,6}).$$

We express each generator of $C(\Gamma)^*$ as a sum (with non-negative coefficients) of the $\{A_{ij}, A_{ij;k}, B_{ijk}\}$. Both cones are stable under the $\mathfrak{S}_6$-action, so it suffices to produce expressions for one representative of each $\mathfrak{S}_6$-orbit. We use the representatives from Table 5:

$$
\begin{array}{rcl}
(1) & = & A_{15}+A_{13}+A_{35}+2B_{246} \quad (2)=A_{34;5}+B_{126} \\
(3) & = & A_{15}+A_{14}+2B_{236} \quad (4)=A_{25}+B_{146}+B_{136} \\
(5) & = & A_{23}+B_{146}+B_{156}+B_{236} \quad (6)=A_{15}+A_{14}+B_{236}+B_{246}+B_{356} \\
(7) & = & A_{13;5}+A_{15}+B_{236}+B_{246} \quad (8)=A_{12;5}+A_{14}+B_{256}+B_{356} \\
(9) & = & A_{24}+A_{34}+B_{126}+B_{136}+B_{156} \\
(10) & = & A_{25}+B_{136}+B_{146}+B_{256}+B_{346} \\
(11) & = & A_{34;5}+A_{35;4}+A_{25;3}+B_{146} \quad (12)=A_{12}+A_{34}+2B_{126}+2B_{346} \\
(13) & = & A_{15}+A_{14}+A_{23}+2B_{146}+2B_{236} \\
(14) & = & A_{23;5}+A_{15}+A_{25}+B_{136}+B_{146}+B_{236} \\
(15) & = & A_{23;5}+A_{24;5}+A_{15}+B_{156}+B_{346} \\
(16) & = & A_{24;5}+A_{15}+B_{136}+B_{156}+B_{236} \\
(17) & = & A_{23}+2A_{25}+2B_{136}+2B_{146} \\
(18) & = & A_{12;3}+A_{34}+B_{126}+B_{136}+A_{36;1} \\
(19) & = & A_{12;5}+A_{15}+A_{25}+B_{136}+B_{246}+B_{346} \\
(20) & = & A_{13;5}+A_{35}+A_{45}+2B_{126}+B_{456} \\
(21) & = & A_{12}+A_{13}+B_{126}+B_{136}+B_{246}+B_{346}+B_{456} \\
(22) & = & A_{15}+A_{23}+A_{34}+B_{126}+B_{146}+B_{156}+2B_{236} \\
(23) & = & A_{13;5}+A_{14;5}+A_{23}+A_{13}+B_{256}+2B_{456} \\
(24) & = & A_{15}+A_{23}+A_{24}+A_{34}+B_{126}+B_{136}+B_{146}+B_{156}+2B_{236} \\
(25) & = & 2A_{14}+A_{24}+2A_{13;5}+2B_{256}+2B_{356}
\end{array}
$$

This completes the proof of Theorem 5.1.□

5.2. Geometric interpretations of coextremal rays. By definition, a *coextremal ray* $\mathbb{R}_+\rho \subset \mathrm{NM}_1(X)$ satisfies the following

- for any nontrivial $\rho_1, \rho_2 \in \mathrm{NM}_1(X)$ with $\rho_1+\rho_2 \in \mathbb{R}_+\rho$, $\rho_1, \rho_2 \in \mathbb{R}_+\rho$;
- $K_X\rho < 0$.

Batyrev ([Ba], Theorem 3.3) shows that, for smooth (or $\mathbb{Q}$-factorial terminal) threefolds, the minimal model program yields a geometric interpretation of coextremal rays. They arise from diagrams

$$
\begin{array}{ccc}
X & \stackrel{\psi}{\dashrightarrow} & Y \\
 & & \downarrow \mu \\
 & & B
\end{array}
$$

where ψ is a sequence of birational contractions and μ is a Mori fiber space. The coextremal ray $\rho = \psi^*[C]$, where C is a curve lying in the general fiber of μ. These interpretations will hold for higher-dimensional varieties, provided the standard conjectures of the minimal model program are true.

It is natural then to write down these Mori fiber space structures explicitly. Our analysis makes reference to the list of orbits of coextremal rays in Table 5 (and uses the same numbering):

(1) The first orbit in the Table is orthogonal to each of the boundary divisors $D_{ij} \subset \overline{M}_{0,6}$. The $\mathbb{Q}$-Fano fibration associated with this coextremal ray must contract these divisors. The anticanonical series $|-K_{\overline{M}_{0,6}}|$ yields a birational morphism

$$\overline{M}_{0,6} \to \mathcal{J} \subset \mathbb{P}^4$$

onto a quartic $\mathbb{Q}$-Fano hypersurface, called the *Igusa quartic* [**Hu**]. The fifteen singular points of the Igusa quartic are the images of the D_{ij}. The coextremal ray has anticanonical degree two and corresponds to curves passing through the generic point, i.e., the conics in $\mathcal{J}$.

