UNIVERSAL TORSORS AND COX RINGS

by

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Abstract. — We study the geometry of universal torsors on Del Pezzo surfaces.

Contents

Introduction

The study of surfaces over nonclosed fields k leads naturally to certain auxiliary varieties, called *universal torsors*. The proofs of the Hasse principle and weak approximation for certain Del Pezzo surfaces required a very detailed knowledge of the projective geometry, in fact, explicit equations, for these torsors [4], [6], [5], [14],[15], [13]. More recently, it was realized that in many cases the study of asymptotics of rational

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points of bounded height also boils down to counting of integral points on universal torsors [11], [12], [2].

Colliot-Thélène and Sansuc gave a general formalism for writing down equations for these torsors. Briefly, their method consists in the following: Let X be a smooth projective variety, $\{D_j\}_{j\in J}$ a finite set of irreducible divisors on X such that $U := X \setminus \cup_{j\in J} D_j$ has trivial Picard group. In practice, one usually chooses D_j to be the generators of the effective cone of X, e.g., the lines on the Del Pezzo surface. Consider the resulting exact sequence:

$$0 \to \frac{\bar{k}[U]^*}{\bar{k}^*} \to \oplus_{j\in J}\, \mathbb{Z}D_j \to \mathrm{Pic}(X_{\bar{k}}) \to 0.$$

Applying $\mathrm{Hom}(-, \bar{k}^*)$ one obtains an exact sequence of tori

$$1 \to T_{\mathrm{NS}} \to T \to R \to 1,$$

which can also be regarded as sheaves over U. Choose rational functions, invertible on U, which generate the torus R. These can be interpreted as giving a section $U \to R$, which induces an T_{NS}-torsor over U. This torsor canonically extends to X. In practice, this extension can be made explicit, yielding equations for the universal torsor.

However, in cases when the effective cone is simplicial, there are *no* relations among the generators and this method gives little information. In this paper we outline an alternative approach to the construction of universal torsors which works even when the effective cone is simplicial and illustrate it in specific examples.

1. Generalities on the Cox ring

Let X be a nonsingular projective variety over an algebraically closed field k characteristic zero. Let $\mathrm{NS}(X) \subset H^2(X, \mathbb{Z})$ denote the Néron-Severi group and $\mathrm{Pic}(X)$ the Picard group. We shall always assume $\mathrm{Pic}(X)$ is torsion free, and in particular,

$$\mathrm{Pic}(X) \simeq \mathrm{NS}(X).$$

Let $\mathrm{NE}^1(X) \subset \mathrm{NS}(X)_{\mathbb{R}}$ denote the cone of (pseudo)effective divisors, i.e., the smallest real closed cone containing all the effective divisors of X. For any finite set $\Xi \subset \mathrm{NS}(X)$, let $\mathrm{Cone}(\Xi)$ be the cone generated by Ξ.

Let $\mathrm{NE}_1(X)$ denote the cone effective curves and $\mathrm{NM}^1(X) \subset \mathrm{NS}(X)_{\mathbb{R}}$ its dual, the cone of nef divisors. By Kleiman's criterion, this is the smallest real closed cone containing all ample divisors of X.

Let $L_1, \ldots, L_r$ be line bundles on X and for $\nu = (n_1, \ldots, n_r) \in \mathbb{N}^r$ write

$$L^\nu := L_1^{\otimes n_1} \otimes \ldots \otimes L_r^{\otimes n_r}.$$

Consider the ring

$$R(X, L_1, \ldots, L_r) := \bigoplus_{\nu \in \mathbb{N}^r} \Gamma(X, L^\nu),$$

which need not be finitely generated in general.

By definition, a line bundle L on X is *semiample* if L^n is globally generated for some $n > 0$:

PROPOSITION 1.1. — *([9] Lemma 2.8) If the line bundles $L_1, \ldots, L_r$ are semiample then $R(X, L_1, \ldots, L_r)$ is finitely generated.*

REMARK 1.2. — If the L_i are ample then, after replacing each L_i by a large multiple, $R(X, L_1, \ldots, L_r)$ is generated by

$$\Gamma(X, L_1) \otimes \ldots \otimes \Gamma(X, L_r).$$

However, this is not generally the case if the L_i are only semiample (despite the assertion in the second part of Lemma 2.8 of [9]). Indeed, let $X \to \mathbb{P}^1 \times \mathbb{P}^1$ be a double cover and L_1 and L_2 be the pull-backs of the polarizations on the $\mathbb{P}^1$'s to X. For suitably large n_1 and n_2, $L_1^{n_1} \otimes L_2^{n_2}$ is very ample and its sections embed X. However,

$$\Gamma(X, L_1^{n_1}) \otimes \Gamma(X, L_2^{n_2}) \simeq \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n_1)) \otimes \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n_2)),$$

and any morphism induced by these sections factors through $\mathbb{P}^1 \times \mathbb{P}^1$.

