ON THE ARITHMETIC OF DEL PEZZO SURFACES OF DEGREE 2

by

Andrew Kresch and Yuri Tschinkel

Date: 8 January 2003

Abstract. — We study the arithmetic of certain Del Pezzo surfaces of degree 2. We produce examples of Brauer-Manin obstruction to the Hasse principle, coming from 2- and 4-torsion elements in the Brauer group.

Contents

1. Introduction 1
2. Geometry 3
3. Galois group - generic case 4
4. Group cohomology 5
5. Computation of $\mathrm{Br}(S)/\mathrm{Br}(\mathbb{Q})$ in the generic case 10
6. The non-generic case 12
7. Examples of Brauer-Manin obstruction 13
8. Appendix: Cyclic Azumaya algebras on diagonal cubics 23
References 27

1. Introduction

A Del Pezzo surface is a smooth projective surface, isomorphic over the algebraic closure to $\mathbb{P}^2$, to $\mathbb{P}^1 \times \mathbb{P}^1$, or to the blow-up of $\mathbb{P}^2$ at up to 8 points

Key words and phrases. — Del Pezzo surfaces, Brauer groups.

The second author was supported by the NSF grant 0100277.

in general position. In the last case the Del Pezzo surface has degree equal to 9 minus the number of points in the blow-up. The arithmetic of Del Pezzo surfaces over number fields is an active area of investigation. It is known that the Hasse principle holds for Del Pezzo surfaces of degree at least 5.

Counterexamples to the Hasse principle were discovered for Del Pezzo surfaces of degrees 3 and 4 (see [12] and [8], respectively). A growing body of evidence (e.g., [3]) led to the question of whether the failure of the Hasse principle for Del Pezzo surfaces is always explained by the Brauer-Manin obstruction; this question is specifically raised by Colliot-Thélène and Sansuc in [5]. Computer verifications for diagonal cubics in [4] and theoretical advances, e.g., [2], [10], [14], lend support to an affirmative answer to this question.

A general smooth Del Pezzo surface of degree 2 can be realized as a double cover of $\mathbb{P}^2$ ramified in a smooth curve of degree 4. In this note we consider surfaces S over $\mathbb{Q}$ of the form

$$w^2 = Ax^4 + By^4 + Cz^4. \tag{1.1}$$

We compute their Galois-theoretic invariant $\mathrm{Br}(S)/\mathrm{Br}(\mathbb{Q})$ and produce examples of obstruction to the Hasse principle (see [4], [9] for background). Of particular interest is Example 7.6, where the obstruction comes from a 4-torsion element in the Brauer group. By [13], only 2- and 3-torsion Brauer group elements occur for Del Pezzo surfaces of degree ≥ 3.

The tool we use is group cohomology. Let F be a Galois extension of $\mathbb{Q}$, and let G denote the Galois group $\mathrm{Gal}(F/\mathbb{Q})$. If $\mathrm{Pic}(S_F)$ is equal to the geometric Picard group $M := \mathrm{Pic}(S_{\overline{\mathbb{Q}}})$ then we have

$$\mathrm{Br}(S)/\mathrm{Br}(\mathbb{Q}) = H^1(G, M). \tag{1.2}$$

More generally, the Hochschild-Serre spectral sequence gives rise to the following exact sequence:

$$\begin{aligned} 0 \longrightarrow \mathrm{Pic}(S) \longrightarrow \mathrm{Pic}(S_F)^G \longrightarrow \ker(\mathrm{Br}(\mathbb{Q}) \to \mathrm{Br}(F)) \\ \longrightarrow \ker(\mathrm{Br}(S) \to \mathrm{Br}(S_F)) \longrightarrow H^1(G, \mathrm{Pic}(S_F)) \longrightarrow H^3(G, F^*). \end{aligned} \tag{1.3}$$

We compute the group in (1.2) and represent lifts of elements to $\mathrm{Br}(S)$ by Azumaya algebras. By (1.3) and cohomological dimension, such lifts exist after possibly enlarging F; in practice we find it is often possible to take $[F : \mathbb{Q}]$ quite small and still have $H^1(G, \mathrm{Pic}(S_F))$ isomorphic to $H^1(G, M)$ and the final map in (1.3) trivial. Lastly, we explain the computation of local invariants and obtain the above-mentioned examples.

In an Appendix we show that in the case of the diagonal cubics considered in [4] the present techniques give rise to cyclic Azumaya algebras. This simplifies the construction of cocycle representatives and the local obstruction analysis, as compared with the original consideration of bicyclic group cohomology.

The authors would like to thank J.-L. Colliot-Thélène for helpful discussions and correspondence.

2. Geometry

Consider the surface S given by the equation

$$w^2 = Ax^4 + By^4 + Cz^4$$

in the weighted projective space $\mathbb{P}(2,1,1,1)$. It is a double cover of $\mathbb{P}^2$, branched over the twisted Fermat quartic curve

$$0 = Ax^4 + By^4 + Cz^4. \tag{2.1}$$

Let a, b, c denote some chosen 4-th roots of A, B, C, respectively. The 56 exceptional curves on S are the pre-images of the bitangents to the quartic. These are given by the following equations

$$\delta ax + by = 0, \quad \delta by + cz = 0, \quad \delta cz + ax = 0, \quad \text{where } \delta^4 = -1, \tag{2.2}$$

$$\alpha ax + \beta by + \gamma cz = 0 \quad (\alpha^4 = \beta^4 = \gamma^4 = 1). \tag{2.3}$$

Multiplying the equations (2.3) by a scalar doesn't change the line it defines, so it is natural to index the line by an element $(\alpha, \beta, \gamma) \in \mu_4^3/\mu_4$. Each bitangent lifts to a pair of exceptional curves in S: for example, the preimage of the line given by $\delta ax + by = 0$ is the pair of curves with equations

$$w = \pm c^2 z^2 \ .$$

These will be denoted by $L_{z,\delta,\pm}$. There are 24 exceptional curves lying over the lines in (2.2). The preimages of the lines in (2.3) are given by

$$w = \pm\sqrt{2}(\alpha\beta abxy + \beta\gamma bcyz + \alpha\gamma acxz) \ .$$

The ambiguity $\pm$ is resolved by scaling the tuple (α, β, γ), so that we can choose $+$ and consider $(\alpha, \beta, \gamma) \in \mu_4^3/\mu_2$. The 56 exceptional curves are denoted as follows:

$$\begin{array}{lll}
L_{z,\delta,\pm} : & \delta ax + by = 0, & w = \pm c^2 z^2, \\
L_{x,\delta,\pm} : & \delta by + cz = 0, & w = \pm a^2 x^2, \\
L_{y,\delta,\pm} : & \delta cz + ax = 0, & w = \pm b^2 y^2, \\
L_{\alpha,\beta,\gamma} : & \alpha ax + \beta by + \gamma cz = 0 & w = \sqrt{2}(\alpha\beta abxy + \beta\gamma bcyz + \alpha\gamma acxz) \ .
\end{array}$$

Each exceptional curve has self-intersection (-1). Each pair of curves lying above a bitangent to the Fermat quartic has intersection number 2. Other intersection numbers are 0 or 1 and are readily determined. From this the intersection matrix can be constructed. Let $\zeta = e^{\pi i/4}$. Then one sees that the Picard group is a free abelian group with generators:

(2.4)
$$\begin{array}{llll} v_1 = [L_{x,\zeta,+}] & v_2 = [L_{x,\zeta^3,-}] & v_3 = [L_{y,\zeta,+}] & v_4 = [L_{y,\zeta^3,-}] \\ v_5 = [L_{z,\zeta,+}] & v_6 = [L_{z,\zeta^3,-}] & v_7 = [L_{i,i,i}] & v_8 = [L_{z,\zeta^7,-}] + [L_{z,\zeta^3,-}] + [L_{i,i,i}] \ . \end{array}$$

Moreover, we have in the Picard group:

(2.5)
$$\begin{array}{ll} [L_{x,\zeta^5,+}] = -v_1 - v_7 + v_8 & [L_{x,\zeta^7,-}] = -v_2 - v_7 + v_8 \\ [L_{y,\zeta^5,+}] = -v_3 - v_7 + v_8 & [L_{y,\zeta^7,-}] = -v_4 - v_7 + v_8 \\ [L_{z,\zeta^5,+}] = -v_5 - v_7 + v_8 & [L_{1,1,i}] = -v_2 - v_3 + v_8 \\ [L_{1,1,-1}] = -v_5 - v_6 + v_8 & [L_{1,1,-i}] = -v_1 - v_4 + v_8 \\ [L_{1,i,1}] = -v_1 - v_6 + v_8 & [L_{1,i,-i}] = -v_3 - v_5 + v_8 \\ [L_{1,-1,1}] = -v_3 - v_4 + v_8 & [L_{1,-1,-1}] = -v_1 - v_2 + v_8 \\ [L_{1,-i,1}] = -v_2 - v_5 + v_8 & [L_{1,-i,i}] = -v_4 - v_6 + v_8 \\ [L_{i,1,1}] = -v_4 - v_5 + v_8 & [L_{i,1,-i}] = -v_2 - v_6 + v_8 \\ [L_{i,-1,-1}] = -v_3 - v_6 + v_8 & [L_{i,-1,-i}] = -v_1 - v_5 + v_8 \\ [L_{i,-i,1}] = -v_1 - v_3 + v_8 & [L_{i,-i,-1}] = -v_2 - v_4 + v_8 \end{array}$$

The anticanonical class is

$$-K_S = -v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7 + 3v_8. \tag{2.6}$$

Since the anticanonical class is the class of any pair of curves lying above a bitangent, the class of any exceptional curve can be read off from (2.4)–(2.6).