(2) Forgetting any of the six marked points

$$\overline{M}_{0,6} \to \overline{M}_{0,5}$$

yields a Mori fiber space, and the fibers are coextremal.

(3) We define a conic bundle structure on $\overline{M}_{0,6}$ by explicit linear series, using the blow-up description of Subsection 3.1. Consider the cubic surfaces in $\mathbb{P}^3$ passing through the lines

$$\ell_{14}, \ell_{15}, \ell_{24}, \ell_{25}, \ell_{34}, \ell_{35}.$$

This linear series has additional base points: Any cubic surface containing the lines $\ell_{14}, \ell_{24}, \ell_{34}$ (resp. $\ell_{15}, \ell_{25}, \ell_{35}$) must be singular at p_4 (resp. p_5), and thus contains the line ℓ_{45} by the Bezout Theorem.

Our linear series has projective dimension two. Indeed, cubic hypersurfaces in $\mathbb{P}^3$ depend on 19 parameters; the singularities at p_4 and p_5 each impose four conditions, the remaining points p_1, p_2, p_3 impose three further conditions, and containing the six lines imposes six more conditions. Thus we obtain a conic bundle structure

$$\mu : \overline{M}_{0,6} \dashrightarrow \mathbb{P}^2$$

collapsing the two-parameter family of conics passing through the six lines above.

(4) For any two disjoint subsets $\{i,j\}, \{k,l\} \subset \{1,2,3,4,5,6\}$ we consider the forgetting maps

$$\phi_{ij} : \overline{M}_{0,6} \to \mathbb{P}^1, \quad \phi_{kl} : \overline{M}_{0,6} \to \mathbb{P}^1.$$

Together, these induce a conic bundle structure

$$(\phi_{ij}, \phi_{kl}) : \overline{M}_{0,6} \to \mathbb{P}^1 \times \mathbb{P}^1.$$

The class of a generic fiber is coextremal.

5.3. The moving cone. Our analysis gives, implicitly, the moving cone of $\overline{M}_{0,6}$:

THEOREM 5.3. *The closed moving cone of $\overline{M}_{0,6}$ is equal to $\mathrm{NM}_1(\Gamma, \overline{M}_{0,6})^*$, where Γ is the set of generators for $\mathrm{NE}^1(\overline{M}_{0,6})$.*

In the terminology of [**Ru**], the 'inductive moving cone' equals the 'moving cone'. Combining Theorem 5.3 with the computation of the ample cones to the boundaries D_{ij} and D_{ijk} and the fixed-point divisors F_σ (Proposition 4.3) we obtain the moving cone. However, finding explicit *generators* for the moving cone is a formidable computational problem.

Proof: Recall that $\overline{M}_{0,6}$ is a log Fano threefold: $-(K_{\overline{M}_{0,6}} + \epsilon \sum_{ij} D_{ij})$ is ample for small $\epsilon > 0$ [**KeMc**]. Using Corollary 2.16 of [**KeHu**], it follows that $\overline{M}_{0,6}$ is a 'Mori Dream Space'. The argument of Theorem 3.4.4 of [**Ru**] shows that an effective divisor on $\overline{M}_{0,6}$ that restricts to an effective divisor on each generator $A_i \in \Gamma$ is in the moving cone. □

REMARK 5.4. Our proof of Theorem 5.1 uses the cone $\mathrm{NM}_1(\Sigma, \overline{M}_{0,6})^*$, rather than the (strictly) smaller moving cone. Of course, if the coextremal rays are in $\mathrm{NM}_1(\Sigma, \overline{M}_{0,6})$, *a fortiori* they are in $\mathrm{NM}_1(\Gamma, \overline{M}_{0,6})$.

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Table 1. Generators for $\mathrm{NM}_1(\{D_{ijk}\}, \overline{M}_{0,6})$