PROPOSITION 1.3. — *Let $L_1, \ldots, L_r$ be a set of line bundles on X such that L_j is generated by sections $s_{j,0}, ..., s_{j,d_j}$. Assume that the induced morphism $X \to \prod_j \mathbb{P}^{d_j}$ is birational into its image. Then the ring generated by the $s_{j,k}$'s has the same fraction field as $R(X, L_1, \ldots, L_r)$.*

Proof. — Both rings have fraction field $k(X)(t_1, ..., t_r)$, where t_j is a nonzero section of L_j. $\square$

Definition 1.4. — [9] *Let X be a nonsingular projective variety so that* $\mathrm{Pic}(X)$ *is a free abelian group of rank r. The* Cox ring *for X is defined*

$$\mathrm{Cox}(X) := R(X, L_1, \dots, L_r)$$

where $L_1, \dots, L_r$ are lines bundles so that

1. *the L_i form a $\mathbb{Z}$-basis of* $\mathrm{Pic}(X)$*;*
2. *the cone* $\mathrm{Cone}(\{L_1, \dots, L_r\})$ *contains* $\mathrm{NE}^1(X)$*.*

This ring is naturally graded by $\mathrm{Pic}(X)$*: for $\nu \in \mathrm{Pic}(X)$ the ν-graded piece is denoted* $\mathrm{Cox}(X)_\nu$*.*

Proposition 1.5. — [9] *The ring* $\mathrm{Cox}(X)$ *does not depend on the choice of generators for* $\mathrm{Pic}(X)$*.*

Proof. — Suppose we have two sets of generators $L_1, \dots, L_r$ and $M_1, \dots, M_r$. Since $\mathrm{Cone}(\{L_i\})$ and $\mathrm{Cone}(\{M_i\})$ contain all the effective divisors, the nonzero graded pieces of both $R(X, L_1, \dots, L_r)$ and $R(X, M_1, \dots, M_r)$ are indexed by the effective divisor classes in $\mathrm{Pic}(X)$. Choose isomorphisms

$$M_j \simeq L^{(a_{1j}, \dots, a_{rj})}, \quad i = 1, \dots, r, A = (a_{ij})$$

which naturally induce isomorphisms

$$\Gamma(M^\nu) \simeq \Gamma(L^{A\nu}), \quad A\nu = (a_{11}\nu_1 + \dots + a_{1r}\nu_r, \dots, a_{r1}\nu_1 + \dots + a_{rr}\nu_r).$$

Thus we find $R(X, L_1, \dots, L_r) \simeq R(X, M_1, \dots, M_r)$. □

As $\mathrm{Cox}(X)$ is graded by $\mathrm{Pic}(X)$, a free abelian group of rank r, the torus

$$T(X) := \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{G}_m)$$

acts on $R(X)$. Indeed, each $\nu \in \mathrm{Pic}(X)$ naturally yields a character χ_ν of $T(X)$, so we have the action

$$t \cdot \xi = \chi_\nu(t)\xi, \quad \xi \in \mathrm{Cox}(X)_\nu, t \in T(X).$$

Thus the isomorphism constructed in Proposition 1.5 is *not* canonical: Two such isomorphisms differ by the action of an element of $T(X)$.

The following conjecture is a special case of 2.14 of [9]:

Conjecture 1.6 (Finiteness of Cox ring). — Let X be a log Fano variety. Then $\mathrm{Cox}(X)$ is finitely generated.

Remark 1.7. — Note that if $\mathrm{Cox}(X)$ is finitely generated it follows trivially that $\mathrm{NE}^1(X)$ is finitely generated. Moreover, the nef cone $\mathrm{NM}^1(X)$ is also finitely generated.

Indeed, the nef cone corresponds to one of the chambers in the group of characters of $T(X)$ governed by the stability conditions for points $v \in \mathrm{Spec}(\mathrm{Cox}(X))$. These chambers are bounded by finitely many hyperplanes (see Theorem 0.2.3 in [7] for more details).