3. Galois group - generic case

Let G be the Galois group of the extension

$$F := \mathbb{Q}(\zeta, a^2, b/a, c/a) \tag{3.1}$$

over $\mathbb{Q}$ (where $\zeta = e^{\pi i/4}$). The subextension $\mathbb{Q}(\zeta)/\mathbb{Q}$ corresponds to a normal subgroup H of index 4. The quotient group is the Klein four-group. In the *generic* case, when $|G| = 128$, we have the generators

$$\sigma, \tau, \iota_a, \iota_b, \iota_c$$

characterized by

$$\begin{array}{llll}
\sigma(a^2)=a^2, & \sigma(b/a)=b/a, & \sigma(c/a)=c/a, & \sigma(\zeta)=\zeta^{-1},\\
\tau(a^2)=a^2, & \tau(b/a)=b/a, & \tau(c/a)=c/a, & \tau(\zeta)=\zeta^{3},\\
\iota_a(a^2)=-a^2, & \iota_a(b/a)=-ib/a, & \iota_a(c/a)=-ic/a, & \iota_a(\zeta)=\zeta,\\
\iota_b(a^2)=a^2, & \iota_b(b/a)=ib/a, & \iota_b(c/a)=c/a, & \iota_b(\zeta)=\zeta,\\
\iota_c(a^2)=a^2, & \iota_c(b/a)=b/a, & \iota_c(c/a)=ic/a, & \iota_c(\zeta)=\zeta\ .
\end{array}$$

The corresponding action of G on exceptional curves is given by

	σ	τ	ι_a	ι_b	ι_c
$L_{z,\delta,s}$	$L_{z,\sigma(\delta),s}$	$L_{z,\tau(\delta),s}$	$L_{z,i\delta,s}$	$L_{z,-i\delta,s}$	$L_{z,\delta,-s}$
$L_{x,\delta,s}$	$L_{x,\sigma(\delta),s}$	$L_{x,\tau(\delta),s}$	$L_{x,\delta,-s}$	$L_{x,i\delta,s}$	$L_{x,-i\delta,s}$
$L_{y,\delta,s}$	$L_{y,\sigma(\delta),s}$	$L_{y,\tau(\delta),s}$	$L_{y,-i\delta,s}$	$L_{y,\delta,-s}$	$L_{y,i\delta,s}$
$L_{\alpha,\beta,\gamma}$	$L_{\alpha^{-1},\beta^{-1},\gamma^{-1}}$	$L_{i\alpha^{-1},i\beta^{-1},i\gamma^{-1}}$	$L_{i\alpha,\beta,\gamma}$	$L_{\alpha,i\beta,\gamma}$	$L_{\alpha,\beta,i\gamma}$

Now we can build the matrices which encode the action of the various generators on the Picard group $\mathrm{Pic}(S_F)$. For instance for ι_a the matrix is:

$$\begin{pmatrix}
-2 & -1 & -1 & -1 & -1 & -1 & -1 & -3\\
-1 & -2 & -1 & -1 & -1 & -1 & -1 & -3\\
-1 & -1 & -1 & -2 & -1 & -1 & -1 & -3\\
-1 & -1 & 0 & -2 & -1 & -1 & 0 & -2\\
-1 & -1 & -1 & -1 & -1 & 0 & 0 & -2\\
-1 & -1 & -1 & -1 & -2 & -1 & -1 & -3\\
-1 & -1 & 0 & -1 & -1 & 0 & -1 & -2\\
3 & 3 & 2 & 3 & 3 & 2 & 2 & 7
\end{pmatrix}.$$

4. Group cohomology

We start with a review of group cohomology. Let G be a finite group and let M a G-module. A standard free resolution of $\mathbb{Z}$ is given as follows:

$$\mathcal{C}_\bullet^G := \ldots \mathbb{Z}[G\times G\times G] \to \mathbb{Z}[G\times G] \to \mathbb{Z}[G]\ , \tag{4.1}$$

where the augmentation map $\mathbb{Z}[G]\to\mathbb{Z}$ is given by $g\mapsto 1$ (for all $g\in G$) and where each map in $\mathcal{C}_\bullet^G$ is of the form

$$(g_1,\ldots g_n)\mapsto \sum_{i=1}^{n}(-1)^{i+1}(g_1,\ldots,\widehat{g_i},\ldots,g_n)\ .$$

The action of $g \in G$ on any of the terms in (4.1) is the diagonal left multiplication action. We may identify

$$\begin{array}{ccc} \mathbb{Z}[G \times G] & \simeq & \bigoplus_{g' \in G} \mathbb{Z}[G] \ , \\ (g, gg') & \mapsto & (0, \dots, g, \dots, 0) \ , \end{array} \tag{4.2}$$

where the unique non-zero entry g is in the g'-th position. We also identify

$$\begin{array}{ccc} \mathbb{Z}[G \times G \times G] & \simeq & \bigoplus_{(g', g'') \in G \times G} \mathbb{Z}[G] \ , \\ (g, gg', gg'g'') & \mapsto & (0, \dots, g, \dots, 0) \ , \end{array} \tag{4.3}$$

where the unique non-zero entry g is in the (g', g'')-th position.

After these identifications, the complex $\mathrm{Hom}(\mathcal{C}^G_\bullet, M)$ is identified with

$$\mathcal{C}^\bullet_{G,M} := M \xrightarrow{d^0} \bigoplus_{g' \in G} M \xrightarrow{d^1} \bigoplus_{(g', g'') \in G \times G} M \dots \ . \tag{4.4}$$

Here the g'-th coordinate of the map d^0 is $m \mapsto m - g' \cdot m$ and the (g', g'')-th coordinate of d^1 is $(\dots, m_g, \dots) \mapsto m_{g'} - m_{g'g''} + g' \cdot m_{g''}$. Of course, $H^i(G, M)$ is identified with the i-th cohomology of (4.4). For instance, the kernel of d^0 is the module M^G of G-invariants of M.

Now let H be a subgroup of G. Since restriction is an exact functor, $\mathcal{C}^G_\bullet$ is a resolution of $\mathbb{Z}$ as an H-module. We choose a set $Q \subset G$ of coset representatives, so $G = \bigcup_{q \in Q} Hq$.

We have an isomorphism of H-modules

$$\begin{array}{ccc} \mathbb{Z}[G] & \simeq & \bigoplus_{q \in Q} \mathbb{Z}[H] \ , \\ hq & \mapsto & (0, \dots, h, \dots, 0) \ , \end{array} \tag{4.5}$$

where h appears in the q-th position ($h \in H, q \in Q$). Also

$$\begin{array}{ccc} \mathbb{Z}[G \times G] & \simeq & \bigoplus_{(q, h', q') \in Q \times H \times Q} \mathbb{Z}[H] \ , \\ (hq, hh'q') & \mapsto & (0, \dots, h, \dots, 0) \ , \end{array} \tag{4.6}$$

where h appears in the (q, h', q') position. We can project the resolution $\mathcal{C}^G_\bullet$ to the standard resolution $\mathcal{C}^H_\bullet$. Under the identification (4.5) the map on the degree zero component is the sum of the $|Q|$ projection maps. Under the identifications (4.2) and (4.6) the map on the degree 1 component sends the element $(0, \dots, h, \dots, 0)$ from (4.6) to $(0, \dots, h, \dots, 0)$ with h in the h' position. Applying $\mathrm{Hom}_H(-, M)$ we get an inclusion of complexes $\mathcal{C}^\bullet_{H,M}$ into

$\mathrm{Hom}_H(\mathcal{C}_\bullet^G, M)$, and via our identifications,

$$\begin{array}{ccccc} M & \longrightarrow & \bigoplus_H M & \longrightarrow & \cdots \\ \downarrow{\scriptstyle \chi^0} & & \downarrow{\scriptstyle \chi^1} & & \\ \bigoplus_Q M & \longrightarrow & \bigoplus_{Q\times H\times Q} M & \longrightarrow & \cdots \ . \end{array} \tag{4.7}$$

This allows us to take elements of $H^i(H, M)$, represented as cocycles via the standard resolution, and realize them as cocycles in the complex $\mathrm{Hom}_H(\mathcal{C}_\bullet^G, M)$.

Now we discuss cohomology of group extensions. Assume that there is an exact sequence of groups

$$1 \to H \to G \to Q \to 1 \ .$$

Then Q acts on the cohomology $H^q(H, M)$ for all q and there is an associated standard spectral sequence

$$E_2^{p,q} = H^p(Q, H^q(H, M)) \Rightarrow H^{p+q}(G, M) \ . \tag{4.8}$$

This leads to a 5-term exact sequence

(4.9)

$$0 \to H^1(Q, M^H) \to H^1(G, M) \to H^1(H, M)^Q \xrightarrow{d_2^{0,1}} H^2(Q, M^H) \to H^2(G, M) \ .$$

For the purpose of computing $d_2^{0,1}$, we describe explicitly the spectral sequence at the E_0 level. There is an action of Q on the complex $\mathrm{Hom}_H(\mathcal{C}_\bullet^G, M)$, which is induced from the G action on this complex that combines the conjugation of G on itself with its action on M. This Q action has invariants $\mathrm{Hom}_G(\mathcal{C}_\bullet^G, M)$. Moreover, each term is acyclic as a Q-module. This leads to the E_0-term of the spectral sequence (4.8)

$$\begin{array}{ccccc} \mathrm{Hom}_H(\mathbb{Z}[G^3], M) & \longrightarrow & \bigoplus_Q \mathrm{Hom}_H(\mathbb{Z}[G^3], M) & \longrightarrow & \cdots \\ \uparrow{\scriptstyle d_0^{0,1}} & & \uparrow{\scriptstyle d_0^{1,1}} & & \uparrow \\ \mathrm{Hom}_H(\mathbb{Z}[G^2], M) & \longrightarrow & \bigoplus_Q \mathrm{Hom}_H(\mathbb{Z}[G^2], M) & \longrightarrow & \cdots \\ \uparrow{\scriptstyle d_0^{0,0}} & & \uparrow{\scriptstyle d_0^{1,0}} & & \uparrow \\ \mathrm{Hom}_H(\mathbb{Z}[G], M) & \longrightarrow & \bigoplus_Q \mathrm{Hom}_H(\mathbb{Z}[G], M) & \longrightarrow & \bigoplus_{Q^2} \mathrm{Hom}_H(\mathbb{Z}[G], M) \end{array}$$

In the special case where G is a *semi-direct* product of H and Q, we have identifications (4.5) and (4.6). Now $\tilde{q} \in Q$ acts on elements of the groups

appearing in the bottom row of (4.7) as follows:

$$\tilde{q}\cdot(\dots,m_q,\dots)=(\dots,\tilde{q}\cdot m_{\tilde{q}^{-1}q},\dots) \tag{4.10}$$

$$\tilde{q}\cdot(\dots,m_{q,h',q'},\dots)=(\dots,\tilde{q}\cdot m_{\tilde{q}^{-1}q,\tilde{q}^{-1}h'\tilde{q},\tilde{q}^{-1}q'},\dots) \tag{4.11}$$

For finite abelian groups G there are more efficient resolutions than the standard resolution. In each of the following representative cases we give an alternative resolution of $\mathbb{Z}$ by free $\mathbb{Z}[G]$-modules together with explicit maps from the standard resolution to the more efficient resolution. This allows us to compute the images of cocycles from the efficient resolution in the standard resolution.