B_{126}	0	0	0	0	0	0	−1	0	0	0	0	0	0	0	0	0
B_{136}	0	0	0	0	0	0	0	−1	0	0	0	0	0	0	0	0
B_{146}	0	0	0	0	0	0	0	0	−1	0	0	0	0	0	0	0
B_{156}	0	0	0	0	0	0	0	0	0	−1	0	0	0	0	0	0
B_{236}	0	0	0	0	0	0	0	0	0	0	−1	0	0	0	0	0
B_{246}	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	0	0
B_{256}	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	0
B_{346}	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0
B_{356}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	0
B_{456}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1
B_{123}	1	0	0	0	1	1	0	0	0	0	0	0	0	0	0	−1
B_{124}	1	0	0	1	0	1	0	0	0	0	0	0	0	0	−1	0
B_{125}	1	0	0	1	1	0	0	0	0	0	0	0	0	−1	0	0
B_{134}	1	0	1	0	0	1	0	0	0	0	0	0	−1	0	0	0
B_{135}	1	0	1	0	1	0	0	0	0	0	0	−1	0	0	0	0
B_{145}	1	0	1	1	0	0	0	0	0	0	−1	0	0	0	0	0
B_{234}	1	1	0	0	0	1	0	0	0	−1	0	0	0	0	0	0
B_{235}	1	1	0	0	1	0	0	0	−1	0	0	0	0	0	0	0
B_{245}	1	1	0	1	0	0	0	−1	0	0	0	0	0	0	0	0
B_{345}	1	1	1	0	0	0	−1	0	0	0	0	0	0	0	0	0

Table 2. Generators for $\mathrm{NM}_1(D_{45}, \overline{M}_{0,6})$

A_{45}	1	0	0	0	0	0	1	1	0	0	1	0	0	0	0	1
$A_{45;1}$	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0
$A_{45;2}$	1	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0
$A_{45;3}$	1	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0
$A_{45;6}$	2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	1
$C_{45;6}$	1	0	0	0	0	0	1	1	0	0	1	0	0	0	0	0
$C_{45;1}$	2	0	1	1	0	0	1	1	0	0	0	0	0	0	0	1
$C_{45;2}$	2	1	0	1	0	0	1	0	0	0	1	0	0	0	0	1
$C_{45;3}$	2	1	1	0	0	0	0	1	0	0	1	0	0	0	0	1
C_{45}	2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0

Table 3. Generators A_{ij} for $\mathrm{NM}_1(\{D_{ij}\}, \overline{M}_{0,6})$

A_{12}	1	0	0	0	0	0	1	0	0	0	0	0	0	1	1	1
A_{13}	1	0	0	0	0	0	0	1	0	0	0	1	1	0	0	1
A_{14}	1	0	0	0	0	0	0	0	1	0	1	0	1	0	1	0
A_{15}	1	0	0	0	0	0	0	0	0	1	1	1	0	1	0	0
A_{16}	0	−2	0	0	0	0	1	1	1	1	0	0	0	0	0	0
A_{23}	1	0	0	0	0	0	0	0	1	1	1	0	0	0	0	1
A_{24}	1	0	0	0	0	0	0	1	0	1	0	1	0	0	1	0
A_{25}	1	0	0	0	0	0	0	1	1	0	0	0	1	1	0	0
A_{26}	0	0	−2	0	0	0	1	0	0	0	1	1	1	0	0	0
A_{34}	1	0	0	0	0	0	1	0	0	1	0	0	1	1	0	0
A_{35}	1	0	0	0	0	0	1	0	1	0	0	1	0	0	1	0
A_{36}	0	0	0	−2	0	0	0	1	0	0	1	0	0	1	1	0
A_{45}	1	0	0	0	0	0	1	1	0	0	1	0	0	0	0	1
A_{46}	0	0	0	0	−2	0	0	0	1	0	0	1	0	1	0	1
A_{56}	0	0	0	0	0	−2	0	0	0	1	0	0	1	0	1	1

TABLE 4. Generators $A_{ij;k}$ for $\mathrm{NM}_1(\{D_{ij}\}, \overline{M}_{0,6})$