It has been conjectured by Batyrev [1] that the pseudo-effective cone of a Fano variety is finitely generated. However, the finiteness of the Cox ring is not a formal consequence of the finiteness of the pseudo-effective cone.

Example 1.8. — Let $p_1, \ldots, p_9 \in H \subset \mathbb{P}^3$ be nine distinct coplanar points given as a complete intersection of two generic cubic curves in the hyperplane H, and let X be the blow-up of $\mathbb{P}^3$ at these points. Then $\mathrm{NE}^1(X)$ is finitely generated but $\mathrm{Cox}(X)$ is not. Indeed, X is an equivariant compactification of the additive group $\mathbb{G}_a^3$, acting by translation on the affine space $\mathbb{P}^3 - H$. The group action can be used to show that $\mathrm{NE}^1(X)$ is generated by the boundary components (see [8]). Similarly, one can show that the cone $\mathrm{NE}_1(X)$ is generated by classes of curves in the boundary components, in particular, in the proper transform $\tilde{H} \subset X$ of H. It is well-known that $\mathrm{NE}_1(\tilde{H})$ is infinite: each section of the induced fibration in cubic plane curves

$$\tilde{H} \rightarrow \mathbb{P}^1$$

is a (-1)-curve and a generator. These sections are also generators of $\mathrm{NE}_1(X)$ (since the sections other than the exceptional divisors intersect $\tilde{H}$ negatively). It follows that $\mathrm{NE}_1(X)$ and $\mathrm{NM}^1(X)$ are not finitely generated and hence $\mathrm{Cox}(X)$ is not finitely generated (see Remark 1.7).

Proposition 1.9. — *Let X be a nonsingular projective variety whose anticanonical divisor $-K_X$ is nef and big. Suppose that D is a nef divisor on X. Then $H^i(X, D) = 0$ for each $i > 0$ and D is semiample.*

Proof: The first assertion is a consequence of Kawamata-Viehweg vanishing [10] §2.5. The second is a special case of the Kawamata Basepoint-freeness Theorem [10] §3.2. □

Proposition 1.9 largely determines the Hilbert function of the Cox ring:

Corollary 1.10. — *Retain the assumptions of Proposition 1.9. Then for nef classes ν we have*

$$\dim \mathrm{Cox}(X)_\nu = \chi(\mathcal{O}_X(\nu)).$$

Remark 1.11. — In practice, this will help us to find generators of $\mathrm{Cox}(X)$.

2. The E_6 cubic surface

By definition, the E_6 *cubic surface* is given by the homogeneous equation

$$S = \{(w, x, y, z) : xy^2 + yw^2 + z^3 = 0\} \subset \mathbb{P}^3. \tag{2.1}$$

We recall some elementary properties:

Proposition 2.1. —

1. *The surface S has a single singularity at the point $p := (0, 1, 0, 0)$, of type E_6.*
2. *S is the unique cubic surface with this property, up to projectivity.*
3. *S contains a unique line, satisfying the equations $y = z = 0$.*

Any smooth cubic surface may be represented as the blow-up of $\mathbb{P}^2$ at six points in 'general position'. There is an analogous property of the E_6 cubic surface:

Proposition 2.2. — *The E_6 cubic surface S is the closure of the image of $\mathbb{P}^2$ under the linear series*

$$w = a^2c \quad x = -(ac^2 + b^3) \quad y = a^3 \quad z = a^2b,$$

where

$$\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) = \langle a, b, c\rangle.$$

This map is the inverse of the projection of S from the double point p. The affine open subset

$$\mathbb{A}^2 := \{a \neq 0\} \subset \mathbb{P}^2$$

is mapped isomorphically onto $S - \ell$. In particular, $S \setminus \ell \simeq \mathbb{A}^2$, so the E_6 cubic surface is a compactification of $\mathbb{A}^2$.

Remark 2.3. — Note that S is not an *equivariant* compactification of $\mathbb{G}_a^2$, so the general theory of [3] does not apply.

Indeed, if S were an equivariant compactification of $\mathbb{G}_a^2$ then the projection from p would be $\mathbb{G}_a^2$-equivariant (see [8]). Therefore, the map $\mathbb{P}^2 \dashrightarrow S$ given above has to be $\mathbb{G}_a^2$-equivariant. The only $\mathbb{G}_a^2$-action on $\mathbb{P}^2$ under which a line is invariant is the standard translation action [8]. However, the linear series above is not invariant under the standard translation action

$$b \mapsto b + \beta a \quad c \mapsto c + \gamma a.$$

We proceed to compute the effective cone of the minimal resolution $\beta : \tilde{S} \to S$. Let $\ell \subset \tilde{S}$ be the proper transform of the line mentioned in Proposition 2.1.