NOTATIONS 4.1. — Let G be a finite abelian group and $g\in G$ an element of order n. Put $N_g:=1+g+\cdots+g^{n-1}$ and $\Delta_g:=1-g$ in $\mathbb{Z}[G]$. For $g_1,\dots,g_\nu\in G$ and $i_1,\dots,i_\nu\in\mathbb{Z}$ the element in $\mathcal{C}_1^G$ which, under the identification (4.2) is the vector $(0,\dots,1,\dots,0)$ with 1 in the $(g_1^{i_1}g_2^{i_2}\cdots g_\nu^{i_\nu})$-th position, is denoted $\alpha_{i_1,\dots,i_\nu}$. Similarly, given $i'_1,\dots,i'_\nu\in\mathbb{Z}$ the element in $\mathcal{C}_2^G$ which, under the identification (4.3) is the vector $(0,\dots,1,\dots,0)$ with 1 in the $(g_1^{i_1}g_2^{i_2}\cdots g_\nu^{i_\nu},g_1^{i'_1}g_2^{i'_2}\cdots g_\nu^{i'_\nu})$-th position is denoted $\alpha_{i_1,\dots,i_\nu,i'_1,\dots,i'_\nu}$.

Case 1: $G=\mathbb{Z}/n$.

$$\mathcal{C}_\bullet^{[n]}:=\cdots\mathbb{Z}[G]\xrightarrow{N_g}\mathbb{Z}[G]\xrightarrow{\Delta_g}\mathbb{Z}[G]\ ,$$

with quasi-isomorphism

$$\sigma_\bullet^{[n]}:\mathcal{C}_\bullet^G\to\mathcal{C}_\bullet^{[n]}$$

given by

$$\begin{aligned}\sigma_1^{[n]}(\alpha_i)&=1+g+\cdots+g^{i-1}\ ,\\ \sigma_2^{[n]}(\alpha_{i,i'})&=\begin{cases}1 & \text{if } i>i'\\ 0 & \text{otherwise}\end{cases}\ ,\end{aligned} \tag{4.12}$$

where g is a generator of G. For instance, if M is a G-module then $H^i(G,M)$ is the ith cohomology of

$$0\longrightarrow M\xrightarrow{\Delta_g}M\xrightarrow{N_g}M\longrightarrow\cdots\ . \tag{4.13}$$

Case 2: $G=\mathbb{Z}/n\oplus\mathbb{Z}/m$.

$$\mathcal{C}_\bullet^{[n,m]}:=\cdots\mathbb{Z}[G]^3\xrightarrow{A^{[g,h]}}\mathbb{Z}[G]^2\xrightarrow{(\Delta_g\ \Delta_h)}\mathbb{Z}[G]\ ,$$

where

$$A^{[g,h]} := \begin{pmatrix} N_g & \Delta_h & 0 \\ 0 & -\Delta_g & N_h \end{pmatrix}$$

with quasi-isomorphism

$$\sigma_\bullet^{[n,m]} : \mathcal{C}_\bullet^G \to \mathcal{C}_\bullet^{[n,m]}$$

given by

$$\sigma_1^{[n,m]}(\alpha_{i,j}) = (1 + g + \cdots + g^{i-1}, g^i + g^i h + \cdots + g^i h^{j-1}) , \tag{4.14}$$

where g (resp. h) is a generator of $\mathbb{Z}/n$ (resp. $\mathbb{Z}/m$).

Case 3: $G = \mathbb{Z}/n \oplus \mathbb{Z}/m \oplus \mathbb{Z}/\ell$.

$$\mathcal{C}_\bullet^{[n,m,\ell]} := \cdots \mathbb{Z}[G]^6 \xrightarrow{A^{[g,h,u]}} \mathbb{Z}[G]^3 \xrightarrow{(\Delta_g\ \Delta_h\ \Delta_u)} \mathbb{Z}[G] ,$$

where

$$A^{[g,h,u]} := \begin{pmatrix} N_g & \Delta_h & 0 & \Delta_u & 0 & 0 \\ 0 & -\Delta_g & N_h & 0 & \Delta_u & 0 \\ 0 & 0 & 0 & -\Delta_g & -\Delta_h & N_u \end{pmatrix}$$

with quasi-isomorphism

$$\sigma_\bullet^{[n,m,\ell]} : \mathcal{C}_\bullet^G \to \mathcal{C}_\bullet^{[n,m,\ell]}$$

given by

$$\sigma_1^{[n,m,\ell]}(\alpha_{i,j,k}) = \tag{4.15}$$
$$(1 + g + \cdots + g^{i-1}, g^i(1 + h + \cdots + h^{j-1}), g^i h^j (1 + \cdots + g^i h^j u^k)) ,$$

where g (resp. h, resp. u) is a generator of $\mathbb{Z}/n$ (resp. $\mathbb{Z}/m$, $\mathbb{Z}/\ell$).

The existence of efficient resolutions is not limited to the case of abelian groups. For instance, taking $G = \mathfrak{D}_n$, the dihedral group with generators g and h and relations $g^n = h^2 = ghgh = e$, it is an exercise in linear algebra to verify that there is a resolution

$$\cdots \mathbb{Z}[G]^4 \xrightarrow{D_n^3} \mathbb{Z}[G]^3 \xrightarrow{D_n^2} \mathbb{Z}[G]^2 \xrightarrow{D_n^1} \mathbb{Z}[G]$$

with

$$D_n^3 = \begin{pmatrix} \Delta_g & 0 & 0 & N_h \\ 0 & \Delta_h & 0 & -N_g \\ 0 & 0 & \Delta_{gh} & -N_g \end{pmatrix}, \qquad D_n^2 = \begin{pmatrix} N_g & 0 & N_{gh} \\ 0 & N_h & -N_{gh} \end{pmatrix},$$

and $D_n^1 = (\Delta_g\ \Delta_h)$.

In each case, given a G-module M we apply $\mathrm{Hom}_G(-, M)$ to every complex above. This provides a practical method for computing group cohomology of finite abelian groups. The dual maps are notated as above but with the super- and subscripts interchanged. For example, $A_{[g,h]} : M^2 \to M^3$ maps the element $(m, 0)$ to $(m + g \cdot m + \cdots + g^{n-1} \cdot m, m - h \cdot m, 0)$.

5. Computation of $\mathrm{Br}(S)/\mathrm{Br}(\mathbb{Q})$ in the generic case

In this section we explain the computation of $H^1(G, M)$, where $M = \mathrm{Pic}(S_F)$, in the generic case. We have an exact sequence

$$1 \to H \to G \to Q \to 1 \tag{5.1}$$

with $H = (\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/2)$ and $Q = (\mathbb{Z}/2)^2$. In principle, $H^1(G, M)$ can be computed using the standard resolution (4.4). However, in this case the map d_1 would be given by a 131072×1024-matrix, which makes direct computations impractical. Exploiting the fact that G is an extension of one abelian group by another, we can use the spectral sequence technique, explained in Section 4, to simpify the computation significantly.

In the following, we will constantly refer to the diagram in Figure 1. First we compute $M^H = M^G = \mathbb{Z}$, spanned by the canonical class. In particular, $H^1(Q, M^H) = 0$. Thus $H^1(G, M)$ is equal to the kernel of the map

$$d_2^{0,1} : H^1(H, M)^Q \to H^2(Q, M^H) .$$

The group $H^1(H, M)$ is computed by the complex on the left side of the diagram. In this diagram the horizontal arrows labeled $\sigma^i_{[4,4,4]}$ and χ^i give quasi-isomorphisms of complexes. The linear algebra required to compute $\mathrm{Ker}(M^3 \to M^6)$ is quite modest and the cohomology group is identified as