$A_{12;3}$	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1
$A_{12;4}$	1	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0
$A_{12;5}$	1	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0
$A_{12;6}$	2	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0
$A_{13;2}$	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1
$A_{13;4}$	1	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0
$A_{13;5}$	1	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0
$A_{13;6}$	2	0	1	0	1	1	0	1	0	0	0	0	0	0	0	0
$A_{14;2}$	1	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0
$A_{14;3}$	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0
$A_{14;5}$	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0
$A_{14;6}$	2	0	1	1	0	1	0	0	1	0	0	0	0	0	0	0
$A_{15;2}$	1	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0
$A_{15;3}$	1	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0
$A_{15;4}$	1	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0
$A_{15;6}$	2	0	1	1	1	0	0	0	0	1	0	0	0	0	0	0
$A_{16;2}$	0	-1	0	0	0	0	1	0	0	0	0	0	0	0	0	0
$A_{16;3}$	0	-1	0	0	0	0	0	1	0	0	0	0	0	0	0	0
$A_{16;4}$	0	-1	0	0	0	0	0	0	1	0	0	0	0	0	0	0
$A_{16;5}$	0	-1	0	0	0	0	0	0	0	1	0	0	0	0	0	0
$A_{23;1}$	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1
$A_{23;4}$	1	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0
$A_{23;5}$	1	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0
$A_{23;6}$	2	1	0	0	1	1	0	0	0	0	1	0	0	0	0	0
$A_{24;1}$	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0
$A_{24;3}$	1	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0
$A_{24;5}$	1	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0
$A_{24;6}$	2	1	0	1	0	1	0	0	0	0	0	1	0	0	0	0
$A_{25;1}$	1	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0
$A_{25;3}$	1	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0
$A_{25;4}$	1	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0
$A_{25;6}$	2	1	0	1	1	0	0	0	0	0	0	0	1	0	0	0
$A_{26;1}$	0	0	-1	0	0	0	1	0	0	0	0	0	0	0	0	0
$A_{26;3}$	0	0	-1	0	0	0	0	0	0	0	1	0	0	0	0	0
$A_{26;4}$	0	0	-1	0	0	0	0	0	0	0	0	1	0	0	0	0
$A_{26;5}$	0	0	-1	0	0	0	0	0	0	0	0	0	1	0	0	0
$A_{34;1}$	1	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0
$A_{34;2}$	1	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0
$A_{34;5}$	1	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0
$A_{34;6}$	2	1	1	0	0	1	0	0	0	0	0	0	0	1	0	0
$A_{35;1}$	1	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0
$A_{35;2}$	1	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0
$A_{35;4}$	1	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0
$A_{35;6}$	2	1	1	0	1	0	0	0	0	0	0	0	0	0	1	0
$A_{36;1}$	0	0	0	-1	0	0	0	1	0	0	0	0	0	0	0	0
$A_{36;2}$	0	0	0	-1	0	0	0	0	0	0	1	0	0	0	0	0
$A_{36;4}$	0	0	0	-1	0	0	0	0	0	0	0	0	0	1	0	0
$A_{36;5}$	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	1	0
$A_{45;1}$	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0
$A_{45;2}$	1	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0
$A_{45;3}$	1	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0
$A_{45;6}$	2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	1
$A_{46;1}$	0	0	0	0	-1	0	0	0	1	0	0	0	0	0	0	0
$A_{46;2}$	0	0	0	0	-1	0	0	0	0	0	0	1	0	0	0	0
$A_{46;3}$	0	0	0	0	-1	0	0	0	0	0	0	0	0	1	0	0
$A_{46;5}$	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	1
$A_{56;1}$	0	0	0	0	0	-1	0	0	0	1	0	0	0	0	0	0
$A_{56;2}$	0	0	0	0	0	-1	0	0	0	0	0	0	1	0	0	0
$A_{56;3}$	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	1	0
$A_{56;4}$	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	1

Table 5. $\mathfrak{S}_6$-orbits of coextremal rays of $\overline{M}_{0,6}$

	$\deg_{-K}$	order																
(1)	2	1	3	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1
(2)	2	6	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
(3)	2	15	2	0	0	0	0	0	0	0	1	1	0	1	1	1	1	0
(4)	2	45	1	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0
(5)	3	60	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
(6)	3	72	2	0	0	0	0	0	0	0	1	1	1	0	1	1	0	0
(7)	3	120	2	0	0	0	0	1	0	0	0	1	0	1	0	1	0	0
(8)	3	120	2	0	0	0	0	1	0	0	1	0	1	0	0	1	0	0
(9)	3	180	2	0	0	0	0	0	0	0	0	1	0	1	1	1	1	0
(10)	4	6	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(11)	4	10	3	0	0	1	1	1	2	0	0	0	0	0	0	0	0	0
(12)	4	30	2	0	0	0	0	0	0	0	0	1	0	0	1	0	1	1
(13)	4	60	3	0	0	0	0	0	0	0	0	2	1	1	1	1	1	1
(14)	4	90	3	0	0	0	0	1	0	0	1	1	0	1	1	2	0	0
(15)	4	90	3	0	0	0	0	2	0	1	1	0	1	1	0	0	0	0
(16)	4	180	2	0	0	0	0	1	0	0	0	0	0	1	0	1	0	0
(17)	4	180	3	0	0	0	0	0	0	0	1	1	1	0	2	2	0	1
(18)	4	360	2	0	0	0	0	0	0	0	0	1	0	0	1	1	0	1
(19)	4	360	3	0	0	0	0	1	0	0	1	1	1	0	1	2	0	0
(20)	4	360	3	0	0	0	0	1	0	1	1	0	1	2	0	0	1	0
(21)	5	120	2	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1
(22)	5	360	3	0	0	0	0	0	0	0	0	2	0	1	1	2	0	1
(23)	5	360	4	0	0	0	0	2	0	1	1	1	2	2	0	0	0	0
(24)	6	360	4	0	0	0	0	0	0	0	0	3	0	2	1	2	1	1
(25)	6	360	5	0	0	0	0	2	0	1	2	1	2	3	0	0	1	0
		3905																