Proposition 2.4. — *The Picard group* $\mathrm{Pic}(\tilde{S})$ *is a free abelian group of rank seven, generated by ℓ and the exceptional curves of β. For a suitable ordering $\{F_1, F_2, F_3, F_4, F_5, F_6\}$ of the exceptional curves, the intersection pairing takes the form*

$$\begin{array}{c|ccccccc} & F_1 & F_2 & F_3 & \ell & F_4 & F_5 & F_6 \\ \hline F_1 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ F_2 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \\ F_3 & 1 & 0 & -2 & 0 & 0 & 0 & 1 \\ \ell & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ F_4 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ F_5 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ F_6 & 0 & 1 & 1 & 0 & 0 & 1 & -2 \end{array} \quad . \tag{2.2}$$

Proposition 2.5. — *The effective cone* $\mathrm{NE}(\tilde{S})$ *is simplicial and generated by $\Phi := \{F_1, F_2, F_3, \ell, F_4, F_5, F_6\}$. The nef cone* $\mathrm{NM}(\tilde{S})$ *is generated* *by*

$$\begin{aligned}
A_1 &= F_2 + F_3 + 2\ell + 2F_4 + 2F_5 + 2F_6 \\
A_2 &= F_1 + F_2 + 2F_3 + 3\ell + 3F_4 + 3F_5 + 3F_6 \\
A_3 &= F_1 + 2F_2 + 2F_3 + 4\ell + 4F_4 + 4F_5 + 4F_6 \\
A_\ell &= 2F_1 + 3F_2 + 4F_3 + 3\ell + 4F_4 + 5F_5 + 6F_6 \\
A_4 &= 2F_1 + 3F_2 + 4F_3 + 4\ell + 4F_4 + 5F_5 + 6F_6 \\
A_5 &= 2F_1 + 3F_2 + 4F_3 + 5\ell + 5F_4 + 5F_5 + 6F_6 \\
A_6 &= 2F_1 + 3F_2 + 4F_3 + 6\ell + 6F_4 + 6F_5 + 6F_6
\end{aligned}$$

and each nef divisor is contained in the monoid generated by these divisors.

Proof. — The intersection form in terms of $A := \{A_1, \dots, \}$ is:

$$(2.3) \qquad \begin{array}{c|ccccccc}
 & A_1 & A_2 & A_3 & A_\ell & A_4 & A_5 & A_6 \\
\hline
A_1 & 0 & 1 & 1 & 2 & 2 & 2 & 2 \\
A_2 & 1 & 1 & 2 & 3 & 3 & 3 & 3 \\
A_3 & 1 & 2 & 2 & 4 & 4 & 4 & 4 \\
A_\ell & 2 & 3 & 4 & 3 & 4 & 5 & 6 \\
A_4 & 2 & 3 & 4 & 4 & 4 & 5 & 6 \\
A_5 & 2 & 3 & 4 & 5 & 5 & 5 & 6 \\
A_6 & 2 & 3 & 4 & 6 & 6 & 6 & 6
\end{array}.$$

This is the inverse of the intersection matrix (2.2) written in terms of the basis Φ, so the A_i generate the dual to $\mathrm{Cone}(\Phi)$. Observe that all the entries of matrix (2.3) are nonnegative and

$$\mathrm{Cone}(A) \subset \mathrm{Cone}(\Phi).$$

Suppose that D is an effective divisor on $\tilde{S}$. We write D as a sum of the fixed components contained in $\{F_1, \dots, F_6, \ell\}$ and the parts moving relative to Φ:

$$D = M_\Phi + a_1 F_1 + \ldots + a_6 F_6 + a_\ell \ell, \quad a_1, \dots, a_6, a_\ell \geq 0.$$

A priori, M_Φ may have fixed components, but they are not contained in Φ (however, see Lemma 2.7). It follows that M_Φ intersects each element of Φ nonnegatively, i.e., it is contained in $\mathrm{Cone}(A)$ and thus in $\mathrm{Cone}(\Phi)$. We conclude that $D \in \mathrm{Cone}(\Phi)$. Since $A_1, ..., A_6, A_\ell$ generate $\mathrm{NS}(\tilde{S})$ over