$$H^1(H, M) = \mathbb{Z}/2 .$$

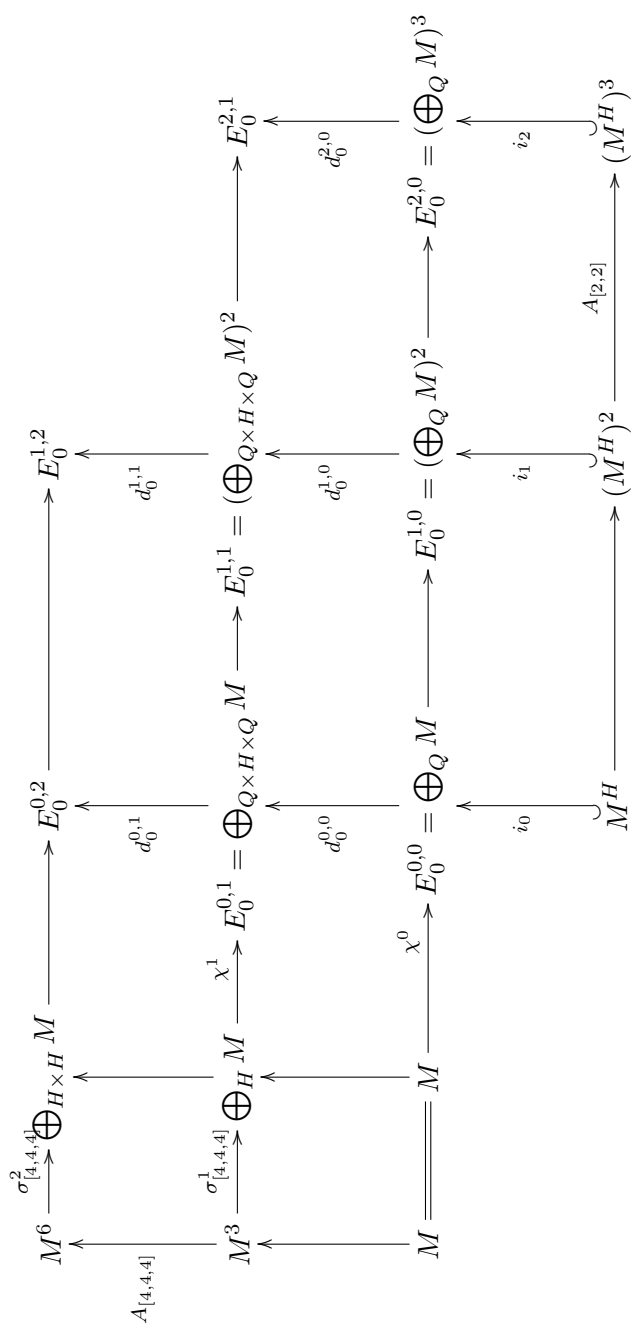
It remains to take a single cocycle representative of the non-zero element of $H^1(H, M)$ and follow it through the diagram to determine whether it lies in the kernel of $d_2^{0,1}$.

Remark 5.1. — In this case the class is automatically Q-invariant since $\mathbb{Z}/2$ has only the identity automorphism. In general, as we point out below, at a certain point in the diagram chase this invariance is naturally tested.

We start with a representative in M^3 for the nontrivial element $\lambda \in H^1(H, M)$, for instance

$$u = ((0, 0, 0, 0, -1, -1, -1, 1), (0, 0, 0, 0, -1, 1, 0, 0), (0, 0, 0, 0, -2, 0, -1, 1)).$$

FIGURE 1. E_0 spectral sequence



Let v denote the image in $E_0^{1,1}$ of v by the composite of three horizontal maps in Figure 1. Now, we claim, v lies in the image of $d_0^{1,0}$ if and only if λ is Q-invariant (obviously true in this case). Indeed, a linear algebra solver can produce

$$v_0 = ((0,0,0,0,-1,1,0,0)^{*4}, (0,0,0,0,-1,-1,-1,1)^{*4}))$$

satisfying $d_0^{1,0}(v_0) = v$, where each vector with superscript $*4$ denotes the element in $\bigoplus_Q M$ with the vector repeated 4 times. Applying the coboundary map $E_0^{1,0} \to E_0^{2,0}$ to v_0 necessarily produces an element in the image of i_2, representing $d_2^{0,1}(\lambda)$ in $H^2(Q, M^H)$. In the present case, we get 0; in general, a linear solver can test whether or not it is a coboundary.

6. The non-generic case

We start by presenting some examples when the Galois group is smaller than in the generic case.

Example 6.1. — Consider the case $(A,B,C) = (-6,-3,2)$. The Galois group of the field F, defined in (3.1), has order 32; it is an extension of the Klein four-group by $(\mathbb{Z}/4) \oplus (\mathbb{Z}/2)$. However, in this way it is not a split extension. On the other hand, we can use the split extension

$$1 \to H \to G \to \mathbb{Z}/2 \to 1$$

where $H = (\mathbb{Z}/4)^2$, generated by $\iota_a\iota_b$ and $\sigma\tau\iota_a\iota_c$, and the image of $\mathbb{Z}/2$ in G is generated by σ. In this case, we compute $H^1(H,M) = 0$. By (4.9), $H^1(G,M)$ is isomorphic to $H^1(\mathbb{Z}/2, M^H)$. We find that M^H has rank 2, spanned by

$$(1,1,1,1,1,1,1,-3), \qquad (1,1,1,1,1,1,0,-2),$$

hence M^H is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}'$, where $\mathbb{Z}'$ is free of rank 1 with non-trivial $\mathbb{Z}/2$-action. So, we have

$$H^1(G,M) = \mathbb{Z}/2.$$

As in the generic case, we have $M^G = \mathbb{Z}$, that is $\mathrm{Pic}(S)$ has rank 1.

Example 6.2. — The case $(A,B,C) = (1,1,-2)$ is interesting because $\mathrm{Pic}(S)$ has rank 2. The Galois group G fits into an exact sequence

$$1 \to \mathbb{Z}/4 \to G \to \mathbb{Z}/2 \to 1$$

with subgroup $H = \mathbb{Z}/4$ generated by $\iota_c\sigma\tau$ and quotient group generated by the image of τ. As in Example 6.1 we have $H^1(H, M) = 0$. Now M^H has rank 3, with generators

$$(1,1,1,1,1,1,1,-3)\ , \qquad (0,0,0,0,1,-1,0,0)\ , \qquad (0,0,0,0,1,1,1,-1)\ ,$$

and the action of τ fixes the first 2 vectors and negates the third. Hence

$$H^1(G, M) = \mathbb{Z}/2$$

and $\mathrm{Pic}(S)$ has rank 2.

EXAMPLE 6.3. — The case $(A, B, C) = (1, 1, 1)$ yields $G = \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$, the Klein four-group, and we directly compute

$$H^1(G, M) = (\mathbb{Z}/2)^3.$$

In this case $\mathrm{Pic}(S)$ has rank 1.

The comprehensive treatment proceeds via a case-by-case computer analysis of all subgroups of the generic Galois group, such that in the exact sequence (5.1) the subgroup maps onto Q. These correspond to field extensions containing the eighth roots of unity. Each such subgroup (it is enough to consider conjugacy classes of subgroups) is a semi-direct product of abelian groups, and hence yields to the sort of analysis of the examples of this section. By means of an exhaustive computer run, we obtain

THEOREM 6.4. — *Let S have the form (1.1). Then $\mathrm{Br}(S)/\mathrm{Br}(\mathbb{Q})$ is isomorphic to one of the following groups:*

$$(1),\ \mathbb{Z}/2,\ \mathbb{Z}/4,\ (\mathbb{Z}/2)\oplus(\mathbb{Z}/2),$$
$$(\mathbb{Z}/4)\oplus(\mathbb{Z}/2),\ (\mathbb{Z}/2)\oplus(\mathbb{Z}/2)\oplus(\mathbb{Z}/2).$$

7. Examples of Brauer-Manin obstruction

Here we compute the Brauer-Manin obstruction to the Hasse principle in several representative cases. In all the examples below the surface S has points in all completions of $\mathbb{Q}$.

EXAMPLE 7.1. — The case $(A, B, C) = (-25, -5, 45)$. The Galois group $G = \mathrm{Gal}(F/\mathbb{Q})$ has order 32 and fits into an exact sequence

$$1 \to H \to G \to \mathbb{Z}/2 \to 1.$$

We have $H = (\mathbb{Z}/4) \oplus (\mathbb{Z}/2)^2$, generated by $\iota_a^2\iota_b\iota_c$, ι_c^2, and $\sigma\tau$. A complement to H in G is $\mathbb{Z}/2 = \langle\sigma\iota_a\iota_b\rangle$. After the sort of computation detailed in the previous section we find

$$H^1(\mathbb{Z}/2, M^H) \xrightarrow{\sim} H^1(G, M) \tag{7.1}$$

in (4.9) with M^H spanned by $(1,1,1,1,1,1,1,-3)$ and $(1,1,1,1,1,1,0,-2)$. Hence, as in Example 6.1, we have

$$H^1(G, M) = \mathbb{Z}/2.$$

In light of (7.1) we are in the nice situation where $\mathrm{Br}(S)$ has elements (that are not in $\mathrm{Br}(\mathbb{Q})$) annihilated by the base extension $\mathbb{Q} \to \mathbb{Q}[i] = F^H$. Specifically, if we let ε be a divisor on $S_{\mathbb{Q}[i]}$ whose class in $M = \mathrm{Pic}(S_F)$ is $(1,1,1,1,1,1,0,-2)$, then the nonzero element of $\mathrm{Br}(S)/\mathrm{Br}(\mathbb{Q})$ is represented by the quaternion algebra

$$(-1, g)$$

where g is any rational function on S satisfying $div(g) = \alpha + \overline{\alpha}$; here the bar denotes complex conjugation. Let D be the curve given by

$$-5x^2 - 2y^2 + 9z^2 = 0 \qquad \text{and} \qquad w = i(3y^2 - 6z^2).$$

Then the cycle

$$\alpha = D - (z = 0)$$

(the sum of the curve and a canonical divisor) has the required class in M. So with

$$g = -5(x/z)^2 - 2(y/z)^2 + 9,$$

we have $(-1, g) \in \mathrm{Br}(S)$.

Observe that the image of Z in $\mathbb{P}^2$ is the conic having four tangencies with the quartic (2.1), at the points $(\pm 1 : \pm\sqrt{2} : 1)$. So we find

$$g > 0$$

at all $\mathbb{Q}$-points of the plane. It is necessary to complete p-adic analysis only at the primes $p = 2$ and $p = 3$ (since 5-adically, $\mathbb{Q}[i]$ is a split extension of $\mathbb{Q}$). For the 2-adic analysis, we assume x, y, and z to be 2-adic integers, not all even, and find by analysis mod 16 that the condition $-25x^4 - 5y^4 + 45z^4$ should be a 2-adic square implies x and z are odd and y is even. So, without loss of generality, we may take $z = 1$. By mod 32 analysis, the only possible values of (x, y) mod 8 are

$$(1,2), \quad (1,6), \quad (3,0), \quad (3,4), \quad (5,0), \quad (5,4), \quad (7,2), \quad (7,6).$$

In each case we find $g = 12 \pmod{16}$, hence $(-1, g)$ is *ramified* at all 2-adic points of S. By a similar analysis mod 27 we find that at any 3-adic point x and y are prime to 3, hence so is g, and $(-1, g)$ is *unramified* at all 3-adic points of S.

Therefore S provides an example of Brauer-Manin obstruction to the Hasse principle.

EXAMPLE 7.2. — Here we show that Example 6.1 fits into an infinite family of examples of Brauer-Manin obstruction to the Hasse principle. Consider

$$(A, B, C) = (-2p, -p, 2),$$

where p is any prime such that

$$p = 3 \pmod{16}.$$

The computation of the group cohomology is exactly as in Example 6.1. So, $H^1(G, M) = H^1(\mathbb{Z}/2, M^H) = \mathbb{Z}/2$. We proceed as in Example 7.1.

By the condition on p we may write

$$p = u^2 + 2v^2$$

for integers u and v. Now we compute the equation of the conic tangent to the quartic at the points $(\pm\sqrt{u/p}, \pm\sqrt{2v/p})$ and find that with the curve D given by

$$-ux^2 - vy^2 + z^2 = 0 \qquad \text{and} \qquad w = i(-2vx^2 + uy^2)$$

the cycle $\alpha = D - (z = 0)$ has the correct class in M. Now we analyze the ramification pattern of $(-1, g)$ where

$$g = -u(x/z)^2 - v(y/z)^2 + 1.$$

The analysis is simplest if we assume

$$u = 1 \pmod 4 \qquad \text{and} \qquad v = 1 \pmod 4$$

(until now, u and v have been determined only up to sign; this fixes the signs). Then $(-1, g)$ is

(i) *unramified* over $\mathbb{R}$ (at any rational point);
(ii) *ramified* at all 2-adic points of S;
(iii) *unramified* at all p-adic points of S;

and there is a Brauer-Manin obstruction to the Hasse principle.

We leave the verification of (i)–(ii) to the reader. For (iii) we need the following lemma.

LEMMA 7.3. — *Let p be a prime with $p = 3 \pmod{16}$. Write $p = u^2 + 2v^2$ for integers u and v with $u = 1 \pmod 4$ and $v = 1 \pmod 4$. Now, if we let y be a solution to $y^4 = -2 \pmod p$ then we have $vy^2 = u \pmod p$.*

Proof. — The two square roots of -2 mod p are $\pm uv^{-1}$. So $y^2 = \pm uv^{-1} \pmod p$ and the lemma is asserting that the correct sign is $+$, or equivalently, that uv is a quadratic residue mod p. By quadratic reciprocity,

$$\left(\frac{u}{p}\right) = \left(\frac{p}{u}\right) \qquad \text{and} \qquad \left(\frac{v}{p}\right) = \left(\frac{p}{v}\right).$$

If p' is a prime dividing v, then p is a quadratic residue mod p'. This and a similar consideration when p' divides u yield

$$\left(\frac{p}{v}\right) = 1 \qquad \text{and} \qquad \left(\frac{2p}{u}\right) = 1.$$

By mod 16 analysis, u is congruent to 1 or 7 mod 8, hence

$$\left(\frac{2}{u}\right) = 1.$$

So, as required, uv is a quadratic residue mod p. □

To establish (iii) we claim that for any p-adic integer solution (w, x, y, z) to (1.1), with not all of w, x, y, and z divisible by p, the p-adic integer $z^2 g = -ux^2 - vy^2 + z^2$ is not divisible by p. Indeed, since 2 is not a quadratic residue mod p we must have p dividing z, hence x and y are nonzero mod p. Without loss of generality we suppose $x = 1$. Now y must be a fourth root of -2 mod p. The claim follows from Lemma 7.3.

EXAMPLE 7.4. — Here we give a recipe for testing the presence of Brauer-Manin obstruction to the Hasse principle in the generic case. This is the case when the Galois group has order 128, or equivalently, when the set

$$\left\{ \varepsilon A^\alpha B^\beta C^\gamma \;\middle|\; \varepsilon \in \{-2,-1,1,2\} \text{ and } (\alpha,\beta,\gamma) \in \{0,1\}^3 \smallsetminus \{(0,0,0)\} \right\}$$

contains no perfect squares.

As we computed in Section 5, we have $\mathrm{Br}(S)/\mathrm{Br}(\mathbb{Q}) = H^1(G, M) = \mathbb{Z}/2$. Since G has a subgroup of index two

$$H = \langle \sigma\tau, \iota_a^2, \iota_a\iota_b, \iota_a\iota_c, \iota_a\sigma \rangle$$

with the property that

$$M^H = \langle\, (1,1,1,1,1,1,1,-3), (1,1,1,1,1,1,0,-2) \,\rangle,$$

we have $H^1(G/H, M^H) \xrightarrow{\sim} H^1(G, M)$. Therefore, we can construct a quaternion algebra as in Example 7.1. In this case,

$$F^H = \mathbb{Q}(\sqrt{-ABC}).$$

Let $\theta = \sqrt{-ABC}$, and let $(r_0 : s_0 : t_0)$ be a $\mathbb{Q}(\theta)$-rational point on the conic

$$Ar^2 + Bs^2 + Ct^2 = 0$$

(this exists by the Hasse principle). Then

$$Ar_0x^2 + Bs_0y^2 + Ct_0z^2 = 0$$

defines a conic over $\mathbb{Q}(\theta)$, meeting the quartic curve (2.1) in 4 tangencies. We have the identity

$$C^2t_0^2(Ax^4 + By^4 + Cz^4) + ABC(s_0x^2 - r_0y^2)^2 \\ + C(Ar_0x^2 + Bs_0y^2 + Ct_0z^2)(Ar_0x^2 + Bs_0y^2 - Ct_0z^2) = 0.$$

Hence there is a curve D on $S_{\mathbb{Q}(\theta)}$ defined by

$$Ar_0x^2 + Bs_0y^2 + Ct_0z^2 = 0 \qquad \text{and} \qquad w = \theta(s_0x^2 - r_0y^2)/(Ct_0)$$

such that the union of D and its conjugate is rationally equivalent to twice the anticanonical class. This rational equivalence is given explicitly by the rational function

$$g := (Ar_1s_1 + A^2BCr_2s_2) + (Bs_1^2 - A^2BCr_2^2)(y/x)^2 + Cs_1t_0(z/x)^2 + ACr_2t_0w/x^2,$$

where we suppose $t_0 \in \mathbb{Q}$ and write

$$r_0 = r_1 + r_2\theta \qquad \text{and} \qquad s_0 = s_1 + s_2\theta.$$

To test the Brauer-Manin obstruction to the Hasse principle for S, one has to analyze the quaternion algebra

$$(-ABC, g)$$

at real- and $\mathbb{Q}_p$-valued points of S (for p dividing $2ABC$).

To see that this case sometimes yields a nontrivial Brauer-Manin obstruction, consider $(A, B, C) = (-126, -91, 78)$. Here we may take $r_0 = -13$, $s_0 = -12$, and $t_0 = 21$, and g is proportional to

$$3 + 3(y/x)^2 + 2(z/x)^2.$$

In this case the quaternion algebra $(-ABC, 3 + 2(y/x)^2 + 3(z/x)^2)$ is ramified at all $\mathbb{Q}_2$-points of S and unramified at all points in other completions.

EXAMPLE 7.5. — The case $(A, B, C) = (34, 34, 34)$. Here $G = \mathrm{Gal}(F/\mathbb{Q})$ is isomorphic to $(\mathbb{Z}/2)^3$:

$$G = \langle \iota_a\iota_b\iota_c\sigma, \tau, \sigma\rangle.$$

We have $H^1(G, M) = (\mathbb{Z}/2)^3$. In fact, for the index-two subgroup

$$H = \langle \iota_a\iota_b\iota_c\sigma, \tau\rangle$$

we have M^H spanned by

$$\begin{array}{l}(1,-1,0,0,0,0,0,0),\\ (0,0,1,-1,0,0,0,0),\\ (0,0,0,0,1,-1,0,0),\\ (1,1,1,1,1,1,1,-3),\end{array} \tag{7.2}$$

and

$$H^1(G/H, M^H) \xrightarrow{\sim} H^1(G, M).$$

Here, of course, σ in G/H acts nontrivially on the first three vectors in (7.2) and trivially on the last. We have

$$F^H = \mathbb{Q}(\sqrt{-17}).$$

Using (4.13) we can identify elements of $\mathrm{Br}(S)/\mathrm{Br}(\mathbb{Q})$ with the image of the (-1)-eigenspace of M^H (under the σ-action). To produce quaternion algebras representing a given element of $\mathrm{Br}(S)/\mathrm{Br}(\mathbb{Q})$ we need to find divisors defined over $\mathbb{Q}(\sqrt{-17})$ representing particular classes in M^H. Notice that the class of any combination of exceptional curves defined over $\mathbb{Q}(\sqrt{-17})$ in M^H is a coboundary of (4.13). Hence, we need additional cycles defined over $\mathbb{Q}(\sqrt{-17})$. We use descent to produce line bundles on $S_{\mathbb{Q}(\sqrt{-17})}$ and obtain the desired cycles as loci of vanishing of rational sections of these line bundles.