$\mathbb{Z}$ each nef divisor can be written as a nonnegative linear combination of these divisors. $\square$

We give explicit descriptions of the sections of the generators of the effective cone and the morphisms they induce. Proposition 1.9 applies to the minimal resolution of the E_6 cubic surface S. Since $\mathcal{O}_S(-K_S) = \mathcal{O}_S(+1)$, the anticanonical class of S is nef and big. Furthermore, rational double points admit crepant resolutions, i.e., $\beta^* K_S = K_{\tilde{S}}$, so $-K_{\tilde{S}}$ is also nef and big.

Choose nonzero sections $\xi_1, \ldots, \xi_\ell$ generating $\Gamma(F_1), \ldots, \Gamma(\ell)$:

$$\Gamma(F_1) = \langle \xi_1 \rangle, \ldots, \Gamma(F_6) = \langle \xi_6 \rangle, \Gamma(\ell) = \langle \xi_\ell \rangle .$$

These are canonical up to scalar multiplication. Each effective divisor

$$D = b_1 F_1 + b_2 F_2 + b_3 F_3 + b_\ell F_\ell + b_4 F_4 + b_5 F_5 + b_6 F_6$$

has a distinguished nonzero section

$$\xi^{(b_1,b_2,b_3,b_\ell,b_4,b_5,b_6)} := \xi_1^{b_1} \ldots \xi_6^{b_6} \xi_\ell^{b_\ell}.$$

The distinguished section of A_j is denoted $\xi^{\alpha(j)}$. Note that we have an injective ring homomorphism

$$k[\xi_1, \ldots, \xi_6, \xi_\ell] \rightarrow \mathrm{Cox}(\tilde{S}). \tag{2.4}$$

There is a partial order on the monoid of effective divisors of $\tilde{S}$: $D_1 \prec D_2$ if $D_2 - D_1$ is effective. The restriction of this order to the generators of the nef cone is illustrated in the tree below:

$$\begin{tikzcd}
& A_6 \arrow[d, dash] & \\
& A_5 \arrow[d, dash] & \\
& A_4 \arrow[dl, dash] \arrow[dr, dash] & \\
A_\ell \arrow[dr, dash] & & A_3 \arrow[dl, dash] \\
& A_2 \arrow[d, dash] & \\
& A_1 &
\end{tikzcd}$$

Whenever $D_1 \prec D_2$ we have a inclusion

$$\Gamma(D_1) \hookrightarrow \Gamma(D_2)$$

which is natural up to scalar multiplication: Indeed, express

$$D_1 - D_2 = b_1 F_1 + b_2 F_2 + \ldots + b_6 F_6 + b_\ell \ell, \quad b_j \geq 0$$

so we have

$$\begin{array}{ccc} s_1 & \mapsto & \xi^{(b_1, b_2, b_3, b_\ell, b_4, b_5, b_6)} s_1 \\ \Gamma(D_1) & \hookrightarrow & \Gamma(D_2). \end{array}$$

The homomorphism (2.4) is not surjective. We apply Proposition 1.9 and Corollary 1.10 to extract the generators for $\mathrm{Cox}(\tilde{S})$ beyond the ξ_j:

$$\begin{array}{rcl} \Gamma(A_1) & = & \left\langle \xi^{\alpha(1)}, \tau_1 \right\rangle \\ \Gamma(A_2) & = & \left\langle \xi^{\alpha(2)}, \xi^{\alpha(2)-\alpha(1)} \tau_1, \tau_2 \right\rangle \\ \Gamma(A_3) & = & \left\langle \xi^{\alpha(3)}, \xi^{\alpha(3)-\alpha(2)} \tau_2, \xi^{\alpha(3)-\alpha(1)} \tau_1, \xi^{\alpha(3)-2\alpha(1)} \tau_1^2 \right\rangle \\ \Gamma(A_\ell) & = & \left\langle \xi^{\alpha(\ell)}, \xi^{\alpha(\ell)-\alpha(2)} \tau_2, \xi^{\alpha(\ell)-\alpha(1)} \tau_1, \tau_\ell \right\rangle \end{array}$$