Here we explicitly carry out the task of representing the class of the first entry of (7.2) in $\mathrm{Br}(S)$. Set $\rho = \iota_a\iota_b\iota_c\sigma$. Over $F = \mathbb{Q}(\sqrt{-17}, \zeta)$ we have

$$[L_{x,\zeta,+}] - [L_{x,\zeta^3,-}] = (1,-1,0,0,0,0,0,0) \tag{7.3}$$

in $\mathrm{Pic}(S_F)$. Consider the line bundle $\mathcal{O}([L_{x,\zeta,+}] - [L_{x,\zeta^3,-}])$ together with isomorphisms

$$\mathcal{O}(L_{x,\zeta,+} - L_{x,\zeta^3,-}) \xrightarrow{\eta} \mathcal{O}(L_{x,\zeta^7,-} - L_{x,\zeta^5,+})$$

and

$$\mathcal{O}(L_{x,\zeta,+} - L_{x,\zeta^3,-}) \xrightarrow{\xi} \mathcal{O}(L_{x,\zeta^3,+} - L_{x,\zeta,-}).$$

These constitute descent data (for the covering $S_F \to S_{\mathbb{Q}(\sqrt{-17})}$) provided that the diagram

$$\begin{array}{ccccc}
\mathcal{O}(L_{x,\zeta,+} - L_{x,\zeta^3,-}) & \xrightarrow{\eta} & \mathcal{O}(L_{x,\zeta^7,-} - L_{x,\zeta^5,+}) & \xrightarrow{\rho(\eta)} & \mathcal{O}(L_{x,\zeta,+} - L_{x,\zeta^3,-}) \\
\downarrow \xi & & \downarrow \rho(\xi) & & \\
\mathcal{O}(L_{x,\zeta^3,+} - L_{x,\zeta,-}) & \xrightarrow{\tau(\eta)} & \mathcal{O}(L_{x,\zeta^5,-} - L_{x,\zeta^7,+}) & & \\
\downarrow \tau(\xi) & & & & \\
\mathcal{O}(L_{x,\zeta,+} - L_{x,\zeta^3,-}) & & & &
\end{array}$$

(with arrows labelled 1 from the top-left entry to the top-right entry and from the top-left entry to the bottom entry)

commutes. The isomorphisms given by

$$\eta = \delta \frac{x^2 - iy^2 + z^2 - \frac{1}{\sqrt{34}} w}{x^2 - iy^2 - z^2 + \frac{1}{\sqrt{34}} w} \qquad \text{and} \qquad \xi = \varepsilon \frac{\zeta y + z}{\zeta^3 y + z},$$

satisfy this condition if and only if δ, $\eta \in F$ satisfy

$$\begin{aligned} \delta\, \rho(\delta) &= -1, \\ \varepsilon\, \tau(\varepsilon) &= 1, \\ \delta\, \rho(\varepsilon) &= \tau(\delta)\, \varepsilon. \end{aligned} \tag{7.4}$$

We find that one solution is

$$\delta = \sqrt{-17}\zeta - 4\zeta^3 \qquad \text{and} \qquad \varepsilon = 4\zeta + \sqrt{-17}\zeta^3.$$

Then, by effective descent, we obtain a line bundle $\mathcal{E}$ on $S_{\mathbb{Q}(\sqrt{-17})}$.

Using (7.4) and descent, we see that

$$f := 1 + \rho(\eta) + \tau(\xi) + \rho(\eta\, \tau(\xi))$$

defines a rational section of $\mathcal{E}$. We write f as a quotient of quartic polynomials and observe that f has (with respect to local trivializations of $\mathcal{E}$) a simple pole along $L_{x,\zeta,-} \cup L_{x,\zeta^3,-} \cup L_{x,\zeta^5,+} \cup L_{x,\zeta^7,+}$ and a zero of order one along some curve Z. Then, by (7.3), we deduce that

$$[Z] = (-3, -1, -2, -2, -2, -2, -2, 6)$$

in the Picard group. Therefore, if $h \in \mathbb{Q}(S)$ defines a rational equivalence between $Z \cup \sigma(Z)$ and some hyperplane sections, then the quaternion algebra $(-17, h)$ represents an element of $\mathrm{Br}(S)$ of the desired class in $\mathrm{Br}(S)/\mathrm{Br}(\mathbb{Q})$.

Denoting by g the numerator of f, we have

$$g = (x^2 + iy^2 + z^2 + \frac{1}{\sqrt{34}}w)[y^2 + iz^2 + (4\zeta - \sqrt{-17}\zeta^3)(y^2 + \sqrt{2}yz + z^2)]$$
$$+ (x^2 + iy^2 - z^2 - \frac{1}{\sqrt{34}}w)[y^2 + \sqrt{2}yz + z^2 + (4\zeta - \sqrt{-17}\zeta^3)(-y^2 + iz^2)].$$

The simultaneous vanishing of g, $\rho(g)$, $\tau(g)$, and $\rho\tau(g)$ defines the curve Z. Equivalently, writing

$$g = p_0 + p_1\zeta + p_2\zeta^2 + p_3\zeta^3$$

with $p_i \in \mathbb{Q}(\sqrt{-17})[w, x, y, z]$ we have Z defined by the vanishing p_i for $i = 0$, …, 3. A unique (up to scale) $\mathbb{Q}(\sqrt{-17})$ linear combination of these is defined over $\mathbb{Q}$, namely

$$\begin{aligned} h_1 &:= \frac{1}{2}p_0 + \frac{4 - \sqrt{-17}}{2}p_1 + \frac{1}{2}p_2 - \frac{4 + \sqrt{-17}}{2}p_3 \\ &= wy^2 + wz^2 + x^2y^2 + 8x^2yz + x^2z^2 + y^4 - z^4. \end{aligned}$$

Then $h = h_1/x^4$ is as desired. Cyclically permuting the variables x, y, and z, we obtain polynomials h_2 and h_3 such that the classes of $(-17, h_i/x^4)$ generate $\mathrm{Br}(S)/\mathrm{Br}(\mathbb{Q})$.

The ramification pattern of an Azumaya algebra is an invariant of its class in $\mathrm{Br}(S)$. However, in practice, the ramification pattern of an algebra $(-17, h_i/x^4)$ is difficult to test on p-adic points where h_i vanishes. Hence, it is helpful to have several rational functions that determine the same class in $\mathrm{Br}(S)/\mathrm{Br}(\mathbb{Q})$. We can obtain additional such functions by repeating the previous construction for different choices of δ and ε satisfying (7.4). In the present case, it is easier to use symmetry to obtain these functions. Allowing all permutations of the variables x, y, and z, we obtain additional polynomials h_4, h_5, and h_6 such that $(-17, h_i/x^4)$ and $(-17, h_{i+3}/x^4)$ represent the same class in $\mathrm{Br}(S)/\mathrm{Br}(\mathbb{Q})$ for all i.

Let $\mathfrak{q}_i \in \mathrm{Br}(S)$ denote the class of $(-17, h_i/x^4)$, for each i. Here are the results of the local analysis, confirming the presence of a Brauer-Manin obstruction:

- $\mathfrak{q}_i$ is unramified on $S(\mathbb{R})$, for all i;
- $S(\mathbb{Q}_2)$ is the disjoint union of two nonempty sets, U and R, such that each $\mathfrak{q}_i$ is unramified on U, and such that for $1 \le i \le 3$ and any $r \in R$, one of $\{\mathfrak{q}_i, \mathfrak{q}_{i+3}\}$ (and *a posteriori* the other as well) is ramified at r;
- $\mathfrak{q}_i$ is unramified on $S(\mathbb{Q}_p)$ for all i and $p \notin \{2, 17\}$, and hence in particular $\mathfrak{q}_i = \mathfrak{q}_{i+3}$ for $1 \le i \le 3$;

– At any point of $S(\mathbb{Q}_{17})$, exactly two of $\{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3\}$ are ramified.

EXAMPLE 7.6. — The case $(A, B, C) = (-9826, -2, 136) = (-2p^3, -2, 8p)$ where $p = 17$ illustrates working with a non-cyclic Azumaya algebra. We have $F = \mathbb{Q}(\zeta, \sqrt[4]{p})$. The Galois group of F over $\mathbb{Q}$ has order 16:

$$G = \langle \iota_a\iota_b\iota_c\sigma\tau, \iota_a^3\iota_c, \iota_b\iota_c^3\sigma\rangle.$$

In this case, $H^1(G, M) = \mathbb{Z}/4$. We leave it to the interested reader to produce a subgroup H of index 2 with $H^1(G/H, M^H) = \mathbb{Z}/2$ and show that

$$(-2, 136 + (y/x)^2 + 18(z/x)^2)$$

represents the unique nontrivial 2-torsion element of $\mathrm{Br}(S)/\mathrm{Br}(\mathbb{Q})$ and is unramified at all points S valued in all completions of $\mathbb{Q}$. So the obstruction analysis requires a representative of a generator of $\mathrm{Br}(S)/\mathrm{Br}(\mathbb{Q})$.

The central element $u := \iota_a\iota_b\iota_c\sigma\tau$ of G satisfies

$$F^u = \mathbb{Q}(i, \sqrt[4]{p})$$

and

$$H^1(G/\langle u\rangle, M^u) = \mathbb{Z}/4.$$

We remark that the exceptional curves $L_{\alpha,\beta,\gamma}$ (α, β, $\gamma \in \mu_4$) are defined over F^u. The quotient $G' := G/\langle u\rangle$ is isomorphic to the dihedral group $\mathfrak{D}_4$; generators $g := \iota_a^3\iota_c$ and $h := \iota_b\iota_c^3\sigma$ satisfy $g^4 = h^2 = ghgh = e$. We use the resolution of Section 4 to identify classes in $H^1(G', M^u)$ with pairs $(v, v') \in (M^u)^2$ satisfying

$$N_g v = N_h v' = 0 \qquad \text{and} \qquad N_{gh} v = N_{gh} v', \tag{7.5}$$

modulo those of the form $(v - gv, v - hv)$. Now a generator of $H^1(G', M^u)$ is the class of $(v_1, 0)$ where

$$v_1 = (-1, 0, 1, 0, 0, 0, 0, 0) = [L_{1,i,1}] - [L_{i,-1,-1}]. \tag{7.6}$$

Another representative for the same cohomology class is $(v_2, 0)$ where

$$v_2 = (-1, 0, -1, 0, -1, -1, -2, 2) = [L_{1,-1,i}] - [L_{i,i,i}]. \tag{7.7}$$

To produce an Azumaya algebra from one of these cocycles $(v_i, 0)$ we must find rational equivalences that reflect the identities (7.5). Luckily, for each of the cycle representatives given in (7.6) and (7.7), the result of applying N_{gh} is equal to zero as a cycle. So it remains only to find rational functions whose divisors are N_g applied to these cycle representatives. For (7.6), a function

that vanishes on $L_{1,i,1} \cup L_{i,-i,-i} \cup L_{1,-i,1} \cup L_{i,i,-i}$ and has a simple pole along $L_{i,-1,-1} \cup L_{1,-1,-i} \cup L_{i,1,-1} \cup L_{1,1,-i}$ is

$$f_1 := \frac{iy^2 + p(1+i)xz - (1/2)w}{iy^2 + p(-1+i)xz + (1/2)w}.$$

The corresponding rational equivalence for (7.7) is

$$f_2 := \frac{iy^2 + p(1-i)xz + (1/2)w}{iy^2 + p(-1-i)xz - (1/2)w}.$$

For $i = 1$ and 2 we have $f_i\, h(f_i) = 1$, hence

$$(f_i, 1, 1) \in (F^u(S)^*)^3$$

is the cocycle data for an Azumaya algebra $\mathfrak{A}_i$ on S.

We claim $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are equal in $\mathrm{Br}(S)$ and are:

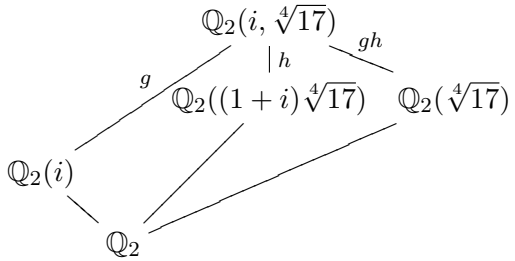
- unramified at all points of $S(\mathbb{Q}_2)$;
- ramified at all points of $S(\mathbb{Q}_{17})$;
- unramified at all points of $S(\mathbb{R})$.

The last of these claims is clear, since $(f_i, 1, 1) \in S^1 \times \{1\} \times \{1\}$ at any point of $S(\mathbb{R})$ (where $S^1 \subset \mathbb{C}^*$ denotes the unit circle) and this is a connected subgroup of the group of cocycles, hence trivial in cohomology.

For the claim regarding 2-adic points, we pause to discuss the cohomology group $H^2(\mathfrak{D}_4, \mathbb{Q}_2(i, \sqrt[4]{17})^*)$, where generators act by

$$g\colon \begin{cases} i \mapsto i \\ \sqrt[4]{17} \mapsto -i\sqrt[4]{17} \end{cases} \qquad h\colon \begin{cases} i \mapsto -i \\ \sqrt[4]{17} \mapsto i\sqrt[4]{17} \end{cases}$$

Consider the diagram of field extensions, where labels indicate fixed fields.



Now by the resolution for $\mathfrak{D}_4$ of Section 4, a 2-cocycle is (r, s, t) with

$$r \in \mathbb{Q}_2(i)^*, \qquad s \in \mathbb{Q}_2((1+i)\sqrt[4]{17})^*, \qquad t \in \mathbb{Q}_2(\sqrt[4]{17})^*$$

satisfying

$$Nr = Ns\,Nt,$$

where in each instance, N denotes the norm from the respective field to $\mathbb{Q}_2$. Coboundaries are triples

$$(N_g c, N_h d, N_{gh}(c/d))$$

for c, $d \in \mathbb{Q}_2(i, \sqrt[4]{17})^*$.

At every 2-adic point of S, at least one of f_1 and f_2 is well-defined; at some points, both are. The only values that occur mod 32 are:

$$\begin{array}{cccccccc} 1{+}0i & 1{+}8i & 1{+}16i & 1{+}24i & 25{+}4i & 25{+}12i & 25{+}20i & 25{+}28i \\ 0{+}31i & 8{+}31i & 16{+}31i & 24{+}31i & 4{+}7i & 12{+}7i & 20{+}7i & 28{+}7i \\ 31{+}0i & 31{+}24i & 31{+}16i & 31{+}8i & 7{+}28i & 7{+}20i & 7{+}12i & 7{+}4i \\ 0{+}i & 24{+}i & 16{+}i & 8{+}i & 28{+}25i & 20{+}25i & 12{+}25i & 4{+}25i \end{array} \tag{7.8}$$

We claim that for any cocycle $(f, 1, 1)$ with f (necessarily in $\mathbb{Z}_2[i]$) taking one of the values mod 32 listed in (7.8), there exists $c \in \mathbb{Q}_2(i, \sqrt[4]{17})^*$ with $N_{gh}c = 1$ and $N_g c = f$. In particular, $(f, 1, 1)$ is a coboundary. Indeed, the image of N_g among $c \in \mathbb{Z}_2[i, \sqrt[4]{17}]$ satisfying $N_{gh}c = 1$ is the set of $f \in \mathbb{Q}(i)^*$ with $Nf = 1$ and f mod 32 equal to one of the values in the first row of (7.8). Also, there exists $c \in \mathbb{Q}_2(i, \sqrt[4]{17})^*$ with $N_{gh}c = 1$ and $N_g c = i$. Since norms are multiplicative, the claim follows and the 2-adic analysis is complete.

The 17-adic analysis is simpler because $\mathbb{Q}_{17}$ has square roots of -1, and hence we are reduced to analyzing norms for the extension $\mathbb{Q}_{17} \to \mathbb{Q}_{17}(\sqrt[4]{17})$. Norms for this extensions are precisely powers of 17 times fourth powers in $\mathbb{Z}_{17}^*$. Evaluating f_1 at points of $S(\mathbb{Q}_{17})$ and substituting a square root of -1 for i, we get only the classes 8 and 15 mod 17, and these are not fourth powers.

8. Appendix: Cyclic Azumaya algebras on diagonal cubics

In [4], there is an analysis of the Brauer-Manin obstruction on a diagonal cubic surface S, given by

$$Ax^3 + By^3 + Cz^3 + Dt^3 = 0, \tag{8.1}$$

with A, B, C, and D positive integers. Let $\theta = e^{2\pi i/3}$; first of all, $S(\mathbb{Q}) = \emptyset$ if and only if $S(\mathbb{Q}(\theta)) = \emptyset$, and hence it suffices to work over the field $k := \mathbb{Q}(\theta)$. The analysis proceeds by constructing Azumaya algebras that are split by a bicyclic extension of k and computing local invariants.

Here we simplify the algorithm proposed in op. cit. by constructing *cyclic* Azumaya algebras on S_k which generate $\mathrm{Br}(S_k)/\mathrm{Br}(k)$. We use descent to exhibit the necessary cycles, as in Example 7.5.

We start by making the following assumption:

$$\begin{gathered}\sqrt[3]{A/B} \notin \mathbb{Q},\ \sqrt[3]{A/C} \notin \mathbb{Q},\ \dots,\ \sqrt[3]{C/D} \notin \mathbb{Q}\\ \sqrt[3]{AB/CD} \notin \mathbb{Q},\ \sqrt[3]{AC/BD} \notin \mathbb{Q},\ \sqrt[3]{AD/BC} \notin \mathbb{Q}\end{gathered} \tag{8.2}$$

(in all other cases, the Hasse principle is known to hold). Then we define

$$\alpha = \sqrt[3]{B/A} \qquad \beta = \sqrt[3]{D/C} \qquad \gamma = \sqrt[3]{AD/BC} = \alpha^{-1}\beta$$
$$\alpha' = \sqrt[3]{C/A} \qquad \beta' = \sqrt[3]{D/B}$$

We assume, further, that $S(\mathbb{Q}_p) \neq \emptyset$ for all primes p. Set $K = k(\gamma, \alpha)$; the assumption (8.2) implies

$$[K : k] = 9. \tag{8.3}$$

We need notation for the following divisors on $S_{\bar{k}}$:

$$L(i)\colon\begin{cases} x+\theta^i\alpha y = 0\\ z+\theta^i\beta t = 0\end{cases} \quad L'(i)\colon\begin{cases} x+\theta^i\alpha y = 0\\ z+\theta^{i+1}\beta t = 0\end{cases} \quad L''(i)\colon\begin{cases} x+\theta^i\alpha y = 0\\ z+\theta^{i+2}\beta t = 0\end{cases}$$

and

$$M(i)\colon\begin{cases} x+\theta^i\alpha' z = 0\\ y+\theta^{i+1}\beta' t = 0\end{cases}$$

Define

$$L = L(0) + L(1) + L(2) \qquad \text{and} \qquad M = M(0) + M(1) + M(2).$$

Now $L + M$ is comprised of 6 pairwise disjoint lines; blowing these down we have $S_{\bar{k}} \to \mathbb{P}^2_{\bar{k}}$. Take ℓ to be the class of a general line in $\mathbb{P}^2_{\bar{k}}$, so

$$3\ell = -K_S + L + M.$$

By results in op. cit., we have

$$\mathbb{Z}/3 = H^1(\mathbb{Z}/3, \mathrm{Pic}(S_{k(\gamma)})) \xrightarrow{\sim} \mathrm{Br}(S_k)/\mathrm{Br}(k),$$

generated by the class in $H^1(\mathbb{Z}/3, \mathrm{Pic}(S_{k(\gamma)}))$ of $\ell - L$ or $\ell - M$ (where we use (4.13) to identify elements with cohomology classes). The authors of op. cit. propose the following procedure to obtain a nontrivial Azumaya algebra on S_k:

(i) Find a divisor D defined over $k(\gamma)$ in the class $\ell - L$ or $\ell - M$,

(ii) Find a function in $k(S)$ whose divisor is the union of D and its Galois conjugates.

Unfortunately, the classes in $\mathrm{Pic}(S_{k(\gamma)})$ of sums of lines defined over $S_{k(\gamma)}$ fail to represent any nonzero elements of $H^1(\mathbb{Z}/3, \mathrm{Pic}(S_{k(\gamma)}))$, and the further field extension required to find suitable sums of lines accounts for much of the complication of the analysis of op. cit.