The sections of A_1 induce a conic bundle structure

$$\phi_1 : \tilde{S} \to \mathbb{P}^1,$$

obtained by projecting S from the line $\ell = \{y = z = 0\}$ (cf. Equation (2.1)). The sections of A_2 induce a blow-up realization

$$\phi_2 : \tilde{S} \to \mathbb{P}^2,$$

obtained by projecting S from the singularity $p = \{w = y = z = 0\}$. The divisor A_ℓ is the anticanonical divisor: Indeed, the adjunction formula implies

$$K_{\tilde{S}} F_i = 0, i = 1, \ldots, 6 \quad K_{\tilde{S}} \ell = -1$$

so the nondegeneracy of the intersection form implies $A_\ell = -K_{\tilde{S}}$. Thus we have

$$\Gamma(A_\ell) = \Gamma(-K_{\tilde{S}}) = \Gamma(-\beta^* K_S) = \Gamma(\beta^* \mathcal{O}_S(+1))$$

so the sections of A_ℓ induce

$$\beta : \tilde{S} \to S.$$

We now analyze the kernel of the homomorphism

$$k[\xi_1, ..., \xi_6, \xi_\ell, \tau_1, \tau_2, \tau_\ell] \to \mathrm{Cox}(\tilde{S}).$$

After renormalizing the ξ_i's and the τ_j's, we may identify

$$y = \xi^{\alpha(\ell)} \quad w = \xi^{\alpha(\ell)-\alpha(2)}\tau_2 \quad z = \xi^{\alpha(\ell)-\alpha(1)}\tau_1 \quad x = \tau_\ell.$$

Equation (2.1) gives the relation

$$\tau_\ell \xi^{2\alpha(\ell)} + \tau_2^2 \xi^{3\alpha(\ell)-2\alpha(2)} + \tau_1^3 \xi^{3\alpha(\ell)-3\alpha(1)} = 0.$$

Dividing by a suitable monomial ξ^β, we obtain

$$\tau_\ell \xi_\ell^3 \xi_4^2 \xi_5 + \tau_2^2 \xi_2 + \tau_1^3 \xi^2 \xi_3.$$

Note that this can be regarded as a dependence relation among sections of A_6.

Proposition 2.6. — *The homomorphism*

$$\varrho : \mathrm{C}(\tilde{S}) := k[\xi_1, ..., \xi_6, \xi_\ell, \tau_1, \tau_2, \tau_\ell]/\langle \tau_\ell \xi_\ell^3 \xi_4^2 \xi_5 + \tau_2^2 \xi_2 + \tau_1^3 \xi^2 \xi_3 \rangle \to \mathrm{Cox}(\tilde{S})$$

is an isomorphism.

Proof. — If ϱ were not injective, its kernel would have nontrivial elements in degrees $\nu = dA_\ell$. These translate into homogeneous polynomials of degree d vanishing on $S \subset \mathbb{P}^3$. All such polynomials are multiples of the cubic form defining S, which itself is a multiple of the relation we already have.

Now we prove that ϱ is surjective. We use the following

Lemma 2.7. — *Let D be an effective divisor on $\tilde{S}$ with fixed part F_D and moving part M_D. Then F_D is supported in $\{F_1, \ldots, F_6, \ell\}$, and M_D is a linear combination of $A_1, \ldots, A_6, A_\ell$ with nonnegative coefficients.*

Proof. — First observe that M_D is nef and therefore semiample with vanishing higher cohomology, by Proposition 1.9. Proposition 2.5 gives the second claim.

Let F be a fixed component of D not supported in $\{F_1, \ldots, F_6, \ell\}$. To arrive at a contradiction, we need to show that $h^0(M_D + F) > h^0(M_D)$. Since M_D has vanishing higher cohomology and

$$h^2(F + M_D) = h^0(K - F - M_D) = 0$$

it suffices to show that
$$\chi(F+M_D) > \chi(M_D).$$
By Riemann-Roch, it suffices to show that
$$F^2 + 2M_D F - K_{\tilde{S}} F > 0$$
or, equivalently,
$$F^2 + K_{\tilde{S}} F + 2M_D F - 2K_{\tilde{S}} F = 2g(F) - 2 + 2M_D F - 2K_{\tilde{S}} F > 0.$$
Since F is irreducible, $g(F) \geq 0$ and $M_D F \geq 0$ and $-K_{\tilde{S}} F > 1$, as M_D is nef and $-K_{\tilde{S}}$ is nonpositive only along the exceptional curves and has degree 1 only on the line ℓ (see Proposition 2.1). □