We show that (i) can be carried out by solving a norm equation. Then (ii) reduces to some linear algebra. For (i), we start with the further field extension $k(\gamma) \to K$ and the divisor $D := L'(2) - L''(0)$ in class $\ell - M$ (cf. op. cit). Denote by σ the element of $\mathrm{Gal}(K/k(\gamma))$ which sends α to $\theta\alpha$. For the line bundle $\mathcal{O}_{S_K}(D)$ to descend to $k(\gamma)$ we must supply an isomorphism

$$\mathcal{O}_{S_K}(L'(2) - L''(0)) \xrightarrow{\xi} \mathcal{O}_{S_K}(L'(0) - L''(1))$$

satisfying

$$\sigma^2(\xi) \circ \sigma(\xi) \circ \xi = 1. \tag{8.4}$$

Looking at the defining equations, we see ξ must be of the form

$$\xi = \varepsilon \frac{z + \beta t}{x + \alpha y}$$

for some $\varepsilon \in k(\gamma)$. Now the condition (8.4) is equivalent to

$$N_{K/k(\gamma)}(\varepsilon) = -C/A. \tag{8.5}$$

Concretely, if

$$\varepsilon = \lambda + \mu\alpha + \nu\alpha^2$$

with λ, μ, $\nu \in k(\gamma)$, then (8.5) expands as

$$\lambda^3 + \frac{B}{A}\mu^3 + \frac{B^2}{A^2}\nu^3 - 3\frac{B}{A}\lambda\mu\nu = -\frac{C}{A}. \tag{8.6}$$

Equation (8.6) has a solution, by the Hasse principle. There is also an *a priori* bound on the size of some solution [11]. An effective algorithm exists; see for example [7]. Algorithms from [1] and [6] have been implemented in `magma`.

Define $k' = k(\gamma)$. By descent we have a line bundle $\mathcal{E}$ on $S_{k'}$. Also by descent, a rational section of $\mathcal{E}$ is given by

$$\begin{aligned} f &= 1 + \sigma^2(\xi) + \sigma(\xi)\sigma^2(\xi) \\ &= \frac{(x+\theta\alpha y)(x+\theta^2\alpha y) + \sigma^2\varepsilon(x+\theta\alpha y)(z+\theta^2\beta t) + \sigma\varepsilon\sigma^2\varepsilon(z+\theta\beta t)(z+\theta^2\beta t)}{(x+\theta\alpha y)(x+\theta^2\alpha y)} \end{aligned}$$

Then, with respect to local trivializations of $\mathcal{E}$, the section f has a simple pole on $L''(0) + L''(1) + L''(2)$ and vanishes to order one along some cubic curve C. Hence

$$C = -2L - M + 4\ell$$

in $\mathrm{Pic}(S_{k'})$, and $C + K_S = -L + \ell$ is a divisor as desired.

We compute $C^2 = 1$ and $C \cdot K_S = -3$, which implies that its genus is zero, so C is geometrically a twisted cubic. Denoting by g the numerator of f, explicit defining equations of $C \subset S$ over K are $g = \sigma(g) = \sigma^2(g) = 0$. It is possible to express

$$g = g_0 + g_1\alpha + g_2\alpha^2$$

for g_0, g_1, $g_2 \in k'[x, y, z, t]$, and after a bit of algebra we find

$$\begin{aligned}
g_0 &= x^2 + \lambda xz + (B/A)\nu xt\gamma + \theta^2(B/A)\mu yt\gamma + \theta^2(B/A)\nu yz \\
&\quad + [\lambda^2 - (B/A)\mu\nu]z^2 + (B/A)(\lambda\nu - \mu^2)zt\gamma + (B/A)[(B/A)\nu^2 - \lambda\mu]t^2\gamma^2 \\
g_1 &= -xy + \theta^2\mu xz + \theta^2\lambda xt\gamma + \theta\lambda yz + \theta(B/A)\nu yt\gamma + [(B/A)\nu^2 - \lambda\mu]z^2 \\
&\quad + [(B/A)\mu\nu - \lambda^2]zt\gamma + (B/A)(\mu^2 - \lambda\nu)t^2\gamma^2 \\
g_2 &= \theta\nu xz + \theta\mu xt\gamma + y^2 + \mu yz + \lambda yt\gamma \\
&\quad + (\mu^2 - \lambda\nu)z^2 + [\lambda\mu - (B/A)\nu^2]zt\gamma + [\lambda^2 - (B/A)\mu\nu]t^2\gamma^2
\end{aligned}$$

Now C is defined over k' as a subvariety of S by the equations

$$g_0 = g_1 = g_2 = 0. \tag{8.7}$$

In fact, we have

$$\begin{aligned}
g_0(Ax - A\lambda z - B\nu\gamma t) + g_1(-B\nu z - B\mu\gamma t) + g_2(By - B\mu z - B\lambda\gamma t) \\
= Ax^3 + By^3 + Cz^3 + Dt^3
\end{aligned}$$

so (8.7) defines C over k' as a subvariety of $\mathbb{P}^3$. We have completed task (i).

For task (ii), we claim there exist linear polynomials ℓ_0, ℓ_1, $\ell_2 \in k'[x, y, z, t]$ such that the polynomial

$$h = g_0\ell_0 + g_1\ell_1 + g_2\ell_2 \tag{8.8}$$

is in $k[x, y, z, t]$ and is not proportional to $(Ax^3 + By^3 + Cz^3 + Dt^3)$. Knowing this, a modern linear algebra solver can effectively produce such ℓ_0, ℓ_1, and ℓ_2. Then the division algebra generated over $k(S)$ by noncommuting variables r and s subject to relations

$$r^3 = AD/BC, \qquad s^3 = h/x^3, \qquad sr = \theta rs,$$

is the restriction of an Azumaya algebra over S_k generating $\mathrm{Br}(S_k)/\mathrm{Br}(k)$.

To justify the claim, notice first that there exists a rational function on S_k whose divisor is $3H - C - \rho C - \rho^2 C$, where H is a hyperplane section and ρ is a generator of $\mathrm{Gal}(k'/k)$. Next, by a dimension computation, we have an isomorphism

$$H^0(\mathbb{P}^3_k, \mathcal{O}(3))/\langle Ax^3 + By^3 + Cz^3 + Dt^3\rangle \to H^0(S, 3H)$$

so this rational function must be of the form h/ℓ^3 (assuming that H is defined by the vanishing of the linear form ℓ). Finally, a syzygy computation shows that h can be expressed in the form (8.8). Indeed, (8.7) defines C in $\mathbb{P}^3$, so we know $\ell^d h$ lies in the ideal (g_0, g_1, g_2) of $k'[x, y, z, t]$, for some d. Suppose $d \geq 1$ and

$$\ell^d h = \sum_{i=0}^{2} g_i r_i,$$

with $r_i \in k'[x, y, z, t]$ for $i = 0, 1, 2$. Now it suffices to show that there exist s_0, s_1, $s_2 \in k'[x, y, z, t]$ such that $\sum_i g_i s_i = 0$, and ℓ divides $r_i - s_i$ for each i; then we have $\ell^{d-1} h = \sum_i g_i (r_i - s_i)/\ell$ and we can proceed inductively. In other words, it suffices to show that the map on Koszul complexes for (g_0, g_1, g_2), induced by the quotient map $k'[x, y, z, t] \to k'[x, y, z, t]/(\ell)$, gives rise to a surjection on the first homology modules. It is enough to verify this over the algebraic closure, and we are reduced to the case of (g_0, g_1, g_2) defining the twisted cubic, for which it is a standard computation.

References

[1] H. Cohen – *Advanced topics in computational number theory*, Graduate Texts in Mathematics, vol. 193, Springer-Verlag, New York, 2000.

[2] J.-L. Colliot-Thélène – "Surfaces rationnelles fibrées en coniques de degré 4", in *Séminaire de Théorie des Nombres, Paris 1988–1989*, Progr. Math., vol. 91, Birkhäuser Boston, Boston, MA, 1990, p. 43–55.

[3] J.-L. Colliot-Thélène, D. Coray and J.-J. Sansuc – "Descente et principe de Hasse pour certaines variétés rationnelles", *J. Reine Angew. Math.* **320** (1980), p. 150–191.

[4] J.-L. Colliot-Thélène, D. Kanevsky and J.-J. Sansuc – "Arithmétique des surfaces cubiques diagonales", in *Diophantine approximation and transcendence theory (Bonn, 1985)*, Lecture Notes in Math., vol. 1290, Springer, Berlin, 1987, p. 1–108.

[5] J.-L. Colliot-Thélène and J.-J. Sansuc – "La descente sur les variétés rationnelles", in *Journées de Géometrie Algébrique d'Angers,*

Juillet 1979/Algebraic Geometry, Angers, 1979, Sijthoff & Noordhoff, Alphen aan den Rijn, 1980, p. 223–237.

[6] C. Fieker – "Über relative Normgleichungen in algebraischen Zahlkörpern", Ph.D. Thesis, Technische Universität Berlin, 1997.

[7] C. Fieker, A. Jurk and M. Pohst – "On solving relative norm equations in algebraic number fields", *Math. Comp.* **66** (1997), no. 217, p. 399–410.

[8] V. A. Iskovskih – "A counterexample to the Hasse principle for systems of two quadratic forms in five variables", *Mat. Zametki* **10** (1971), p. 253–257.

[9] Y. I. Manin – *Cubic forms*, second ed., North-Holland Publishing Co., Amsterdam, 1986.

[10] P. Salberger – "Zero-cycles on rational surfaces over number fields", *Invent. Math.* **91** (1988), no. 3, p. 505–524.

[11] C. L. Siegel – "Normen algebraischer Zahlen", *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II* (1973), p. 197–215.

[12] P. Swinnerton-Dyer – "Two special cubic surfaces", *Mathematika* **9** (1962), p. 54–56.

[13] ———, "The Brauer group of cubic surfaces", *Math. Proc. Cambridge Philos. Soc.* **113** (1993), no. 3, p. 449–460.

[14] ———, "The solubility of diagonal cubic surfaces", *Ann. Sci. École Norm. Sup. (4)* **34** (2001), no. 6, p. 891–912.