Consider the subrings
$$\mathrm{C}_a(\tilde{S}) := \bigoplus_{\nu \in \mathrm{NM}(\tilde{S})} \mathrm{C}(\tilde{S})_\nu \quad \text{and} \quad \mathrm{Cox}_a(\tilde{S}) = \bigoplus_{\nu \in \mathrm{NM}(\tilde{S})} \mathrm{Cox}(\tilde{S})_\nu$$
obtained by restricting to degrees corresponding to nef (and thus semi-ample) line bundles on $\tilde{S}$. The lemma implies that any element s_D of the Cox ring can be written in the form
$$\mathsf{s}_D = \mu \xi_1^{a_1} \cdots \xi_6^{a_6} \xi_\ell^{a_\ell}$$
with nonnegative exponents, where μ is in $\mathrm{Cox}_a(\tilde{S})$. Hence $\mathrm{Cox}_a(\tilde{S})$ and $\mathrm{Cox}(\tilde{S})$ have the same fraction field, as do $\mathrm{C}_a(\tilde{S})$ and $\mathrm{C}(\tilde{S})$.

Since A_ℓ induces a birational morphism $\tilde{S} \rightarrow S$, Proposition 1.3 implies that the rings $\mathrm{C}_a(\tilde{S})$ and $\mathrm{Cox}_a(\tilde{S})$ have the same fraction field, and it follows that $\mathrm{C}(\tilde{S})$ and $\mathrm{Cox}(\tilde{S})$ have the same fraction field. Proposition 1.1 implies $\mathrm{Cox}(\tilde{S})$ is in the integral closure of $\mathrm{C}(\tilde{S})$ in its fraction field. However, this ring itself is integrally closed, as it's the quotient of a polynomial ring by an irreducible polynomial. □

3. The D_4 cubic surface

The strategy of the previous section can applied to other surfaces as well. Here we illustrate it in the case of a cubic surface given by the homogeneous equation
$$S = \{(x_1, x_2, x_3, w) \,:\, w(x_1 + x_2 + x_3)^2 = x_1 x_2 x_3\} \subset \mathbb{P}^3.$$

We summarize its properties:

1. S has a single singularity at the point $(0,0,0,1)$ of type D_4.
2. S contains 6 lines with the equations

$$\begin{array}{ll} \ell_1' := \{w = x_1 = 0\} & m_1' := \{x_1 = x_2 + x_3 = 0\} \\ \ell_2' := \{w = x_2 = 0\} & m_2' := \{x_2 = x_1 + x_3 = 0\} \\ \ell_3' := \{w = x_3 = 0\} & m_3' := \{x_3 = x_1 + x_2 = 0\} \end{array}$$

3. S is given as a blow-up of $\mathbb{P}^2$ by the linear series

$$x_1 = u_1(u_1+u_2+u_3)^2,\ x_2 = u_2(u_1+u_2+u_3)^2,\ x_3 = u_3(u_1+u_2+u_3)^2,$$

$$w = u_1u_2u_3,$$

where $\langle u_1, u_2, u_3\rangle = \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$.

Let $\tilde{S} \to S$ denote the minimal desingularization of S and

$$\ell_1, \ell_2, \ell_3, m_1, m_2, m_3$$

the strict transforms of the lines. The rational map $S \dashrightarrow \mathbb{P}^2$ induces a morphism $\tilde{S} \to \mathbb{P}^2$ and let L denote the pullback of the hyperplane class. Let E_0, E_1, E_2, E_3 be the exceptional divisors, ordered so that the we have the following intersection matrix:

$$(3.1) \qquad \begin{array}{c|ccccccc} & L & E_1 & E_2 & E_3 & m_1 & m_2 & m_3 \\ \hline L & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ E_1 & 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ E_2 & 0 & 0 & -2 & 0 & 0 & 1 & 0 \\ E_3 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \\ m_1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ m_2 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ m_3 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{array}.$$

This is a rank 7 unimodular matrix, so that the Picard group $\mathrm{Pic}(\tilde{S})$ is generated by $L, ..., m_3$. We can write

$$E_0 = L - (E_1 + E_2 + E_3 + m_1 + m_2 + m_3) \quad \text{and} \quad \ell_j = L - E_j - 2m_j.$$

The anticanonical class is given by

$$-K_{\tilde{S}} = 3L - (E_1 + E_2 + E_3) - 2(m_1 + m_2 + m_3) = \ell_1 + \ell_2 + \ell_3.$$

PROPOSITION 3.1. — *The effective cone* $\mathrm{NE}(\tilde{S})$ *is generated by*

$$\Xi := \{E_0, E_1, E_2, E_3, m_j, \ell_j\}.$$

Proof. — Each effective divisor D can be expressed as a sum

$$D = M_\Xi + b_{E_0}E_0 + b_{E_1}E_1 + \ldots + b_{\ell_3}\ell_3,$$

with nonnegative coefficients, where M_Ξ intersects each of the elements in Ξ nonnegatively. The dual cone to $\mathrm{Cone}(\Xi)$ has generators

$$L, L - E_i - m_i, 2L - E_i - 2m_i, 2L - E_i - E_j - 2m_i - 2m_j,$$
$$2L - E_i - E_j - m_i - 2m_j.$$

Each of these is contained in $\mathrm{Cone}(\Xi)$:

$$\begin{array}{rcl} L & = & \ell_i + E_i + 2m_i, \\ 2L - E_i - 2m_i & = & 2\ell_i + E_i + 2m_i, \\ 2L - E_i - E_j - m_i - 2m_j & = & \ell_i + \ell_j + m_i, \\ L - E_i - m_i & = & \ell_i + m_i, \\ 2L - E_i - E_j - 2m_i - 2m_j & = & \ell_i + \ell_j. \end{array}$$

However, M_Ξ is contained in the dual cone, so that D is a sum of elements in Ξ with nonnegative coefficients. □

Each of the line bundles m_i, ℓ_i and E_i has a distinguished nonzero section (up to constants) denoted by μ_i, λ_i and η_i, respectively. We have

$$\{\lambda_i\eta_i\mu_i^2, \eta_0\eta_1\eta_2\eta_3\mu_1\mu_2\mu_3\} \subset \Gamma(L),$$

where we may identify

$$a_i = \lambda_i\eta_i\mu_i^2 \quad \text{and} \quad a_1 + a_2 + a_3 = \eta_0\eta_1\eta_2\eta_3\mu_1\mu_2\mu_3.$$

The dependence relation translates to

$$\lambda_1\eta_1\mu_1^2 + \lambda_2\eta_2\mu_2^2 + \lambda_3\eta_3\mu_3^2 = \eta_0\eta_1\eta_2\eta_3\mu_1\mu_2\mu_3. \tag{3.2}$$

An argument similar to the one given at the end of Section 2 proves that the natural homomorphism

$$k[\eta_0, ..., \eta_3, \mu_i, \lambda_i]/\langle \lambda_1\eta_1\mu_1^2 + \lambda_2\eta_2\mu_2^2 + \lambda_3\eta_3\mu_3^2 - \eta_0\eta_1\eta_2\eta_3\mu_1\mu_2\mu_3\rangle \to \mathrm{Cox}(\tilde{S})$$

is an isomorphism.

Whereas the E_6-cubic surface has no nontrivial automorphisms the D_4-cubic admits an $\mathfrak{S}_3$-action on the coordinates x_1, x_2, x_3. In particular, the D_4-cubic has nonsplit forms over nonclosed ground fields. They can be expressed as follows: let K/k be a cubic extension with generator γ, and Galois closure E/k. Elements $Y \in K$ can be represented as

$$Y = y\gamma + y'\gamma' + y''\gamma''$$

with $y, y', y'' \in k$ and $\gamma', \gamma'' \in K$. Then

$$w \cdot \mathrm{Tr}_{K/k}(Y)^2 = \mathrm{N}_{K/k}(Y)$$

is isomorphic, over E, to the D_4-cubic surface. The isomorphism is given by

$$\begin{array}{rcl} x_1 & = & y\gamma + y'\gamma' + y''\gamma'' \\ x_2 & = & y\sigma(\gamma) + y'\sigma(\gamma') + y''\sigma(\gamma'') \\ x_3 & = & y\sigma^2(\gamma) + y'\sigma^2(\gamma') + y''\sigma^2(\gamma'') \end{array}$$

where σ and σ^2 are coset representatives for $\mathrm{Gal}(E/k)$ modulo $\mathrm{Gal}(E/K)$.

Given elements $U, V, W \in K$ associated to η_i, μ_i and λ_i, respectively, and $\eta_0 \in k$, the torsor equation (3.2) takes the form

$$\mathrm{Tr}_{K/k}(UV^2W) = \eta_0 \mathrm{N}_{K/k}(UV).$$

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